GR 28

metric is given by

$$ds^2 = h(r)\, c^2 dt^2 - g(r)\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

→ generalised Schwarzschild $x^\mu = (t, r, \theta, \varphi)$

EoM for particle = geodesic

$$\frac{d^2 x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \overset{!}{=} 0$$

$$\frac{dx^\mu}{d\lambda} = \dot{x}^\mu$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{h'}{2h}$$

$$\Gamma^1_{00} = \frac{h'}{2g} \qquad \Gamma^1_{11} = \frac{g'}{2g} \qquad \Gamma^1_{22} = -\frac{r}{g}$$

$$\Gamma^1_{33} = -\frac{r\sin^2\theta}{g}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \qquad \Gamma^3_{23} = \Gamma^3_{23} = \cot\theta$$

coordinates by coordinates

$$p=0: \quad \ddot{t} + \Gamma^0_{01}\dot{t}\dot{r} + \Gamma^0_{10}\dot{r}\dot{t} = 0$$

$$\ddot{t} + \frac{h'}{h}\dot{t}\dot{r}$$

$$p=1; \quad 0 = \ddot{r} + \Gamma^1_{00}\dot{t}\dot{t} + \Gamma^1_{11}\dot{r}\dot{r} + \Gamma^1_{22}\dot{\theta}\dot{\theta} + \Gamma^1_{33}\dot{\varphi}\dot{\varphi}$$

$$= \ddot{r} + \frac{1}{2}\frac{h'}{g}\dot{t}^2 + \frac{g'}{2g}\dot{r}^2 - \frac{r}{g}\dot{\theta}^2 - \frac{r\sin^2\theta}{g}\dot{\varphi}^2$$

$$\rho = 2 : 0 = \ddot{\theta} + \Gamma^{2}_{12}\,\dot{r}\dot{\theta} + \Gamma^{2}_{21}\,\dot{\theta}\dot{r} + \Gamma^{2}_{33}\,\dot{\varphi}^{2}$$

$$= \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\,\dot{\varphi}^{2}$$

$$\rho = 3 : 0 = \ddot{\varphi} + 2\Gamma^{3}_{13}\,\dot{r}\dot{\varphi} + 2\Gamma^{3}_{23}\,\dot{\theta}\dot{\varphi}$$

$$= \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cos\theta\,\dot{\theta}\dot{\varphi}$$

GR 29

Schwarzschild metric

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2$$
$$- r^2 d\theta^2 - \sin^2\theta \, r^2 d\varphi$$
$$= g_{\mu\nu} dx^\mu dx^\nu$$

Killing vectors lead to conserved quantity along geodesic. Similarly action

$$S = \int_\gamma ds^2 = \int d\lambda \; g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \int d\lambda \, L$$

angular momentum : $l = r^2 \dot{\varphi}$ $\left(\frac{\partial L}{\partial \varphi} = 0 \right)$

energy : $F = \left(1-\frac{2M}{r}\right)\dot{t}$ $\left(\frac{dL}{d\lambda} = 0 \right)$

since $\frac{d}{d\lambda} L = 0 \Rightarrow L = -\varepsilon$

since $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, $\varepsilon \in \{\underset{\text{space}}{-1}, \underset{\text{light}}{0}, \underset{\text{time}}{1}\}$

