Introduction to Integrability — Problem Set 4

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

Due: 25. 06. 2019

4.1. Heisenberg Magnet: Circle Solutions (3 points)

The Heisenberg magnet is described by two fields $\vartheta(t,x)$, $\varphi(t,x)$ (altitude and azimuth on the sphere), with equations of motion

$$\dot{\vartheta} = 2\cos(\vartheta)\vartheta'\varphi' + \sin(\vartheta)\varphi''\,, \qquad \dot{\varphi} = \cos(\vartheta)\varphi'^2 - \frac{\vartheta''}{\sin(\vartheta)}\,. \tag{4.1}$$

The momentum P, energy E, and angular momentum Q are given by

$$P = \int (1-\cos\vartheta)\varphi'\,\mathrm{d}x\,, \quad E = \frac{1}{2}\int \bigl(\vartheta'^2 + \sin^2(\vartheta)\varphi'^2\bigr)\,\mathrm{d}x\,, \quad Q = \int \cos(\vartheta)\,\mathrm{d}x\,. \tag{4.2}$$

a) Find the most general solution $\varphi(t,x)$ when $\vartheta(t,x) = \vartheta_0$ is a constant ($0 < \vartheta_0 < \pi$).

b) Impose periodic boundary conditions $\varphi(t,x+L) = \varphi(t,x)$. Note that the condition only needs to be satisfied modulo the equivalence $\varphi \equiv \varphi + 2\pi\mathbb{Z}$.

c) Compute the momentum P, energy E, and angular momentum Q of these solutions.

4.2. Spectral Curve for the Heisenberg Magnet (4 points)

The simplest finite-gap solution of the Heisenberg magnet has a spectral curve with a single branch cut. A suitable ansatz for the quasi-momentum $q(u)$ is

$$q'_\pm(u) = \pm\frac{au+b}{u^2\sqrt{u^2+cu+d}}\,. \tag{4.3}$$

The $\pm$ labels the two branches of the function, which are connected by a branch cut stretching between two branch points at the zeros of the square root. Let A be a counterclockwise cycle around the branch cut, and B a path going from $u = \infty_-$ on the one branch trough the cut and back to $u = \infty_+$ on the other branch. Then $q'(u)$ should satisfy

$$\oint_A q'_+(u)\,\mathrm{d}u = 0\,, \qquad \frac{1}{2\pi}\int_B q'(u)\,\mathrm{d}u = n \in \mathbb{Z}\,, \qquad I = \frac{1}{2\pi i}\oint_A u\,q'_+(u)\,\mathrm{d}u\,, \tag{4.4}$$

where I is called the "filling" of the cut. Moreover, the length L, momentum P, energy E, and angular momentum Q appear in series expansions of $q_+(u)$ as

$$u\to 0:\quad q_+(u) = \frac{L}{u} - \frac{P}{2} + \frac{uE}{4} + \mathcal{O}(u^2)\,, \qquad u\to\infty_+:\quad q_+(u) = \frac{Q}{u} + \mathcal{O}(u^{-2})\,. \tag{4.5}$$

a) Express the coefficients a, b, and c in terms of d, L, and I using the A-cycle conditions and series expansions. *Hint:* A-cycle integrals are sums of residues at $u = 0, \infty$.

b) Integrate $q'(u)$ to $q(u)$, and find d in terms of n and L by the B-cycle condition. Fix the integration constant by the vanishing of $q_+(u)$ at $u = \infty$. *Hint:* Compute $\bigl(\sqrt{A-2Bu+Du^2}/u\bigr)'$. The square root has different signs on the two branches.

c) Expand $q_+(u)$ at $u = 0, \infty$, and find expressions for P, Q, and E by matching (4.5). Compare the results to your results of 4.1 c)

⟶

4.3. Zamolodchikov–Faddeev Algebra and Sine–Gordon (5 points)

Consider operators $A_i(u)$ that satisfy the commutation relations

$$A_i(u_2)A_j(u_1) = \sum_{k,\ell} S_{ij}^{kl}(u_{12})A_k(u_1)A_\ell(u_2)\,, \qquad u_{12} = u_1 - u_2\,. \tag{4.6}$$

