5.1. Heisenberg Spin Chain: Direct Diagonalization (4 points)

Consider the Heisenberg spin chain with periodic boundary conditions and Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{L} (\mathcal{I}_{j,j+1} - \mathcal{P}_{j,j+1}) \,. \tag{5.1}$$

Compute the spectrum of eigenvalues of $\mathcal{H}$ (energies) by direct diagonalization for the cases specified below. M denotes the number of up spins.

a) Compute the spectrum for a spin chain of length $L = 3$ and arbitrary number M of spin flips. How do the eigenstates organize into $\mathfrak{su}(2)$ multiplets? The generators of $\mathfrak{su}(2)$ are $Q^\alpha = \sum_i \sigma_i^\alpha/2$, $\alpha = x, y, z$, and $Q^\pm = Q^x \pm iQ^y$.

b) Restrict to cyclic states, i. e. identify all states that are equivalent under cyclic permutations of the spin chain sites. Compute the spectrum for the states with $L = 4$, $M = 2$, and for $L = 6$, $M = 2, 3$.

5.2. Heisenberg Spin Chain: Bethe Equations (4 points)

The Bethe equations for the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain read

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \,, \qquad k = 1, \dots, M \,. \tag{5.2}$$

Each solution to these equations (such that all finite u_k are distinct) defines an eigenstate of the Heisenberg Hamiltonian with M up spins (magnons). The energy E and momentum P are given by

$$E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right), \qquad \mathrm{e}^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} \,. \tag{5.3}$$

a) Use the Bethe equations to compute the energy spectrum for $L = 3$ and $M \leq 1$. States with $M > L/2$ are obtained from states with $M \leq L/2$ by flipping all spins. Compare to the results of problem 5.1 a). How are the $\mathfrak{su}(2)$ multiplets realized?

In the following, restrict to cyclic states, i. e. require $\mathrm{e}^{iP} = 1$.

b) Compute the energy spectrum for $L = 4$, $M = 2$, and for $L = 6$, $M = 2$. Compare to the results of problem 5.1 b).

c) Compute the energy spectrum from the Bethe equations for any L and $M = 2$.

d) The solution for $L = 6$, $M = 3$ is singular. Show that the regularized rapidities

$$u_1 = \frac{i}{2} + \varepsilon + c\,\varepsilon^6 \,, \qquad u_2 = -\frac{i}{2} + \varepsilon \,, \qquad u_3 = \frac{1 - 4u_1u_2}{4(u_1 + u_2)} + d(\varepsilon) \tag{5.4}$$

solve the Bethe equations and the condition $\mathrm{e}^{iP} = 1$ in the limit $\varepsilon \to 0$ for a suitable constant c and function $d(\varepsilon)$. Compare to your result of problem 5.1 b).

⟶

5.3. Coordinate Bethe Ansatz for the XXZ Spin Chain (4 points)

Consider the Hamiltonian $\mathcal{H}$ of the XXZ spin chain with periodic boundary conditions:

$$\mathcal{H} = \sum_{j=1}^{L} \mathcal{H}_{j,j+1}\,, \qquad \mathcal{H}_{j,k} = \frac{1}{2}\left[\sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y + \Delta(\sigma_j^z\sigma_k^z - \mathbf{1}_j\mathbf{1}_k)\right], \qquad \sigma_{L+1} \equiv \sigma_1\,, \quad (5.5)$$

which acts on a spin chain of length L with spins $|{\downarrow}\rangle = (0,1)^\mathsf{T}$ and $|{\uparrow}\rangle = (1,0)^\mathsf{T}$. Here, σ_j^i, $i \in \{x,y,z\}$ are the Pauli matrices acting on the spin state at site j.

a) Consider a general state with a single up spin

$$|\psi_1\rangle = \sum_{k=1}^{L} f(k)\,|k\rangle\,, \qquad |k\rangle = |{\downarrow\downarrow}\ldots{\downarrow\downarrow}\overset{k}{\uparrow}{\downarrow\downarrow}\ldots{\downarrow\downarrow}\rangle\,. \quad (5.6)$$

Convert the eigenvalue equation $\mathcal{H}|\psi_1\rangle = e_1|\psi_1\rangle$ to a finite difference equation for $f(k)$. Show that the one-magnon ansatz $f(k) = \mathrm{e}^{ipk}$ solves the equation, and that the dispersion relation becomes $e_1(p) = 2(\cos(p) - \Delta)$.

b) Now consider states with two up spins:

$$|\psi_2\rangle = \sum_{1\le k<\ell\le L} f(k,\ell)\,|k,\ell\rangle\,, \qquad |k,\ell\rangle = |{\downarrow}\ldots{\downarrow}\overset{k}{\uparrow}{\downarrow}\ldots{\downarrow}\overset{\ell}{\uparrow}{\downarrow}\ldots{\downarrow}\rangle\,. \quad (5.7)$$

Starting with the eigenvalue equation $\langle k,\ell|\mathcal{H}|\psi_2\rangle = e_2\,f(k,\ell)$, derive two difference equations for $f(k,\ell)$ by considering the two cases $k+1<\ell$ and $k+1=\ell$. Using the two-magnon ansatz

$$f(k,\ell) = \mathrm{e}^{ipk+iq\ell} + S(p,q)\,\mathrm{e}^{iqk+ip\ell}\,, \quad (5.8)$$

show that the dispersion relation is $e_2(p,q) = e_1(p) + e_1(q)$, and that the scattering phase $S(p,q)$ must satisfy

$$S(p,q) = -\frac{1+\mathrm{e}^{i(p+q)} - 2\Delta\,\mathrm{e}^{iq}}{1+\mathrm{e}^{i(p+q)} - 2\Delta\,\mathrm{e}^{ip}}\,. \quad (5.9)$$

Hint: First compute the action of $\mathcal{H}$ on neighboring spins $|{\downarrow\downarrow}\rangle$, $|{\uparrow\uparrow}\rangle$, $|{\downarrow\uparrow}\rangle$, and $|{\uparrow\downarrow}\rangle$.

c) Express $S(p,q)$ in terms of rapidities u, v, which are related to the momenta p, q via

$$\mathrm{e}^{ip} = \frac{u+i/2}{u-i/2}\,. \quad (5.10)$$

Taking the limit $\Delta \to 1$, show that the Bethe equations $\mathrm{e}^{ip_kL} = \prod_{j=1,j\neq k}^{M} S(p_j,p_k)$ for an M-magnon state become

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,. \quad (5.11)$$