BLACK HOLE STATES IN QUANTUM SPIN CHAINS

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GOALS:

- DEFINE BLACK HOLE STATES IN A(N) (INTEGRABLE) SPIN CHAIN;
- STUDY THEIR PROPERTIES;
- CONNECT WITH AdS/CFT;

THE PROTOTYPE MODEL

$$H = \sum_{i=1}^{L} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$$

- SU(2) invariant
- INTEGRABLE

Heisenberg Spin Chain

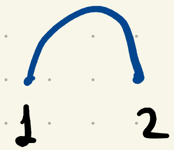
PLAUSIBILITY ARGUMENTS:

1) BLACK-HOLE STATE SHOULD BE A SPIN SINGLET
(BLACK-HOLES HAVE NO HAIR)

2) IT SHOULD BE SUFFICIENTLY GENERIC TO
REFLECT BLACK-HOLES CHAOTIC NATURE

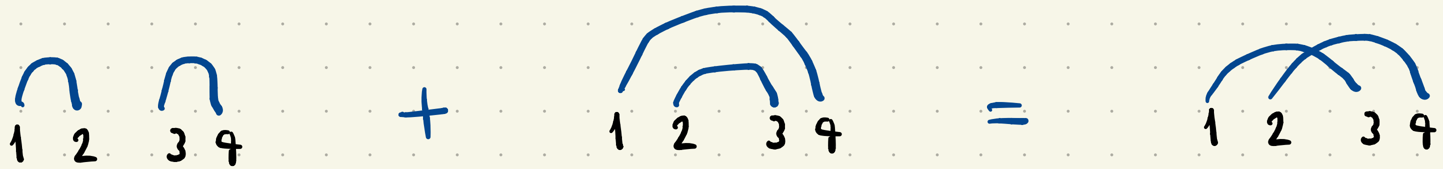
FIRST TASK: ENUMERATE SINGLETS

L=2 anti-symmetric combination $\varepsilon_{\alpha_1 \alpha_2}$, where $\alpha_i \in \{\uparrow, \downarrow\}$



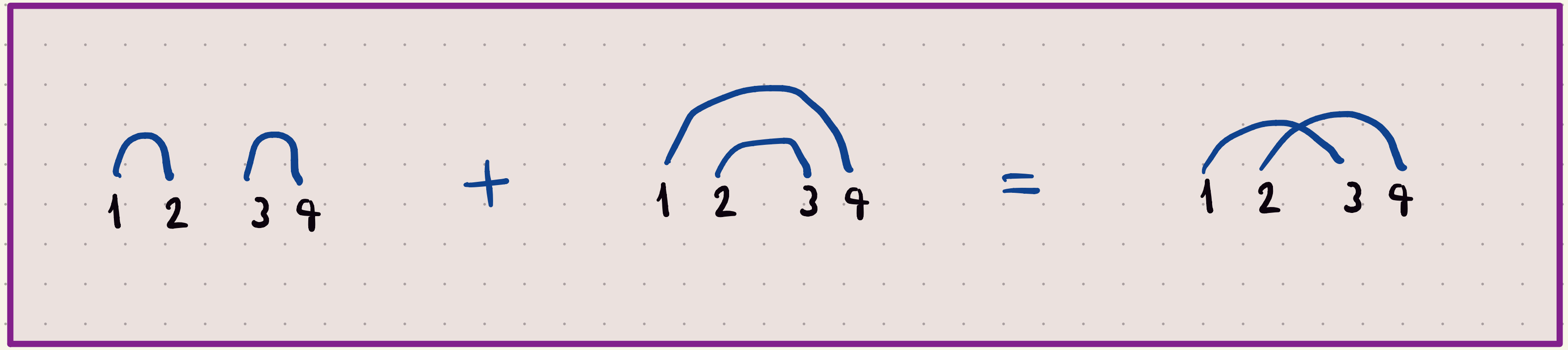
$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

L=4 three ways to distribute 4 spins in pairs



$$\varepsilon_{\alpha_1 \alpha_2} \varepsilon_{\alpha_3 \alpha_4} + \varepsilon_{\alpha_1 \alpha_4} \varepsilon_{\alpha_2 \alpha_3} = \varepsilon_{\alpha_1 \alpha_3} \varepsilon_{\alpha_2 \alpha_4}$$

RUMER 1932

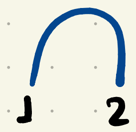


RUMER'S IDENTITY can be used to eliminate crossings for any number of spins

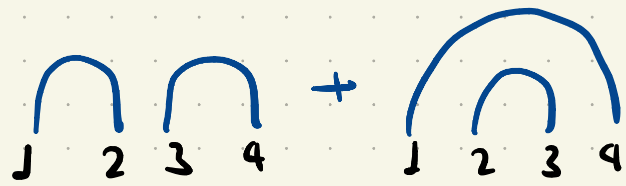
The non-crossing pairings are linearly independent and form a complete basis of singlets

