Introduction to Integrability **Problem Set 1**

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1.1. Kepler Problem and Action-Angle Variables (6 points)

Consider a mass m in a centrally symmetric potential $V(r)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In the Kepler problem, $V(r) = C/r$, but we can keep $V(r)$ general. The Hamiltonian reads

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(r)\,, \tag{1.1}$$

where p_i is the momentum conjugate to x_i.

a) The phase space has six dimensions. Hence the system is integrable if there are three integrals of motion in involution. Show that the components $J_i = \varepsilon_{ijk} x_j p_k$ of the angular momentum vector $J = (J_1, J_2, J_3)$ are conserved.

b) Compute the Poisson bracket $\{J_i, J_j\}$ to show that the components J_i are *not* in involution. Do you recognize their algebra?

c) The Hamiltonian H itself is another conserved quantity. Show that the combination $J^2 = J_1^2 + J_2^2 + J_3^2$ commutes with both H and J_i. Hence $P := (H, J^2, J_3)$ are three integrals of motion in involution.

d) Due to the rotational symmetry, it is useful to employ spherical coordinates r, ϑ, φ,

$$x_1 = r \sin\vartheta \cos\varphi\,, \quad x_2 = r \sin\vartheta \sin\varphi\,, \quad x_3 = r\cos\vartheta\,. \tag{1.2}$$

What are the momenta $p := (p_r, p_\vartheta, p_\varphi)$ conjugate to the positions $q := (r, \vartheta, \varphi)$?

e) Rewrite the conserved quantities P in terms of spherical coordinates (q, p). Invert these relations to express the momenta p in terms of P and q. Observe that the variables q are *separated*: p_r only depends on r, p_ϑ only depends on ϑ, and p_φ only depends on φ.

f) The generating function for a canonical transformation from (q, p) to (Q, P) is

$$S(q, P) = \int^r p_r(q', P)\,\mathrm{d}r' + \int^\vartheta p_\vartheta(q', P)\,\mathrm{d}\vartheta' + \int^\varphi p_\varphi(q', P)\,\mathrm{d}\varphi' \tag{1.3}$$

The positions conjugate to the momenta P_i are $Q_i = \partial S/\partial P_i$. Verify that $\dot{Q}_{J^2} = 0$, $\dot{Q}_{J_3} = 0$, and $\dot{Q}_H = 1$. Use the last equation to derive the standard solution of the Kepler problem

$$t - t_0 = \int_{r_0}^{r} \frac{m\,\mathrm{d}r'}{\sqrt{2m(H - V(r')) - J^2/r'^2}}\,. \tag{1.4}$$

Hint: Write the time derivative as $d/dt = \dot{r}\, d/dr + \dot{\vartheta}\, d/d\vartheta + \dot{\varphi}\, d/d\varphi$, and use Hamiltons equation of motion $\dot{r} = \partial H/\partial p_r$ etc.

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1.2. Euler Top and Lax Pair (6 points)

Consider a rotating body attached to a fixed point without external forces. In the co-moving frame, where the coordinate axes are aligned with the principal moments of inertia I_i of the body, the Hamiltonian reads

$$H = \sum_{i=1}^{3} \frac{J_i^2}{2I_i}\,, \tag{1.5}$$

where $J_i = I_i \omega_i$ are the components of the angular momentum $J = (J_1, J_2, J_3)$, with $\omega = (\omega_1, \omega_2, \omega_3)$ the rotation vector of the comoving frame.

a) Using the Poisson structure $\{J_i, J_j\} = \varepsilon_{ijk} J_k$, derive the equation of motion for J:

$$\frac{dJ_i}{dt} = \varepsilon_{ijk}\omega_j J_k\,. \tag{1.6}$$

Hint: Use the Hamilton equation of motion $dF/dt = -\{H, F\}$.

b) Show that the square of the angular momentum $J^2 = \sum_i J_i^2$ is conserved. Since the system has only two degrees of freedom, and the Hamiltonian H itself is also conserved, the system is integrable. Express J_2 and J_3 in terms of H, J^2, and J_1.

c) As a candidate Lax pair, consider the matrices $L_{ij} = \varepsilon_{ijk} J_k$ and $M_{ij} = -\varepsilon_{ijk}\omega_k$. Show that the equation of motion (1.6) can be written as

$$\frac{dL}{dt} = [M, L]\,. \tag{1.7}$$

d) Calculate the first few conserved quantities $\mathrm{tr}(L^n)$, and observe that the Hamiltonian is not among them. Why not?

e) Show that there exist rescaled variables $\mathcal{J}_i := \alpha_i J_i$ such that the equations of motion take the form

$$\frac{d\mathcal{J}_i}{dt} = 2\mathcal{J}_j\mathcal{J}_k\,, \tag{1.8}$$

where (i, j, k) is any cyclic permutation of $(1, 2, 3)$.

f) Show that the Lax equation $dL(z)/dt = [M(z), L(z)]$ for the following 2×2 matrices $L(z)$ and $M(z)$ is also equivalent to the equations of motion:

$$\begin{aligned} L(z) &= (1 - z^2)\mathcal{J}_1\sigma_1 + (1 + z^2)\mathcal{J}_2\, i\sigma_2 - 2z\mathcal{J}_3\sigma_3\,, \\ M(z) &= z\mathcal{J}_1\sigma_1 - z\mathcal{J}_2\, i\sigma_2 + \mathcal{J}_3\sigma_3\,, \end{aligned} \tag{1.9}$$

where σ_i are the Pauli matrices.

g) Compute all the independent integrals of motion $\mathrm{tr}(L(z)^n)$. Verify that they are indeed conserved, and express them in terms of H and J^2. This shows that $L(z)$, $M(z)$ form a sufficient Lax pair for the system, with spectral parameter z.

Hint: Make use of the Pauli matrix algebra $\sigma_k\sigma_\ell = i\varepsilon_{k\ell m}\sigma_m + \delta_{k\ell}$.