# Introduction to Integrability

Leibniz University Hannover, Summer 2019

Dr. Till Bargheer

## Contents

Many of the following problems are derived and adapted from a lecture course by Niklas Beisert at ETH Zürich (from 2016), which can be found at Niklas' webpage: link

Some problems are derived from a lecture course by Florian Löbbert at Humboldt-University Berlin (also from 2016), see https://qft.physik.hu-berlin.de/integrable-models/

## 1.1. Kepler Problem and Action-Angle Variables (6 points)

Consider a mass $m$ in a centrally symmetric potential $V(r)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In the Kepler problem, $V(r) = C/r$, but we can keep $V(r)$ general. The Hamiltonian reads

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V(r)\,, \tag{1.1}$$

where $p_i$ is the momentum conjugate to $x_i$.

**a)** The phase space has six dimensions. Hence the system is integrable if there are three integrals of motion in involution. Show that the components $J_i = \varepsilon_{ijk} x_j p_k$ of the angular momentum vector $J = (J_1, J_2, J_3)$ are conserved.

**b)** Compute the Poisson bracket $\{J_i, J_j\}$ to show that the components $J_i$ are *not* in involution. Do you recognize their algebra?

**c)** The Hamiltonian $H$ itself is another conserved quantity. Show that the combination $J^2 = J_1^2 + J_2^2 + J_3^2$ commutes with both $H$ and $J_i$. Hence $P := (H, J^2, J_3)$ are three integrals of motion in involution.

**d)** Due to the rotational symmetry, it is useful to employ spherical coordinates $r$, $\vartheta$, $\varphi$,

$$x_1 = r \sin\vartheta \cos\varphi\,, \quad x_2 = r \sin\vartheta \sin\varphi\,, \quad x_3 = r\cos\vartheta\,. \tag{1.2}$$

What are the momenta $p := (p_r, p_\vartheta, p_\varphi)$ conjugate to the positions $q := (r, \vartheta, \varphi)$?

**e)** Rewrite the conserved quantities $P$ in terms of spherical coordinates $(q, p)$. Invert these relations to express the momenta $p$ in terms of $P$ and $q$. Observe that the variables $q$ are *separated*: $p_r$ only depends on $r$, $p_\vartheta$ only depends on $\vartheta$, and $p_\varphi$ only depends on $\varphi$.

**f)** The generating function for a canonical transformation from $(q, p)$ to $(Q, P)$ is

$$S(q, P) = \int^r p_r(q', P)\,\mathrm{d}r' + \int^\vartheta p_\vartheta(q', P)\,\mathrm{d}\vartheta' + \int^\varphi p_\varphi(q', P)\,\mathrm{d}\varphi' \tag{1.3}$$

The positions conjugate to the momenta $P_i$ are $Q_i = \partial S/\partial P_i$. Verify that $\dot{Q}_{J^2} = 0$, $\dot{Q}_{J_3} = 0$, and $\dot{Q}_H = 1$. Use the last equation to derive the standard solution of the Kepler problem

$$t - t_0 = \int_{r_0}^{r} \frac{m\,\mathrm{d}r'}{\sqrt{2m(H - V(r')) - J^2/r'^2}}\,. \tag{1.4}$$

*Hint:* Write the time derivative as $d/dt = \dot{r}\, d/dr + \dot{\vartheta}\, d/d\vartheta + \dot{\varphi}\, d/d\varphi$, and use Hamiltons equation of motion $\dot{r} = \partial H/\partial p_r$ etc.

$\longrightarrow$

## 1.2. Euler Top and Lax Pair (6 points)

Consider a rotating body attached to a fixed point without external forces. In the co-moving frame, where the coordinate axes are aligned with the principal moments of inertia $I_i$ of the body, the Hamiltonian reads

$$H = \sum_{i=1}^{3} \frac{J_i^2}{2I_i}\,, \tag{1.5}$$

where $J_i = I_i\omega_i$ are the components of the angular momentum $J = (J_1, J_2, J_3)$, with $\omega = (\omega_1, \omega_2, \omega_3)$ the rotation vector of the comoving frame.

**a)** Using the Poisson structure $\{J_i, J_j\} = \varepsilon_{ijk}J_k$, derive the equation of motion for $J$:

$$\frac{dJ_i}{dt} = \varepsilon_{ijk}\omega_j J_k\,. \tag{1.6}$$

*Hint:* Use the Hamilton equation of motion $dF/dt = -\{H, F\}$.