$|BH\rangle$: SUM OF ALL NON-CROSSING PAIRINGS WITH EQUAL WEIGHTS

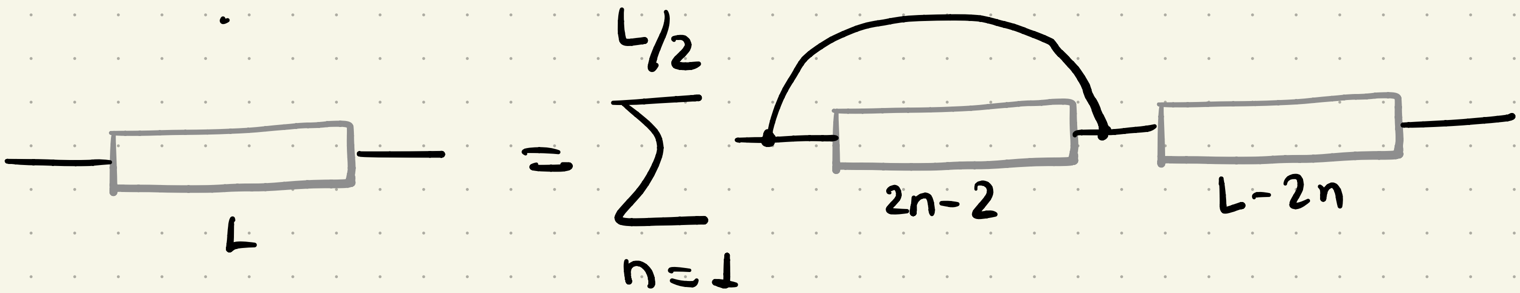
$$|BH_2\rangle = \sum_{\alpha,\beta=\{\uparrow,\downarrow\}} \varepsilon_{\alpha\beta}\,|\alpha\rangle\otimes|\beta\rangle$$



$$|BH_4\rangle = \sum_{\alpha,\beta}\Big(\varepsilon_{\alpha\beta}\,|\alpha\rangle\otimes|\beta\rangle\otimes|BH_2\rangle$$
$$+\ \varepsilon_{\alpha\beta}\,|\alpha\rangle\otimes|BH_2\rangle\otimes|\beta\rangle\Big)$$



$\vdots$



$$\boxed{|BH_L\rangle = \sum_{n=1}^{L/2}\sum_{\alpha\beta} \varepsilon_{\alpha\beta}\,|\alpha\rangle\otimes|BH_{2n-2}\rangle\otimes|\beta\rangle\otimes|BH_{L-2n}\rangle}$$

WHERE IS INTEGRABILITY BROKEN ?

$$Q_3 = i \sum_{\ell} \vec{\sigma}_\ell \cdot (\vec{\sigma}_{\ell+1} \times \vec{\sigma}_{\ell+1})$$

$$[Q_3, H] = 0$$

TO PRESERVE INTEGRABILITY A BOUNDARY STATE SATISFIES

$$Q_3 |\text{state}\rangle = 0$$

BUT $\boxed{Q_3 |BH\rangle \neq 0}$ $\Rightarrow$ EXPECT CHAOTIC FEATURES

WHAT NOW ?

1) $Q_3 |BH\rangle \neq 0$

VS

2) $VBS^{\alpha_1 \cdots \alpha_L} = \varepsilon^{\alpha_1 \alpha_2} \cdots \varepsilon^{\alpha_{L-1} \alpha_L}$

$Q_3 \, VBS^{\alpha_1 \cdots \alpha_L} = 0$

STUDY :

* ENTANGLEMENT ENTROPY
* THERMALIZATION
* KRYLOV COMPLEXITY

ENTANGLEMENT ENTROPY

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L \text{ sites}} = (\mathbb{C}^2)^{\otimes \ell} \otimes (\mathbb{C}^2)^{L-\ell}$$

$S(\ell)$ entanglement entropy of the subspace of length ℓ of the spin chain with respect to the full $(\mathbb{C}^2)^{\otimes L}$ chain

FOR CRITICAL (GAPLESS) SYSTEMS $S(\ell)$ for the ground state:

