From Partons to Strings:

Scattering on the Coulomb Branch of $\mathcal{N}=4$ SYM

25.10. 19909

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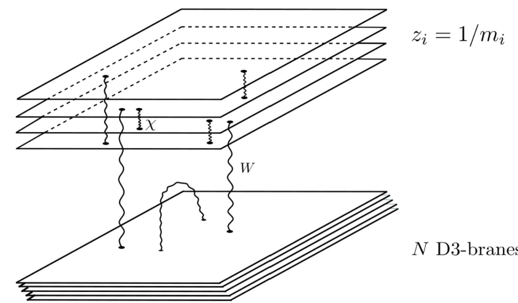
Scattering in $\mathcal{N}=4$ SYM on the Coulomb branch

$$U(N+4) \to U(N) \times U(1)^4$$



$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_{a,b} & \mathcal{O}_{a,N+j} \\ \mathcal{O}_{N+i,b} & \mathcal{O}_{N+i,N+j} \end{pmatrix}$$

where $\mathcal{O} = \{A_\mu, \Psi, \Phi\}$, $a, b = 1, ..., N$ and $i, j = 1, ..., 4$.

$\mathcal{O}_{a,b}$: massless fields

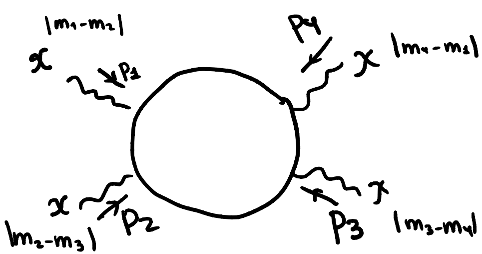
$\mathcal{O}_{a,N+i}$: 'heavy' massive W-bosons

$\mathcal{O}_{N+i,N+j}$: 'light' meson fields $\chi_{i,j} \equiv \Phi_{N+i,N+j}$

The amplitude: color-ordered four point:

$$iA(s,t,m_i) = \langle (\Phi_I)_{N+1,N+2}(p_1)(\Phi_J)_{N+2,N+3}(p_2)(\Phi_I)_{N+3,N+4}(p_3)(\Phi_J)_{N+4,N+1}(p_4)\rangle, \quad I, J \neq 9 \;, \tag{2.6}$$

$$s = -(p_1+p_2)^2, \; t = -(p_2+p_3)^2$$



$$A(s,t,m_i) = g_{YM}^2 M,$$
$$M = 1 + g^2 M^{(1)} + g^4 M^{(2)} + \cdots .$$

The amplitude $A(s,t,m_i)$ is only a function of two cross-ratios: (dual conformal invariance)

$$u = \frac{4m_1 m_3}{-s+(m_1-m_3)^2}, \quad v = \frac{4m_2 m_4}{-t+(m_2-m_4)^2} .$$

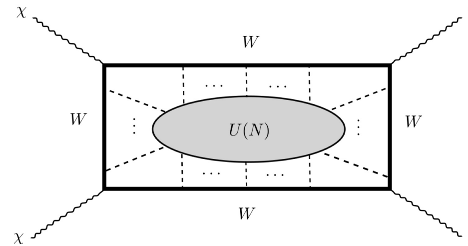


Figure 4: Planar diagrams with external χ fields (wavy lines) attached to a frame made of W-bosons (thick lines), filled with massless fields from the unbroken gauge group $U(N)$ (dashed lines).

Massless external particles: $m_i = m$

$$A(s,t,m) = s^2 f(s,t) \quad \text{with} \quad f(s,t) = -\frac{M(-4m^2/s, -4m^2/t)}{st}$$

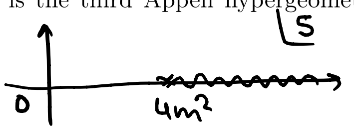
Main goal: explore $f(s,t)$ at finite 't Hooft coupling g.

Perturbative results

Weak coupling:

$$f(s,t) = -\frac{1}{st} + \frac{g^2}{6m^4} F_3\left(1,1,1,1;5/2\middle|\frac{s}{4m^2},\frac{t}{4m^2}\right) + \mathcal{O}(g^4)\ , \quad g \ll 1\ .$$

where F_3 is the third Appell hypergeometric function.



two-particle cut

Strong coupling:

$$f(s,t) = \frac{g}{\pi m^2}\frac{1}{s+t}\int_0^1 dx\ x^{-1-\frac{gs}{m^2\pi}}(1-x)^{-1-\frac{gt}{m^2\pi}} + \ldots$$

$$= -\frac{g^2}{\pi^2 m^4}\frac{\Gamma\left(-\frac{gs}{m^2\pi}\right)\Gamma\left(-\frac{gt}{m^2\pi}\right)}{\Gamma\left(1-\frac{gs}{m^2\pi}-\frac{gt}{m^2\pi}\right)} + \ldots\ , \quad g \gg 1\ ,$$

Poles: $\frac{gs}{m^2\pi} = \mathbb{Z}_+$

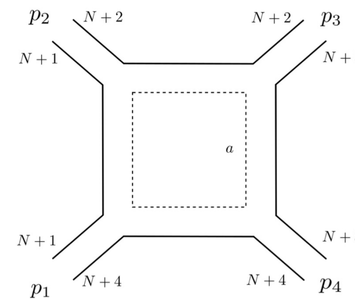


Figure 5: One-loop box diagram in double-line notation.