**b)** Show that the square of the angular momentum $J^2 = \sum_i J_i^2$ is conserved. Since the system has only two degrees of freedom, and the Hamiltonian $H$ itself is also conserved, the system is integrable. Express $J_2$ and $J_3$ in terms of $H$, $J^2$, and $J_1$.

**c)** As a candidate Lax pair, consider the matrices $L_{ij} = \varepsilon_{ijk}J_k$ and $M_{ij} = -\varepsilon_{ijk}\omega_k$. Show that the equation of motion (1.6) can be written as

$$\frac{dL}{dt} = [M, L]\,. \tag{1.7}$$

**d)** Calculate the first few conserved quantities $\mathrm{tr}(L^n)$, and observe that the Hamiltonian is not among them. Why not?

**e)** Show that there exist rescaled variables $\mathcal{J}_i := \alpha_i J_i$ such that the equations of motion take the form

$$\frac{d\mathcal{J}_i}{dt} = 2\mathcal{J}_j\mathcal{J}_k\,, \tag{1.8}$$

where $(i, j, k)$ is any cyclic permutation of $(1, 2, 3)$.

**f)** Show that the Lax equation $dL(z)/dt = [M(z), L(z)]$ for the following $2 \times 2$ matrices $L(z)$ and $M(z)$ is also equivalent to the equations of motion:

$$\begin{aligned} L(z) &= (1 - z^2)\mathcal{J}_1\sigma_1 + (1 + z^2)\mathcal{J}_2\, i\sigma_2 - 2z\mathcal{J}_3\sigma_3\,, \\ M(z) &= z\mathcal{J}_1\sigma_1 - z\mathcal{J}_2\, i\sigma_2 + \mathcal{J}_3\sigma_3\,, \end{aligned} \tag{1.9}$$

where $\sigma_i$ are the Pauli matrices.

**g)** Compute all the independent integrals of motion $\mathrm{tr}(L(z)^n)$. Verify that they are indeed conserved, and express them in terms of $H$ and $J^2$. This shows that $L(z)$, $M(z)$ form a sufficient Lax pair for the system, with spectral parameter $z$.

*Hint:* Make use of the Pauli matrix algebra $\sigma_k\sigma_\ell = i\varepsilon_{k\ell m}\sigma_m + \delta_{k\ell}$.

# Introduction to Integrability — Problem Set 2

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

09. 07. 2019

## 2.1. The Neumann Model (5 points)

The Neumann model describes a particle on a sphere $\mathrm{S}^{N-1}$ subject to harmonic potentials of different magnitudes $a_k$ in the various dimensions $k = 1, \ldots, N$. The Newton equations of motion are

$$\ddot{x}_k = -a_k x_k + \sum_\ell \bigl(a_\ell x_\ell^2 - \dot{x}_\ell^2\bigr) x_k \,, \tag{2.1}$$

where $(x_1, \ldots, x_N) \in \mathrm{S}^{N-1}$. In the Hamiltonian formulation, the phase space has coordinates $x_k$, $y_k$, $k = 1, \ldots, N$, with canonical Poisson structure and Hamiltonian

$$\{x_i, y_j\} = \delta_{ij} \,, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0 \,, \tag{2.2}$$

$$H = \frac{1}{2} \sum_k a_k F_k \,, \qquad F_k = x_k^2 + \sum_{\ell \neq k} \frac{J_{k\ell}^2}{a_k - a_\ell} \,, \qquad J_{k\ell} = x_k y_\ell - x_\ell y_k \,. \tag{2.3}$$

The quantities $F_k$ are individually conserved, and satisfy $\sum_k F_k = 1$.

**a)** Show that the Hamiltonian equations of motion $\dot{x}_i = \partial H / \partial y_i$, $\dot{y}_i = -\partial H / \partial x_i$ take the form

$$\dot{X} = -J \cdot X \,, \qquad \dot{Y} = -J \cdot Y - L_0 \cdot X \,, \tag{2.4}$$

where $X = (x_1, \ldots, x_N)^\mathsf{T}$, $Y = (y_1, \ldots, y_N)^\mathsf{T}$, $J = XY^\mathsf{T} - YX^\mathsf{T}$, and $(L_0)_{ij} = \delta_{ij} a_i$.