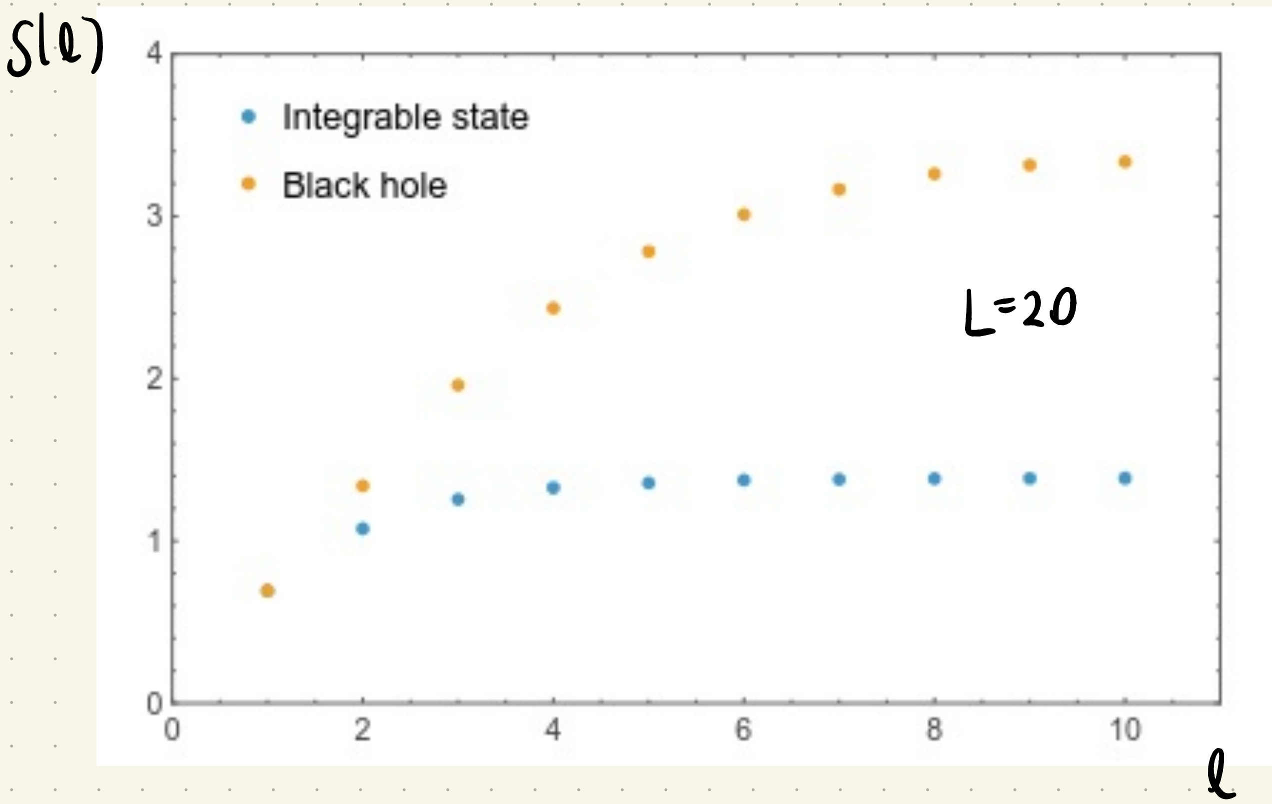
$$S(\ell) = \frac{c}{3} \log\left(\frac{L}{\pi} \sin \frac{\pi \ell}{L}\right) + c_1'$$

FOR GAPPED STATES:

S constant

FOR THERMAL STATES:

S linear in ℓ.



* $S(\ell)$ increases logarithmically

but with $c = 5.2 \gg$ than integrable models

for Heisenberg $c = 1$

✱ KRYLOV COMPLEXITY

Measures to what extent an operator or a state explores a system's Hilbert space under time evolution

Krylov space of order K:

space spanned by $\underbrace{\{|\Psi_0\rangle, H|\Psi_0\rangle, H^2|\Psi_0\rangle, \dots, H^{K-1}|\Psi_0\rangle\}}$

Lanczos algorithm

orthonormal basis

$\{\psi_0, \psi_1, \dots, \psi_{K-1}\}$

$$H = \begin{pmatrix} a_0 & b_1 & & & & \\ b_1 & a_1 & b_2 & & & \\ & b_2 & a_2 & b_3 & & \\ & & b_3 & a_3 & \ddots & \\ & & & \ddots & \ddots & b_{K-1} \\ & & & & b_{K-1} & a_{K-1} \end{pmatrix}.$$

Study the time evolution of $|\psi_0\rangle$ in this new basis

$$|\psi(t)\rangle = \sum_{n=0}^{K-1} \psi_n(t)\,|\psi_n\rangle\,, \qquad |\psi(0)\rangle = |\psi_0\rangle$$

$$\Big\downarrow \text{ Schrödinger equation}$$

$$i\,\frac{\partial \psi_n(t)}{\partial t} = b_n \psi_{n-1}(t) + a_n \psi_n(t) + b_{n+1}\psi_{n+1}(t) \qquad \psi_n(t)\Big|_{t=0} = \delta_{n,0}$$