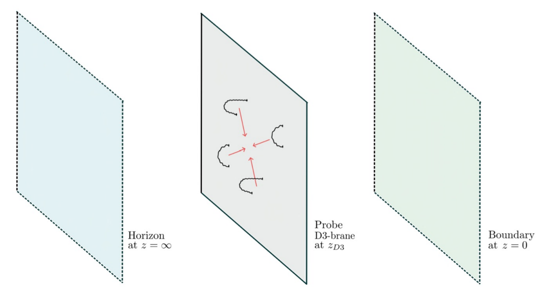
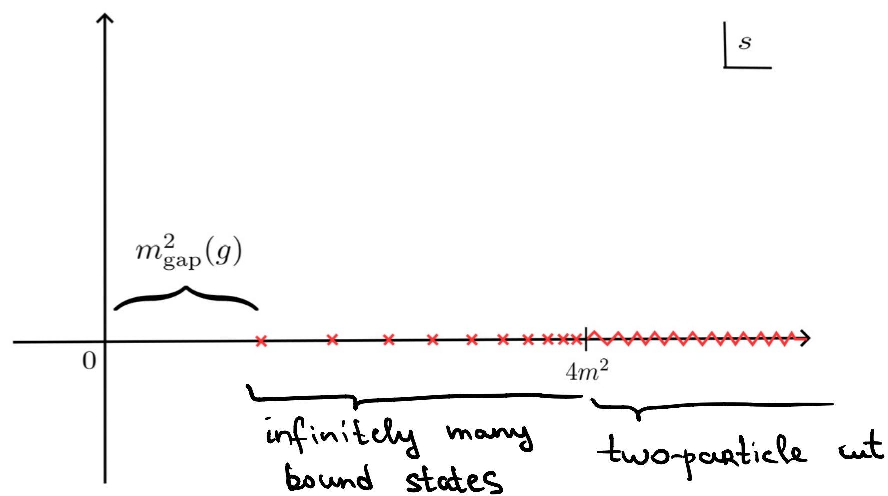


Figure 6: Scattering of open strings on a probe D3-brane at fixed radial position z_{D3} in AdS.

Analyticity

Let us look at the analytic structure of $f(s,t)$:



Spectrum of bound states

Let us consider the Regge expansion of $f(s,t)$ in the large-t limit

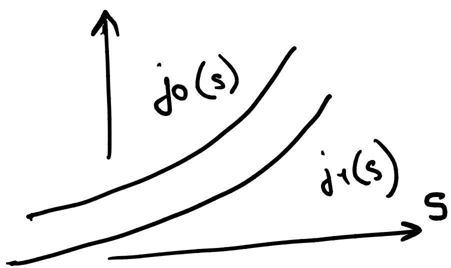
$$f(s,t) = -\frac{1}{s\sqrt{t(t+\frac{4m^2 s}{4m^2-s})}}\sum_i r_i(s) Y^{-j_i(s)-1}, \quad Y = e^{i\tilde{\theta}_s} \; .$$

with

$$\frac{1}{Y} \simeq \frac{4m^2-s}{s}\frac{t}{m^2} + \dots$$

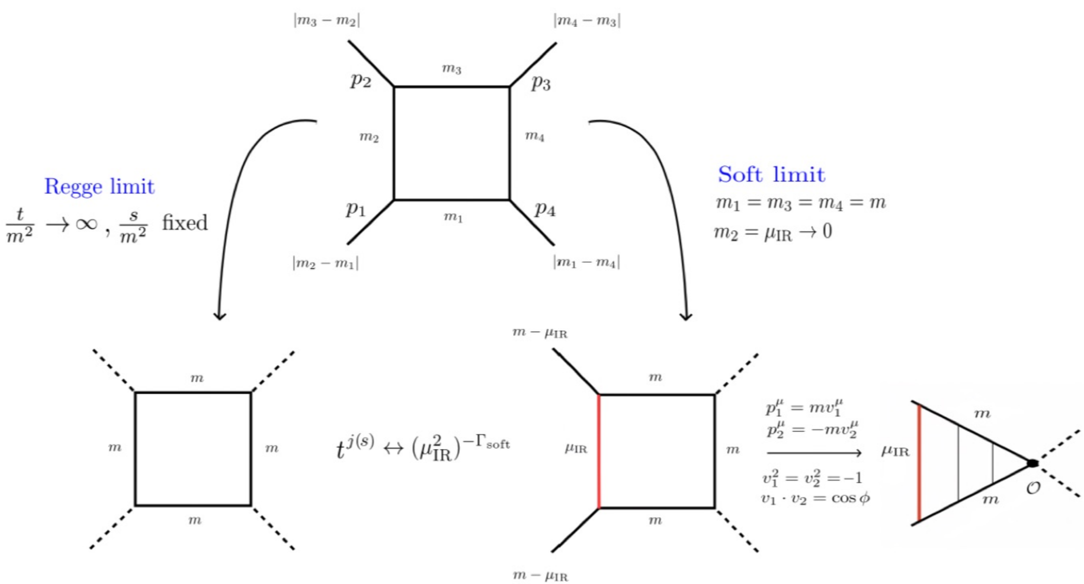
$\Rightarrow$ $j_0(s)$ ntrol the large-t behaviour