**b)** Show that with $K = XX^\mathsf{T}$, the equations of motion can also be written as

$$\dot{K} = -[J, K] \,, \qquad \dot{J} = [L_0, K] \,, \tag{2.5}$$

and that these are equivalent to the Lax equation $\dot{L} = [M, L]$ for the matrices

$$L(z) = L_0 + zJ - z^2 K \,, \qquad M(z) = -zK \,. \tag{2.6}$$

**c)** Show that the spectral curve equation $0 = \det(L(z) - \lambda)$ with a suitable rank-two matrix $P$ can be written as $0 = \det(L_0 - \lambda) \det(1 + P)$. Also show that

$$\det(1 + P) = 1 - z^2 \bigl(V^2 + U(1 - W)\bigr) \,, \tag{2.7}$$

with $U = \sum_k x_k^2 / (a_k - \lambda)$, $V = \sum_k x_k y_k / (a_k - \lambda)$, and $W = \sum_k y_k^2 / (a_k - \lambda)$.
*Hint:* Express $P$ in the basis $\{v_1 = (L_0 - \lambda)^{-1} \cdot X,\ v_2 = (L_0 - \lambda)^{-1} \cdot Y\}$.

**d)** Use the relation $\det(1 + P) = 1 + z^2 \sum_k F_k / (\lambda - a_k)$ as well as the birational transformation $z' = z^{-1} \prod_{i=1}^N (\lambda - a_i)$ to show that there are parameters $b_i$ (that are functions of $a_k$ and $F_k$) such that the spectral curve equation can be written as

$$z'^2 = -\prod_{i=1}^{N} (\lambda - a_i) \prod_{j=1}^{N-1} (\lambda - b_i) \,. \tag{2.8}$$

**e)** The relation (2.8) describes a hyperelliptic curve $z' = \pm(\ldots)^{1/2}$. How many branch points / branch cuts connect the two $z'$ branches? What is the genus of the curve?
*Hint:* One branch point is at $\lambda = \infty$. Figure each of the two $z'$ branches as a ball, and each branch cut as a cylindrical tube connecting the two balls.

$\longrightarrow$

## 2.2. Conservation Laws for the KdV Equation (4 points)

We want to find conserved local charges for the KdV equation

$$\dot{h} = 6hh' - h''' \,, \qquad h = h(t,x) \,. \tag{2.9}$$

**a)** Consider the change of field variable $h \to w$, with

$$h = w + i\varepsilon w' - \varepsilon^2 w^2 \,. \tag{2.10}$$

Show that $h$ satisfies (2.9) if $w$ satisfies

$$\dot{w} = \frac{\partial}{\partial x}\left(3w^2 - w'' - 2\varepsilon^2 w^3\right) . \tag{2.11}$$

**b)** Now let $w$ be a power series in $\varepsilon$,

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n \,, \qquad w_n = w_n(t,x) \,. \tag{2.12}$$

By expanding (2.10) in $\varepsilon$, obtain a recursion relation for $w_n(t,x)$.

**c)** Show that the relation (2.11) implies

$$F_n := \int_{-\infty}^{+\infty} w_n \mathrm{d}x \,, \qquad \dot{F}_n = 0 \,, \tag{2.13}$$

where we assume that $h$ and all its derivatives decay at $|x| \to \infty$. Observe that $F_n$ is only non-zero for even $n$, and write $F_0$, $F_2$, and $F_4$ as integrals over polynomials in $h$ and its derivatives.

## 2.3. KdV Solitons (3 points)

Look for solutions to the KdV equation $\dot{h} = 6hh' - h'''$ with constant velocity $v$ by assuming $h(t,x) = f(x - vt)$. Show that $f$ satisfies the equation

$$\tfrac{1}{2} f'^2 = f^3 + \tfrac{1}{2} v f^2 + \alpha f + \beta \,, \tag{2.14}$$

with $\alpha$ and $\beta$ constant. Assuming that $h$ vanishes at $|x| \to \infty$, what must be the values of $\alpha$ and $\beta$? Solve the differential equation (2.14) to recover the one-soliton solution to the KdV equation.

# Introduction to Integrability — Problem Set 3

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

09. 07. 2019

## 3.1. Flat Connection and Parallel Transport (2 points)

Let $A(x) = A_\mu(x)\,\mathrm{d}x^\mu$ be a matrix-valued connection one-form with corresponding derivative operator $D_\mu = \partial_\mu - A_\mu(x)$, where $\partial_\mu = \partial/\partial x^\mu$ is the ordinary derivative operator. Define the parallel transport operator

$$U^{10} = \overleftarrow{\mathrm{P}} \exp \int_{x_0}^{x_1} A = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_1} \overleftarrow{\mathrm{P}}\big[A_{\nu_1}(y_1) \dots A_{\nu_n}(y_n)\big]\,\mathrm{d}y_1^{\nu_1} \dots \mathrm{d}y_n^{\nu_n}\,, \tag{3.1}$$

where $\overleftarrow{\mathrm{P}}[\dots]$ orders all products by position on the the path from $x_0$ to $x_1$, with factors closer to $x_1$ to the left.