Define Krylov space position operator

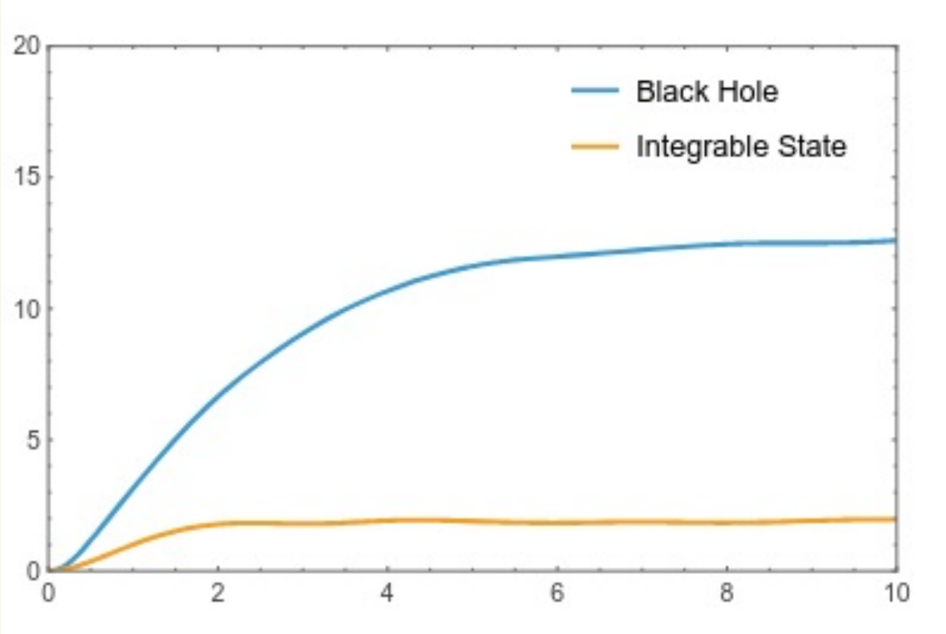
$$\hat{n} = \sum_{n=1}^{K-1} |\psi_n\rangle\langle\psi_n|$$

Krylov Complexity

$$C_K(t) = \langle \psi(t) | \hat{n} | \psi(t) \rangle = \sum_{n=0}^{K-1} n |\psi_n(t)|^2$$

To remove short scale oscillations

$$\boxed{\overline{C_K}(t) = \int_0^t \frac{C_K(s)\,ds}{t}}$$



CONCLUSION: THE INTEGRABLE MODEL EXPLORES A SMALLER REGION OF THE HILBERT SPACE IN COMPARISON WITH $|BH\rangle$

THERMAL ONE-POINT FUNCTIONS

FULL SET OF SCALAR OPERATORS IN $\mathcal{N}=4$ SYM

$$\mathcal{O} = \Psi^{i_1 \cdots i_L} \operatorname{tr} \Phi_{i_1} \cdots \Phi_{i_L} \qquad i_1, \dots, i_L \in \{1, \dots, 6\}$$

ONE-LOOP DILATATION OPERATOR $\sim$ SO(6) INTEGRABLE SPIN CHAIN

IN THE THERMAL ENSEMBLE, ONE-POINT FUNCTION IN LINEAR ORDER IN PERTURBATION THEORY CAN BE COMPUTED BY

— CONTRACT IDENTICAL FIELDS PAIRWISE WITH THE THERMAL PROPAGATOR

$$D_\beta(x) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{(t+n\beta)^2 + x}$$

THERMAL EFFECTS ENCODED IN $D_\beta(x) - D_\infty(x)$

THE FULL CONTRIBUTION TO THE ONE-POINT FUNCTION IS

$$\langle \mathcal{O} \rangle = \frac{1}{L} \left(\frac{\pi T}{\sqrt{3}} \right)^L \frac{\langle BH | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}}$$

WHERE

$$| BH_L \rangle = \sum_{n=1}^{L/2} \sum_{i=1}^{6} | i \rangle \otimes | BH_{2n-2} \rangle \otimes | i \rangle \otimes | BH_{L-2n} \rangle$$

* BREAKS INTEGRABILITY
* PLANAR DO NOT CONSTITUTE A FULL BASIS OF SINGLETS FOR THE SO(6) SPIN CHAIN

SUMMARY

- DEFINED A BLACK-HOLE STATE IN A SPIN CHAIN

- DEMONSTRATED CHAOTIC BEHAVIOUR SIGNS VIA THREE DIFFERENT METHODS

- OPEN QUESTION: UNDERSTAND THE LOGARITHM GROWTH IN THE ENTANGLEMENT ENTROPY

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THANK YOU!