if $\lambda = \tau$ $\quad g_{\mu\nu} u^\mu u^\nu = -\varepsilon$

$\theta = \frac{\pi}{2}$

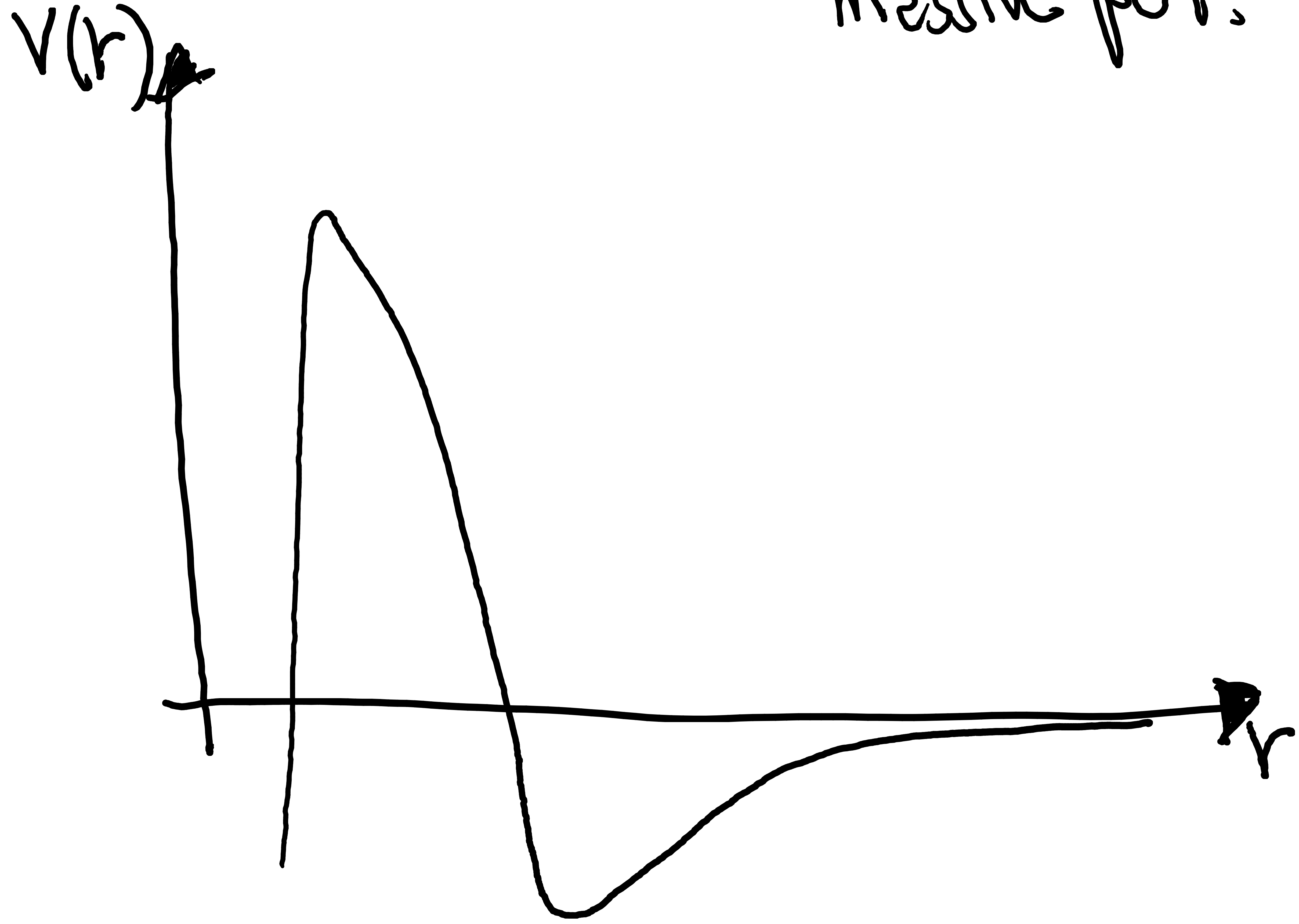
solve $\dot{\varphi} = f(\ell)$ $\quad \dot{t} = \tilde{f}(E)$, assume $\dot{\theta} = 0$

and plug back

$$\underbrace{\frac{E^2 + \varepsilon}{2}}_{= \mathcal{E}} = \underbrace{\frac{\dot{r}^2}{2}}_{E_{kin}} + \underbrace{\varepsilon \frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}}_{V_{eff}}$$

geodesic in Schwarzschild $\Leftrightarrow$ Newtonian particle in V_{eff}

massive part.