**a)** Show that the flatness condition $[D_\mu, D_\nu] = 0$ is equivalent to $\mathrm{d}A = A \wedge A$.

**b)** Assuming that the connection $A(x)$ is flat, show that

$$\frac{\partial}{\partial x_1^\mu} U^{10} = A_\mu(x_1)\, U^{10}\,, \qquad \frac{\partial}{\partial x_0^\mu} U^{10} = U^{10}\, A_\mu(x_0)\,. \tag{3.2}$$

## 3.2. Inverse Scattering Method for the KdV Equation (4 points)

We will use the inverse scattering method to find solutions to the KdV equation. The GLM equation reads

$$K(x,y) + \hat{r}(x+y) + \int_x^\infty K(x,z)\,\hat{r}(z+y)\,\mathrm{d}z = 0\,, \qquad h(x) = -2\frac{\partial}{\partial x} K(x,x)\,. \tag{3.3}$$

For purely solitonic solutions, the potential $h(x)$ is reflectionless, and $\hat{r}(x)$ reduces to

$$\hat{r}(x) = \sum_{j=1}^{N} \lambda_j(t)\,\mathrm{e}^{-\kappa_j x}\,, \qquad \lambda_j(t) = \lambda_j(0)\,\mathrm{e}^{8\kappa_j^3 t}\,, \qquad \kappa_j > 0\,. \tag{3.4}$$

First, consider the single-soliton case $N = 1$, with $\kappa_1 \equiv \kappa$ and $\lambda_j \equiv \lambda$.

**a)** Obtain the solution $h(x)$ by solving the GLM equation (3.3) using the separation ansatz $K(x,y) = K(x)\,\mathrm{e}^{-\kappa y}$

**b)** Find the minimum $x_0(t)$ of $h(x)$ as a function of $\kappa$ and $\lambda(t)$. Express $h(x)$ in terms of $x$, $t$, the velocity $v$ of $x_0(t)$, and $x_0(0)$.

Now consider the two-soliton case $N = 2$. Assume that $K(x,y)$ separates to

$$K(x,y) = K_1(x)\,\mathrm{e}^{-\kappa_1 y} + K_2(x)\,\mathrm{e}^{-\kappa_2 y}\,. \tag{3.5}$$

**c)** Show that the GLM equation can be written as a matrix equation $AK + L = 0$, with

$$A_{i,j} = \delta_{i,j} + \lambda_i \frac{\mathrm{e}^{-(\kappa_i+\kappa_j)x}}{\kappa_i + \kappa_j}\,, \qquad K_i = K_i(x)\,, \qquad L_i = \lambda_i\,\mathrm{e}^{-\kappa_i x}\,. \tag{3.6}$$

**d)** Derive that

$$K(x,x) = K_1(x)\,\mathrm{e}^{-\kappa_1 x} + K_2(x)\,\mathrm{e}^{-\kappa_2 x} = \frac{\partial}{\partial x}\log\det A(x)\,. \tag{3.7}$$

Using this formula, show that $h(x,t) \to 0$ for $x \to \pm\infty$ and for any value of $t$.

$\longrightarrow$

### 3.3. Asymptotics of Two KdV Solitons (3 points)

Consider the two-soliton solution (3.7) with the matrix $A$ in (3.6), and with $0 < \kappa_1 < \kappa_2$.

**a)** Compare the magnitudes of the quantities $\lambda_1(t)\,\mathrm{e}^{-2\kappa_1 x}$, $\lambda_2(t)\,\mathrm{e}^{-2\kappa_2 x}$, and 1 as $x$ varies from $-\infty$ to $\infty$, for the two cases $t \ll 1$ and $t \gg 1$.

**b)** For $t \ll 0$, compute the leading term $\det A_i^-$ of $\det A$ near $x \approx 4\kappa_i^2 t$, for $i = 1, 2$. Drop all terms that are irrelevant for $h(x) \sim (\partial/\partial x)^2 \log\det A$. In the same way, compute the leading terms $\det A_i^+$ for $t \gg 0$.

**c)** Show that the resulting expressions $h_i^\pm(x) = -2(\partial/\partial x)^2 \log\det A_i^\pm$ all take the form of single solitons. Compute the parameters $\kappa_i^\pm$, $\lambda_i^\pm$ of those single-soliton solutions in terms of the two-soliton parameters $(\kappa_1, \kappa_2, \lambda_1, \lambda_2)$.

