The Non-Planar Four-Point Integrand and Konishi Dimension in $\mathcal{N} = 4$ Super Yang–Mills Theory at Five Loops

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Four-Point Correlator

- Lightest half-BPS scalar operators: $\mathcal{O}(x,y) = \mathrm{tr}(y \cdot \phi(x))^2$
- Four-point correlator

$$G_4 = \langle \mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2)\mathcal{O}(x_3,y_3)\mathcal{O}(x_4,y_4)\rangle$$

- Found up to 12 loops in the planar theory
- Non-planar integrand at 4 loops

Strategy

Constrain integrand as much as possible → Fix remaining freedom with twistor calculation

Superconformal Constraints

- Lagrangian insertion method:

$$G_4 = \sum_{\ell \geq 0} a^\ell \int \frac{d^4 x_5 \ldots d^4 x_{4+\ell}}{(-1)^\ell \, \ell!} \, G_4^{(\ell)}(x_i, y_i)$$

Integrand: $G_4^{(\ell)}(x_i, y_i) = \langle \mathcal{O}(x_1, y_1) \ldots \mathcal{O}(x_4, y_4) \, \mathcal{L}(x_5) \ldots \mathcal{L}(x_{4+\ell}) \rangle_{\text{tree}}$

- Partial non-renormalization theorem:

$$G_4^{(\ell > 0)}(x_i, y_i) = R_4(x_i, y_i) \times (x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2) \times F^{(\ell)}(x_i)$$

$$F^{(1)}(x_i) = \frac{1}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{25}^2 x_{34}^2 x_{35}^2 x_{45}^2}$$

$$F^{(2)}(x_i) = \frac{1}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{35}^2 x_{36}^2 x_{45}^2 x_{46}^2 x_{56}^2} + (S_6 \text{ perms.})$$

f-Graphs

ℓ	$\mathcal{N}$	genus 0	genus 1	genus 2	Gram
1	1	–	1	–	–
2	1	1	–	–	–
3	4	1	3	–	1
4	32	3	29	–	4
5	930	7	833	90	208
6	189 341	36	...	...	...

$$F^{(\ell)}(x_i) = \sum_{g\geq 0} \frac{1}{N_{\mathrm{c}}^{2g}} \sum_{k=1}^{\mathcal{N}_\ell} c_k^{(g,\ell)}\, f_k^{(\ell)}$$

The $(1/N_{\mathrm{c}}^{2g})$-correction to $G_4^{(\ell)}$ only depend on f-graphs up to genus g:

$$c_k^{(g,\ell)} = 0 \quad \text{if} \quad \mathrm{genus}(f_k^{(\ell)}) > g$$

Further Constraints

Planar integrand:

- Euclidean/Minkowskian OPE constraints (triangle rule/double-triangle rule)
- Correlator/amplitude-duality (square/pentagon rule)

$\to$ Implementable graph by graph, fixed up to 12 loops

Full integrand:

$$\log\left(1+2\sum_{\ell\geq 1} a^{\ell}\int d^4x_5\ldots d^4x_{4+\ell}\,\mathcal{F}^{(\ell)}(x_i)\right)\Bigg|_{a^{\ell}} \sim (\log u)^{\ell}$$

- No non-planar corrections up to 3 loops
- For $\ell = 4$: $29\,(+3) \longrightarrow 4\,(+3)$
- For $\ell = 5$: $722\,(+208) \longrightarrow 23\,(+208)$

$\to$ Fix remaining coefficients with perturbative twistor computation

Konishi anomalous dimension

- OPE of two $\mathbf{20'}$ operators:

$$\mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2) = c_{\mathbb{1}}\frac{y_{12}^4}{x_{12}^4}\mathbb{1} + c_{\mathcal{O}}\frac{y_{12}^2}{x_{12}^2}\mathcal{O}(x_1,y_1) + c_{\mathcal{K}}(a)\frac{y_{12}^4}{(x_{12}^2)^{1-\gamma_{\mathcal{K}}(a)/2}}\mathcal{K}(x_1) + \ldots$$

$$\mathcal{K}(x) = \mathrm{tr}(\phi(x)\cdot\phi(x)) \quad \text{and} \quad \Delta_{\mathcal{K}}(a) = 2 + \sum_{\ell=1}^{\infty} a^\ell \gamma_{\mathcal{K}}^{(\ell)}$$

- Double-pinching limit $x_2 \to x_1$, $x_4 \to x_3$:

$$\sum_{\ell=1}^{\infty} a^\ell F^{(\ell)}(x_i) \longrightarrow \frac{1}{6x_{13}^4}\left(c_{\mathcal{K}}^2(a)u^{\gamma_{\mathcal{K}}(a)/2} - 1\right) \times (1 + \mathcal{O}(u))$$

$$\log\left(1 + 6x_{13}^4\sum_{\ell=1}^{\infty} a^\ell F^{(\ell)}(x_i)\right) \longrightarrow \frac{1}{2}\gamma_{\mathcal{K}}(a)\log u + \log\left(c_{\mathcal{K}}^2(a)\right) + \mathcal{O}(u)$$

Konishi anomalous dimension

- Effectively, 2855 two-point four-loop massless integrals to compute five-loop genus-one anomalous dimension

$$2\int\left(\prod_{k=6}^{9} d^{4-2\epsilon}x_k\right)\hat{I}^{(5,1)} = -\frac{\gamma_{\mathcal{K}}^{(5,1)}}{\pi^2}(x_{15}^2)^{-2-4\epsilon}$$

- Using integration-by-part relation with `FIRE`, arrive at 24 master integrals
- 22 of them are known, all can be computed with `HyperInt` (Panzer) / `HyperlogProcedures` (Schnetz)

$$\boxed{\gamma_{\mathcal{K}}^{(5,1)} = 135\zeta_5 - \frac{243}{4}\zeta_3^2 + \frac{11907}{32}\zeta_7}$$