For i different from j, $A_i(u)$ and $A_j(u)$ are independent operators. $A_j(u)$ can be thought of as creating a particle of type (flavor) j with rapidity u: $A_j(u)|0\rangle = |A_j(u)\rangle$. The scattering factors $S_{ij}^{k\ell}(u)$ are scalar functions of u that form the scatterin matrix $S(u)$. Obtain consistency conditions for the matrix $S(u)$ from the commutation relations by

a) taking the limit $u_{12} \to 0$.

b) iterating (4.6) twice. Draw a diagram for the resulting condition.

c) relating the product $A_i(u_3)A_j(u_2)A_k(u_1)$ back to a sum over $A_r(u_1)A_p(u_2)A_q(u_3)$ by iteratively applying (4.6) in two different ways. Draw a diagram for the resulting condition. Interpret triple products of $A_i(u_k)$ as states of a three-site system, and write the condition as an equation for $S_{12}(u)$ and $S_{23}(u)$, where the indices denote the sites on which the respective matrix acts.

Specialize to a model with particles A and $\bar{A}$, and three different scattering amplitudes:

$$S_\mathrm{I}(u) = S_{AA}^{AA}(u) = S_{\bar{A}\bar{A}}^{\bar{A}\bar{A}}(u)\,, \quad S_\mathrm{T}(u) = S_{A\bar{A}}^{\bar{A}A}(u) = S_{\bar{A}A}^{A\bar{A}}(u)\,, \quad S_\mathrm{R}(u) = S_{\bar{A}A}^{\bar{A}A}(u) = S_{A\bar{A}}^{A\bar{A}}(u)\,.$$

d) Write the relation (4.6) for all combinations of A and $\bar{A}$. Show that the conditions from c) with $S_x \equiv S_x(u_{12})$, $S'_x \equiv S_x(u_{13})$, and $S''_x \equiv S_x(u_{23})$, $x \in \{\mathrm{I}, \mathrm{T}, \mathrm{R}\}$ amount to

$$S_\mathrm{I}S'_\mathrm{T}S''_\mathrm{R} = S_\mathrm{T}S'_\mathrm{I}S''_\mathrm{R} + S_\mathrm{R}S'_\mathrm{R}S''_\mathrm{T}\,, \qquad S_\mathrm{I}S'_\mathrm{R}S''_\mathrm{I} = S_\mathrm{R}S'_\mathrm{I}S''_\mathrm{R} + S_\mathrm{T}S'_\mathrm{R}S''_\mathrm{T}\,,$$
$$S_\mathrm{R}S'_\mathrm{T}S''_\mathrm{I} = S_\mathrm{R}S'_\mathrm{I}S''_\mathrm{T} + S_\mathrm{T}S'_\mathrm{R}S''_\mathrm{R}\,. \tag{4.7}$$

e) Show that the existence of a non-trivial solution to (4.7) requires that the quantity

$$\Delta = \frac{S_\mathrm{I}^2(u) + S_\mathrm{T}^2(u) - S_\mathrm{R}^2(u)}{2S_\mathrm{I}(u)S_\mathrm{T}(u)} \tag{4.8}$$

is independent of u.

Hint: Write (4.7) in matrix form $M \cdot \left(S''_\mathrm{I}, S''_\mathrm{R}, S''_\mathrm{T}\right)^\mathsf{T} = 0$.

Remark: This type of system is realized in the Sine–Gordon model with Lagrange density $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{m^2}{\beta^2}\cos(\beta\phi)$. In that case, $A(u)$ and $\bar{A}(u)$ are the soliton and anti-soliton solutions, with scattering factors

$$S_\mathrm{I}(u) = \sinh\left[\frac{8\pi}{\eta}\,(i\pi - u)\right]f(u)\,, \quad S_\mathrm{T}(u) = \sinh\left[\frac{8\pi}{\eta}\,u\right]f(u)\,, \quad S_\mathrm{R}(u) = i\sin\left[\frac{8\pi^2}{\eta}\right]f(u)\,,$$

and with $\Delta = -\cos(8\pi^2/\eta)$, where $1/\eta = 1/\beta^2 - 1/(8\pi)$.