**d)** Compare the minima $x_{0,i}^\pm$ of $h_i^\pm(x)$ at $t \ll 0$ and at $t \gg 0$. Interpret the result: What is the effect of the scattering on the two solitons? Sketch the result.

### 3.4. KdV Solitons: Verification and Visualization (3 points)

Via $h(x) = -2(\partial/\partial x)^2 \log\det A$, the $N \times N$ matrix $A$ defined by (3.6) actually yields a valid $N$-soliton solution $h(x)$ for any $N$.

This is a Mathematica exercise. Please hand in a printout of your Mathematica notebook as well as the digital file.

**a)** Validate the $N = 2$ solution by verifying with Mathematica that $\dot{h} - 6hh' + h'''$ indeed vanishes.

*Hint:* Functions are defined as `f[x_]:=`(expression involving `x`). Matrices are best defined with `Table[.., {i,2}, {j,2}]` (for a $2 \times 2$ matrix). There is a built-in function `KroneckerDelta[i,j]`. `Det[..]` computes the determinant, `Log[..]` is the natural logarithm. Derivatives can be computed with `D[..]`, e. g. `D[f[x], {x,2}]` would be $f''(x)$. Use `Simplify[..]` to simplify expressions. An extremely useful construct is `expr /. X -> Y`, which replaces all occurences of `X` in `expr` with `Y`. Get help with the F1 key.

**b)** Plot the $N = 2$ solution for $\kappa_1 = 1/2$, $\kappa_2 = 1/\sqrt{2}$, $\lambda_i(0) = 2\kappa_i$. Plots can be generated with `Plot[h[x,t], {x,-50,50}]`. Repeat for various values of `t`. To show the full plotting region, use `PlotRange -> ...`

Try `Manipulate[Plot[h[x,t], {x,-50,50}], {t,-20,20}]`, play with the slider.

*Advanced:* Include also the single-soliton solutions of problem 3.3 c) in the plot, possibly with different colors or line styles (using `PlotStyle -> ..`).

**c)** Validate the $N = 3$ solution by evaluating the expression $\dot{h} - 6hh' + h'''$. The result is bulky and not easily simplified. Instead, replace $\kappa_i$, $\lambda_i$, and $x$ by random numbers (generated with `RandomReal[]`) to check that the expression vanishes.

**d)** Plot the $N = 3$ solution for two different choices of parameters: One where all three solitons scatter at once, and one where the three solitons only scatter pairwise. Use `Manipulate[..]` as in b) to visualize the result.

# Introduction to Integrability — Problem Set 4

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

09. 07. 2019

## 4.1. Heisenberg Magnet: Circle Solutions (3 points)

The Heisenberg magnet is described by two fields $\vartheta(t,x)$, $\varphi(t,x)$ (altitude and azimuth on the sphere), with equations of motion

$$\dot{\vartheta} = 2\cos(\vartheta)\vartheta'\varphi' + \sin(\vartheta)\varphi'' \,, \qquad \dot{\varphi} = \cos(\vartheta)\varphi'^2 - \frac{\vartheta''}{\sin(\vartheta)} \,. \tag{4.1}$$

The momentum $P$, energy $E$, and angular momentum $Q$ are given by

$$P = \int (1-\cos\vartheta)\varphi' \,\mathrm{d}x \,, \quad E = \frac{1}{2}\int \bigl(\vartheta'^2 + \sin^2(\vartheta)\varphi'^2\bigr)\,\mathrm{d}x \,, \quad Q = \int \cos(\vartheta)\,\mathrm{d}x \,. \tag{4.2}$$

**a)** Find the most general solution $\varphi(t,x)$ when $\vartheta(t,x) = \vartheta_0$ is a constant ($0 < \vartheta_0 < \pi$).

**b)** Impose periodic boundary conditions $\varphi(t,x+L) = \varphi(t,x)$. Note that the condition only needs to be satisfied modulo the equivalence $\varphi \equiv \varphi + 2\pi\mathbb{Z}$.

