3.1. Flat Connection and Parallel Transport (2 points)

Let $A(x) = A_\mu(x)\,\mathrm{d}x^\mu$ be a matrix-valued connection one-form with corresponding derivative operator $D_\mu = \partial_\mu - A_\mu(x)$, where $\partial_\mu = \partial/\partial x^\mu$ is the ordinary derivative operator. Define the parallel transport operator

$$U^{10} = \overleftarrow{\mathrm{P}} \exp \int_{x_0}^{x_1} A = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_1} \overleftarrow{\mathrm{P}}\big[A_{\nu_1}(y_1) \ldots A_{\nu_n}(y_n)\big]\mathrm{d}y_1^{\nu_1} \ldots \mathrm{d}y_n^{\nu_n}\,, \tag{3.1}$$

where $\overleftarrow{\mathrm{P}}[\ldots]$ orders all products by position on the the path from x_0 to x_1, with factors closer to x_1 to the left.

a) Show that the flatness condition $[D_\mu, D_\nu] = 0$ is equivalent to $\mathrm{d}A = A \wedge A$.

b) Assuming that the connection $A(x)$ is flat, show that

$$\frac{\partial}{\partial x_1^\mu} U^{10} = A_\mu(x_1)\, U^{10}\,, \qquad \frac{\partial}{\partial x_0^\mu} U^{10} = U^{10}\, A_\mu(x_0)\,. \tag{3.2}$$

3.2. Inverse Scattering Method for the KdV Equation (4 points)

We will use the inverse scattering method to find solutions to the KdV equation. The GLM equation reads

$$K(x,y) + \hat{r}(x+y) + \int_x^\infty K(x,z)\, \hat{r}(z+y)\, \mathrm{d}z = 0\,, \qquad h(x) = -2\,\frac{\partial}{\partial x} K(x,x)\,. \tag{3.3}$$

For purely solitonic solutions, the potential $h(x)$ is reflectionless, and $\hat{r}(x)$ reduces to

$$\hat{r}(x) = \sum_{j=1}^{N} \lambda_j(t)\, \mathrm{e}^{-\kappa_j x}\,, \qquad \lambda_j(t) = \lambda_j(0)\, \mathrm{e}^{8\kappa_j^3 t}\,, \qquad \kappa_j > 0\,. \tag{3.4}$$

First, consider the single-soliton case $N = 1$, with $\kappa_1 \equiv \kappa$ and $\lambda_j \equiv \lambda$.

a) Obtain the solution $h(x)$ by solving the GLM equation (3.3) using the separation ansatz $K(x,y) = K(x)\, \mathrm{e}^{-\kappa y}$

b) Find the minimum $x_0(t)$ of $h(x)$ as a function of κ and $\lambda(t)$. Express $h(x)$ in terms of x, t, the velocity v of $x_0(t)$, and $x_0(0)$.

Now consider the two-soliton case $N = 2$. Assume that $K(x,y)$ separates to

$$K(x,y) = K_1(x)\, \mathrm{e}^{-\kappa_1 y} + K_2(x)\, \mathrm{e}^{-\kappa_2 y}\,. \tag{3.5}$$

c) Show that the GLM equation can be written as a matrix equation $AK + L = 0$, with

$$A_{i,j} = \delta_{i,j} + \lambda_i \frac{\mathrm{e}^{-(\kappa_i+\kappa_j)x}}{\kappa_i + \kappa_j}\,, \qquad K_i = K_i(x)\,, \qquad L_i = \lambda_i\, \mathrm{e}^{-\kappa_i x}\,. \tag{3.6}$$

d) Derive that

$$K(x,x) = K_1(x)\, \mathrm{e}^{-\kappa_1 x} + K_2(x)\, \mathrm{e}^{-\kappa_2 x} = \frac{\partial}{\partial x} \log\det A(x)\,. \tag{3.7}$$

Using this formula, show that $h(x,t) \to 0$ for $x \to \pm\infty$ and for any value of t.

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3.3. Asymptotics of Two KdV Solitons (3 points)

Consider the two-soliton solution (3.7) with the matrix A in (3.6), and with $0 < \kappa_1 < \kappa_2$.

a) Compare the magnitudes of the quantities $\lambda_1(t)\,\mathrm{e}^{-2\kappa_1 x}$, $\lambda_2(t)\,\mathrm{e}^{-2\kappa_2 x}$, and 1 as x varies from $-\infty$ to ∞, for the two cases $t \ll 1$ and $t \gg 1$.

b) For $t \ll 0$, compute the leading term $\det A_i^-$ of $\det A$ near $x \approx 4\kappa_i^2 t$, for $i = 1, 2$. Drop all terms that are irrelevant for $h(x) \sim (\partial/\partial x)^2 \log\det A$. In the same way, compute the leading terms $\det A_i^+$ for $t \gg 0$.

c) Show that the resulting expressions $h_i^\pm(x) = -2(\partial/\partial x)^2 \log\det A_i^\pm$ all take the form of single solitons. Compute the parameters $\kappa_i^\pm$, $\lambda_i^\pm$ of those single-soliton solutions in terms of the two-soliton parameters $(\kappa_1, \kappa_2, \lambda_1, \lambda_2)$.

d) Compare the minima $x_{0,i}^\pm$ of $h_i^\pm(x)$ at $t \ll 0$ and at $t \gg 0$. Interpret the result: What is the effect of the scattering on the two solitons? Sketch the result.

3.4. KdV Solitons: Verification and Visualization (3 points)

Via $h(x) = -2(\partial/\partial x)^2 \log\det A$, the $N \times N$ matrix A defined by (3.6) actually yields a valid N-soliton solution $h(x)$ for any N.

This is a MATHEMATICA exercise. Please hand in a printout of your MATHEMATICA notebook as well as the digital file.

a) Validate the $N = 2$ solution by verifying with MATHEMATICA that $\dot{h} - 6hh' + h'''$ indeed vanishes.

Hint: Functions are defined as `f[x_]:=`(expression involving `x`). Matrices are best defined with `Table[.., {i,2}, {j,2}]` (for a 2×2 matrix). There is a built-in function `KroneckerDelta[i,j]`. `Det[..]` computes the determinant, `Log[..]` is the natural logarithm. Derivatives can be computed with `D[..]`, e. g. `D[f[x], {x,2}]` would be $f''(x)$. Use `Simplify[..]` to simplify expressions. An extremely useful construct is `expr /. X -> Y`, which replaces all occurences of `X` in `expr` with `Y`. Get help with the F1 key.

b) Plot the $N = 2$ solution for $\kappa_1 = 1/2$, $\kappa_2 = 1/\sqrt{2}$, $\lambda_i(0) = 2\kappa_i$. Plots can be generated with `Plot[h[x,t], {x,-50,50}]`. Repeat for various values of `t`. To show the full plotting region, use `PlotRange -> ...`

Try `Manipulate[Plot[h[x,t], {x,-50,50}], {t,-20,20}]`, play with the slider.

Advanced: Include also the single-soliton solutions of problem 3.3 c) in the plot, possibly with different colors or line styles (using `PlotStyle -> ..`).

c) Validate the $N = 3$ solution by evaluating the expression $\dot{h} - 6hh' + h'''$. The result is bulky and not easily simplified. Instead, replace κ_i, λ_i, and x by random numbers (generated with `RandomReal[]`) to check that the expression vanishes.

d) Plot the $N = 3$ solution for two different choices of parameters: One where all three solitons scatter at once, and one where the three solitons only scatter pairwise. Use `Manipulate[..]` as in b) to visualize the result.