Introduction to Integrability — Problem Set 2

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

Due: 21. 05. 2019

2.1. The Neumann Model (5 points)

The Neumann model describes a particle on a sphere S^{N-1} subject to harmonic potentials of different magnitudes a_k in the various dimensions $k = 1, \dots, N$. The Newton equations of motion are

$$\ddot{x}_k = -a_k x_k + \sum_\ell \big(a_\ell x_\ell^2 - \dot{x}_\ell^2\big) x_k \,, \tag{2.1}$$

where $(x_1, \dots, x_N) \in \mathrm{S}^{N-1}$. In the Hamiltonian formulation, the phase space has coordinates x_k, y_k, $k = 1, \dots, N$, with canonical Poisson structure and Hamiltonian

$$\{x_i, y_j\} = \delta_{ij} \,, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0 \,, \tag{2.2}$$

$$H = \frac{1}{2} \sum_k a_k F_k \,, \qquad F_k = x_k^2 + \sum_{\ell \neq k} \frac{J_{k\ell}^2}{a_k - a_\ell} \,, \qquad J_{k\ell} = x_k y_\ell - x_\ell y_k \,. \tag{2.3}$$

The quantities F_k are individually conserved, and satisfy $\sum_k F_k = 1$.

a) Show that the Hamiltonian equations of motion $\dot{x}_i = \partial H / \partial y_i$, $\dot{y}_i = -\partial H / \partial x_i$ take the form

$$\dot{X} = -J \cdot X \,, \qquad \dot{Y} = -J \cdot Y - L_0 \cdot X \,, \tag{2.4}$$

where $X = (x_1, \dots, x_N)^\mathsf{T}$, $Y = (y_1, \dots, y_N)^\mathsf{T}$, $J = XY^\mathsf{T} - YX^\mathsf{T}$, and $(L_0)_{ij} = \delta_{ij} a_i$.

b) Show that with $K = XX^\mathsf{T}$, the equations of motion can also be written as

$$\dot{K} = -[J, K] \,, \qquad \dot{J} = [L_0, K] \,, \tag{2.5}$$

and that these are equivalent to the Lax equation $\dot{L} = [M, L]$ for the matrices

$$L(z) = L_0 + zJ - z^2 K \,, \qquad M(z) = -zK \,. \tag{2.6}$$

c) Show that the spectral curve equation $0 = \det(L(z) - \lambda)$ with a suitable rank-two matrix P can be written as $0 = \det(L_0 - \lambda)\det(1 + P)$. Also show that

$$\det(1 + P) = 1 - z^2 \big(V^2 + U(1 - W)\big) \,, \tag{2.7}$$

with $U = \sum_k x_k^2 / (a_k - \lambda)$, $V = \sum_k x_k y_k / (a_k - \lambda)$, and $W = \sum_k y_k^2 / (a_k - \lambda)$.
Hint: Express P in the basis $\{v_1 = (L_0 - \lambda)^{-1} \cdot X,\ v_2 = (L_0 - \lambda)^{-1} \cdot Y\}$.

d) Use the relation $\det(1 + P) = 1 + z^2 \sum_k F_k / (\lambda - a_k)$ as well as the birational transformation $z' = z^{-1} \prod_{i=1}^N (\lambda - a_i)$ to show that there are parameters b_i (that are functions of a_k and F_k) such that the spectral curve equation can be written as

$$z'^2 = -\prod_{i=1}^{N} (\lambda - a_i) \prod_{j=1}^{N-1} (\lambda - b_i) \,. \tag{2.8}$$

e) The relation (2.8) describes a hyperelliptic curve $z' = \pm(\dots)^{1/2}$. How many branch points / branch cuts connect the two z' branches? What is the genus of the curve?
Hint: One branch point is at $\lambda = \infty$. Figure each of the two z' branches as a ball, and each branch cut as a cylindrical tube connecting the two balls.

⟶

2.2. Conservation Laws for the KdV Equation (4 points)

We want to find conserved local charges for the KdV equation

$$\dot{h} = 6hh' - h''' \,, \qquad h = h(t, x) \,. \tag{2.9}$$

a) Consider the change of field variable $h \to w$, with

$$h = w + i\varepsilon w' - \varepsilon^2 w^2 \,. \tag{2.10}$$

Show that h satisfies (2.9) if w satisfies

$$\dot{w} = \frac{\partial}{\partial x}\left(3w^2 - w'' - 2\varepsilon^2 w^3\right) . \tag{2.11}$$

b) Now let w be a power series in ε,

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n \,, \qquad w_n = w_n(t, x) \,. \tag{2.12}$$

By expanding (2.10) in ε, obtain a recursion relation for $w_n(t, x)$.

c) Show that the relation (2.11) implies

$$F_n := \int_{-\infty}^{+\infty} w_n \mathrm{d}x \,, \qquad \dot{F}_n = 0 \,, \tag{2.13}$$

where we assume that h and all its derivatives decay at $|x| \to \infty$. Observe that F_n is only non-zero for even n, and write F_0, F_2, and F_4 as integrals over polynomials in h and its derivatives.

2.3. KdV Solitons (3 points)

Look for solutions to the KdV equation $\dot{h} = 6hh' - h'''$ with constant velocity v by assuming $h(t, x) = f(x - vt)$. Show that f satisfies the equation

$$\tfrac{1}{2} f'^2 = f^3 + \tfrac{1}{2} v f^2 + \alpha f + \beta \,, \tag{2.14}$$

with α and β constant. Assuming that h vanishes at $|x| \to \infty$, what must be the values of α and β? Solve the differential equation (2.14) to recover the one-soliton solution to the KdV equation.