**c)** Compute the momentum $P$, energy $E$, and angular momentum $Q$ of these solutions.

## 4.2. Spectral Curve for the Heisenberg Magnet (4 points)

The simplest finite-gap solution of the Heisenberg magnet has a spectral curve with a single branch cut. A suitable ansatz for the quasi-momentum $q(u)$ is

$$q'_\pm(u) = \pm\frac{au+b}{u^2\sqrt{u^2+cu+d}} \,. \tag{4.3}$$

The $\pm$ labels the two branches of the function, which are connected by a branch cut stretching between two branch points at the zeros of the square root. Let $A$ be a counterclockwise cycle around the branch cut, and $B$ a path going from $u = \infty_-$ on the one branch trough the cut and back to $u = \infty_+$ on the other branch. Then $q'(u)$ should satisfy

$$\oint_A q'_+(u)\,\mathrm{d}u = 0 \,, \qquad \frac{1}{2\pi}\int_B q'(u)\,\mathrm{d}u = n \in \mathbb{Z} \,, \qquad I = \frac{1}{2\pi i}\oint_A u\,q'_+(u)\,\mathrm{d}u \,, \tag{4.4}$$

where $I$ is called the "filling" of the cut. Moreover, the length $L$, momentum $P$, energy $E$, and angular momentum $Q$ appear in series expansions of $q_+(u)$ as

$$u \to 0: \quad q_+(u) = \frac{L}{u} - \frac{P}{2} + \frac{uE}{4} + \mathcal{O}(u^2) \,, \qquad u \to \infty_+: \quad q_+(u) = \frac{Q}{u} + \mathcal{O}(u^{-2}) \,. \tag{4.5}$$

**a)** Express the coefficients $a$, $b$, and $c$ in terms of $d$, $L$, and $I$ using the $A$-cycle conditions and series expansions. *Hint:* $A$-cycle integrals are sums of residues at $u = 0, \infty$.

**b)** Integrate $q'(u)$ to $q(u)$, and find $d$ in terms of $n$ and $L$ by the $B$-cycle condition. Fix the integration constant by the vanishing of $q_+(u)$ at $u = \infty$. *Hint:* Compute $\bigl(\sqrt{A - 2Bu + Du^2}/u\bigr)'$. The square root has different signs on the two branches.

**c)** Expand $q_+(u)$ at $u = 0, \infty$, and find expressions for $P$, $Q$, and $E$ by matching (4.5). Compare the results to your results of 4.1 c)

⟶

## 4.3. Zamolodchikov–Faddeev Algebra and Sine–Gordon (5 points)

Consider operators $A_i(u)$ that satisfy the commutation relations

$$A_i(u_2)A_j(u_1) = \sum_{k,\ell} S_{ij}^{kl}(u_{12})A_k(u_1)A_\ell(u_2)\,, \qquad u_{12} = u_1 - u_2\,. \tag{4.6}$$

For $i$ different from $j$, $A_i(u)$ and $A_j(u)$ are independent operators. $A_j(u)$ can be thought of as creating a particle of type (flavor) $j$ with rapidity $u$: $A_j(u)|0\rangle = |A_j(u)\rangle$. The scattering factors $S_{ij}^{k\ell}(u)$ are scalar functions of $u$ that form the scatterin matrix $S(u)$. Obtain consistency conditions for the matrix $S(u)$ from the commutation relations by

**a)** taking the limit $u_{12} \to 0$.

**b)** iterating (4.6) twice. Draw a diagram for the resulting condition.

**c)** relating the product $A_i(u_3)A_j(u_2)A_k(u_1)$ back to a sum over $A_r(u_1)A_p(u_2)A_q(u_3)$ by iteratively applying (4.6) in two different ways. Draw a diagram for the resulting condition. Interpret triple products of $A_i(u_k)$ as states of a three-site system, and write the condition as an equation for $S_{12}(u)$ and $S_{23}(u)$, where the indices denote the sites on which the respective matrix acts.

Specialize to a model with particles $A$ and $\bar{A}$, and three different scattering amplitudes:

$$S_\mathrm{I}(u) = S_{AA}^{AA}(u) = S_{\bar{A}\bar{A}}^{\bar{A}\bar{A}}(u)\,, \quad S_\mathrm{T}(u) = S_{A\bar{A}}^{\bar{A}A}(u) = S_{\bar{A}A}^{A\bar{A}}(u)\,, \quad S_\mathrm{R}(u) = S_{\bar{A}A}^{\bar{A}A}(u) = S_{A\bar{A}}^{A\bar{A}}(u)\,.$$