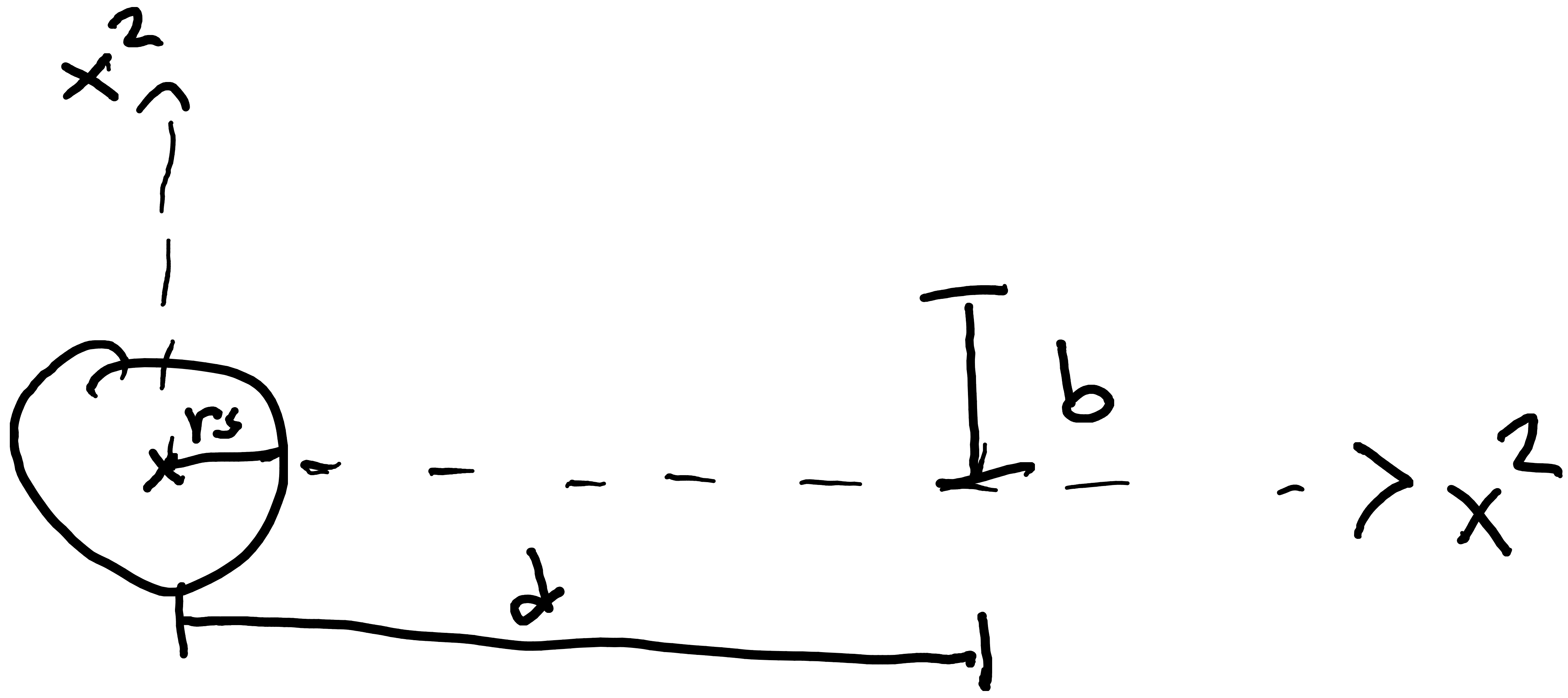


at $r \to \infty$ $\quad ds^2 \sim dt^2$

$$F = \left(1 - \frac{2M}{r}\right)\dot{t} \text{ cst} \Rightarrow F = F\big|_{r\to\infty} = \dot{t}_\infty$$

if $\lambda = \tau$: $\dfrac{dt}{d\tau} = \gamma_\infty = \left(1 - v_\infty^2\right)^{-1/2}$

Therefore $E = \dfrac{F^2 - \varepsilon}{2} > 0$



define $\dot{x}^1 = -v$

so that at $x^0 = t = 0$

$x^1 = d \qquad \frac{dx^1}{dx^0} = -\frac{v}{c} \qquad x^1 = r\cos\theta$

$x^2 = b \qquad \frac{dx^2}{dt} = 0 \qquad x^2 = r\sin\theta$

pola coord: $r^2 = d^2 + b^2 \simeq d^2$

$\theta = \mathrm{atan}\left(\frac{x_2}{x_1}\right)$

$$\dot{\theta} = \frac{d\varphi}{d\tau} = \frac{d\varphi}{dt}\,\frac{dt}{d\tau} = \gamma\,\frac{d}{dt}\left(\operatorname{atan}\frac{x^2}{x^1}\right)$$

$$= \gamma\,\frac{1}{1+\left(\frac{x^2}{x^1}\right)^2}\left(-\frac{x^2}{(x^1)^2}\right)\frac{dx^1}{dt}$$

$$= \gamma\,\frac{-x^2}{(x^1)^2+(x^2)^2}\,\frac{dx^1}{dt} = \gamma\,\frac{(-vb)}{d^2+b^2}$$

$$= -\frac{\gamma v b}{r^2}$$

Since

$$V_{eff} = 0$$

$$\Rightarrow r_\pm = \frac{L^2}{2r_s} \pm \sqrt{\underbrace{\frac{L^4}{4r_s^2} - L^2}_{\geq 0 \Rightarrow L > r_s}}$$

$$\Rightarrow 4r_s^2 > \frac{1}{1-v^2} b^2 v^2$$

$$\Rightarrow b^2 v^2 < 4r_s (1-v^2)$$