**d)** Write the relation (4.6) for all combinations of $A$ and $\bar{A}$. Show that the conditions from c) with $S_x \equiv S_x(u_{12})$, $S'_x \equiv S_x(u_{13})$, and $S''_x \equiv S_x(u_{23})$, $x \in \{\mathrm{I}, \mathrm{T}, \mathrm{R}\}$ amount to

$$S_\mathrm{I}S'_\mathrm{T}S''_\mathrm{R} = S_\mathrm{T}S'_\mathrm{I}S''_\mathrm{R} + S_\mathrm{R}S'_\mathrm{R}S''_\mathrm{T}\,, \qquad S_\mathrm{I}S'_\mathrm{R}S''_\mathrm{I} = S_\mathrm{R}S'_\mathrm{I}S''_\mathrm{R} + S_\mathrm{T}S'_\mathrm{R}S''_\mathrm{T}\,,$$
$$S_\mathrm{R}S'_\mathrm{T}S''_\mathrm{I} = S_\mathrm{R}S'_\mathrm{I}S''_\mathrm{T} + S_\mathrm{T}S'_\mathrm{R}S''_\mathrm{R}\,. \tag{4.7}$$

**e)** Show that the existence of a non-trivial solution to (4.7) requires that the quantity

$$\Delta = \frac{S_\mathrm{I}^2(u) + S_\mathrm{T}^2(u) - S_\mathrm{R}^2(u)}{2S_\mathrm{I}(u)S_\mathrm{T}(u)} \tag{4.8}$$

is independent of $u$.

*Hint:* Write (4.7) in matrix form $M \cdot \left(S''_\mathrm{I}, S''_\mathrm{R}, S''_\mathrm{T}\right)^\mathsf{T} = 0$.

**Remark:** This type of system is realized in the Sine–Gordon model with Lagrange density $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{m^2}{\beta^2}\cos(\beta\phi)$. In that case, $A(u)$ and $\bar{A}(u)$ are the soliton and anti-soliton solutions, with scattering factors

$$S_\mathrm{I}(u) = \sinh\left[\frac{8\pi}{\eta}\,(i\pi - u)\right]f(u)\,, \quad S_\mathrm{T}(u) = \sinh\left[\frac{8\pi}{\eta}\,u\right]f(u)\,, \quad S_\mathrm{R}(u) = i\sin\left[\frac{8\pi^2}{\eta}\right]f(u)\,,$$

and with $\Delta = -\cos(8\pi^2/\eta)$, where $1/\eta = 1/\beta^2 - 1/(8\pi)$.

# Introduction to Integrability — Problem Set 5

Leibniz University Hannover, Summer 2019 — Dr. Till Bargheer

09. 07. 2019

## 5.1. Heisenberg Spin Chain: Direct Diagonalization (4 points)

Consider the Heisenberg spin chain with periodic boundary conditions and Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{L} (\mathcal{I}_{j,j+1} - \mathcal{P}_{j,j+1}) \,. \tag{5.1}$$

Compute the spectrum of eigenvalues of $\mathcal{H}$ (energies) by direct diagonalization for the cases specified below. $M$ denotes the number of up spins.

**a)** Compute the spectrum for a spin chain of length $L = 3$ and arbitrary number $M$ of spin flips. How do the eigenstates organize into $\mathfrak{su}(2)$ multiplets? The generators of $\mathfrak{su}(2)$ are $Q^\alpha = \sum_i \sigma_i^\alpha/2$, $\alpha = x, y, z$, and $Q^\pm = Q^x \pm iQ^y$.

**b)** Restrict to cyclic states, i. e. identify all states that are equivalent under cyclic permutations of the spin chain sites. Compute the spectrum for the states with $L = 4$, $M = 2$, and for $L = 6$, $M = 2, 3$.

## 5.2. Heisenberg Spin Chain: Bethe Equations (4 points)

The Bethe equations for the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain read

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad k = 1, \ldots, M\,. \tag{5.2}$$

Each solution to these equations (such that all finite $u_k$ are distinct) defines an eigenstate of the Heisenberg Hamiltonian with $M$ up spins (magnons). The energy $E$ and momentum $P$ are given by

$$E = \sum_{k=1}^{M} \left(\frac{i}{u_k + i/2} - \frac{i}{u_k - i/2}\right), \qquad \mathrm{e}^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2}\,. \tag{5.3}$$

**a)** Use the Bethe equations to compute the energy spectrum for $L = 3$ and $M \leq 1$. States with $M > L/2$ are obtained from states with $M \leq L/2$ by flipping all spins. Compare to the results of problem 5.1 a). How are the $\mathfrak{su}(2)$ multiplets realized?

In the following, restrict to cyclic states, i. e. require $\mathrm{e}^{iP} = 1$.

**b)** Compute the energy spectrum for $L = 4$, $M = 2$, and for $L = 6$, $M = 2$. Compare to the results of problem 5.1 b).

**c)** Compute the energy spectrum from the Bethe equations for any $L$ and $M = 2$.

**d)** The solution for $L = 6$, $M = 3$ is singular. Show that the regularized rapidities

$$u_1 = \frac{i}{2} + \varepsilon + c\,\varepsilon^6\,, \qquad u_2 = -\frac{i}{2} + \varepsilon\,, \qquad u_3 = \frac{1 - 4u_1u_2}{4(u_1 + u_2)} + d(\varepsilon) \tag{5.4}$$

solve the Bethe equations and the condition $\mathrm{e}^{iP} = 1$ in the limit $\varepsilon \to 0$ for a suitable constant $c$ and function $d(\varepsilon)$. Compare to your result of problem 5.1 b).

$\longrightarrow$

## 5.3. Coordinate Bethe Ansatz for the XXZ Spin Chain (4 points)

Consider the Hamiltonian $\mathcal{H}$ of the XXZ spin chain with periodic boundary conditions:

$$\mathcal{H} = \sum_{j=1}^{L} \mathcal{H}_{j,j+1}\,, \qquad \mathcal{H}_{j,k} = \frac{1}{2}\left[\sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y + \varDelta(\sigma_j^z\sigma_k^z - \mathbf{1}_j\mathbf{1}_k)\right], \qquad \sigma_{L+1} \equiv \sigma_1\,, \tag{5.5}$$

which acts on a spin chain of length $L$ with spins $|{\downarrow}\rangle = (0,1)^\mathsf{T}$ and $|{\uparrow}\rangle = (1,0)^\mathsf{T}$. Here, $\sigma_j^i$, $i \in \{x,y,z\}$ are the Pauli matrices acting on the spin state at site $j$.

**a)** Consider a general state with a single up spin

$$|\psi_1\rangle = \sum_{k=1}^{L} f(k)\,|k\rangle\,, \qquad |k\rangle = |{\downarrow\downarrow}\ldots{\downarrow\downarrow}\overset{k}{\uparrow}{\downarrow\downarrow}\ldots{\downarrow\downarrow}\rangle\,. \tag{5.6}$$

Convert the eigenvalue equation $\mathcal{H}|\psi_1\rangle = e_1|\psi_1\rangle$ to a finite difference equation for $f(k)$. Show that the one-magnon ansatz $f(k) = \mathrm{e}^{ipk}$ solves the equation, and that the dispersion relation becomes $e_1(p) = 2(\cos(p) - \varDelta)$.

**b)** Now consider states with two up spins:

$$|\psi_2\rangle = \sum_{1\le k<\ell\le L} f(k,\ell)\,|k,\ell\rangle\,, \qquad |k,\ell\rangle = |{\downarrow}\ldots{\downarrow}\overset{k}{\uparrow}{\downarrow}\ldots{\downarrow}\overset{\ell}{\uparrow}{\downarrow}\ldots{\downarrow}\rangle\,. \tag{5.7}$$

Starting with the eigenvalue equation $\langle k,\ell|\mathcal{H}|\psi_2\rangle = e_2\,f(k,\ell)$, derive two difference equations for $f(k,\ell)$ by considering the two cases $k+1<\ell$ and $k+1=\ell$. Using the two-magnon ansatz

$$f(k,\ell) = \mathrm{e}^{ipk+iq\ell} + S(p,q)\,\mathrm{e}^{iqk+ip\ell}\,, \tag{5.8}$$

show that the dispersion relation is $e_2(p,q) = e_1(p) + e_1(q)$, and that the scattering phase $S(p,q)$ must satisfy

$$S(p,q) = -\frac{1+\mathrm{e}^{i(p+q)} - 2\varDelta\,\mathrm{e}^{iq}}{1+\mathrm{e}^{i(p+q)} - 2\varDelta\,\mathrm{e}^{ip}}\,. \tag{5.9}$$

*Hint:* First compute the action of $\mathcal{H}$ on neighboring spins $|{\downarrow\downarrow}\rangle$, $|{\uparrow\uparrow}\rangle$, $|{\downarrow\uparrow}\rangle$, and $|{\uparrow\downarrow}\rangle$.

**c)** Express $S(p,q)$ in terms of rapidities $u$, $v$, which are related to the momenta $p$, $q$ via

$$\mathrm{e}^{ip} = \frac{u+i/2}{u-i/2}\,. \tag{5.10}$$

Taking the limit $\varDelta \to 1$, show that the Bethe equations $\mathrm{e}^{ip_kL} = \prod_{j=1,j\neq k}^{M} S(p_j,p_k)$ for an $M$-magnon state become

$$\left(\frac{u_k+i/2}{u_k-i/2}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,. \tag{5.11}$$