$j_1(s), j_2(s) \dots$ are the subleading trajectories

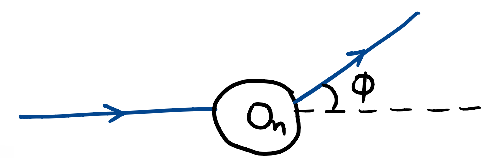


The Regge/cusp correspondence

$$u = \frac{4m_1 m_3}{-s+(m_1-m_3)^2}, \quad v = \frac{4m_2 m_4}{-t+(m_2-m_4)^2}$$



$$j_n(s) + 1 = -\Gamma_{\mathrm{cusp},O_n}(\phi) \ , \quad s = 4m^2 \sin^2(\phi/2)$$

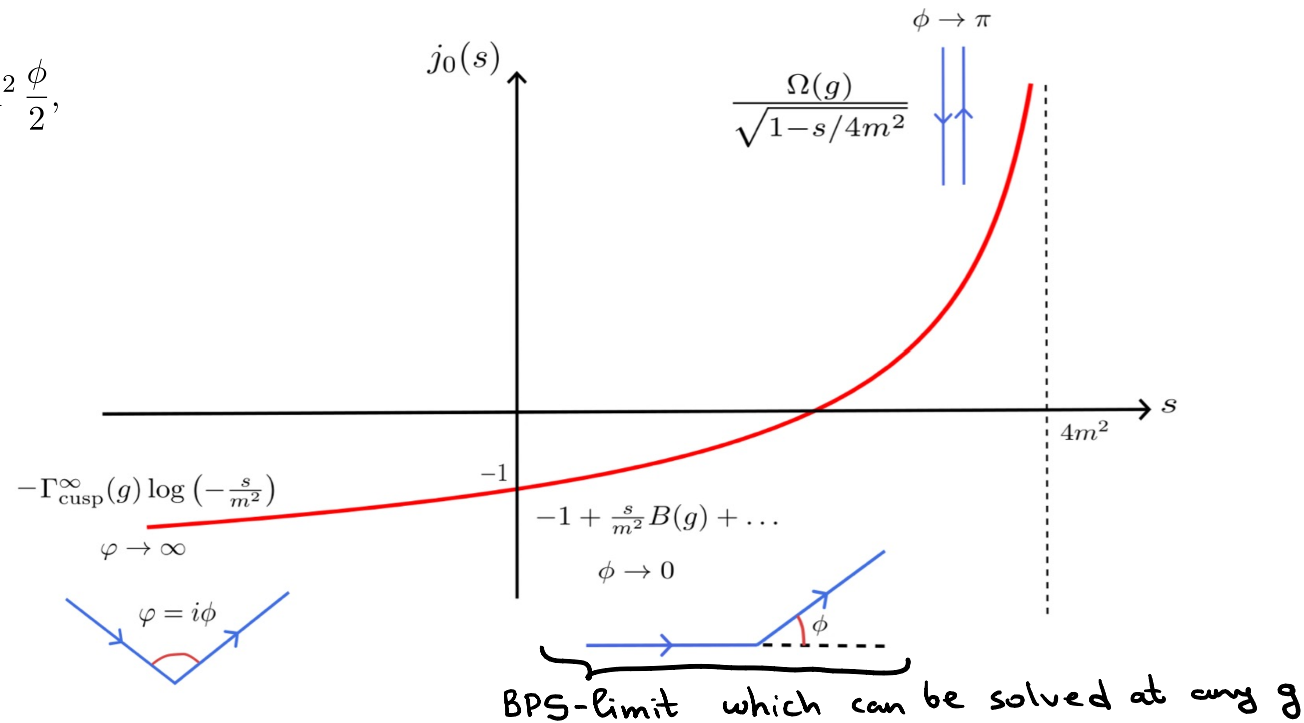


$\Gamma_{\mathrm{cusp},O_n}(\phi)$

$$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Gamma_{\mathrm{cusp}}(g,\phi)}$$

How does the leading Regge trajectory expected to look like?

$$s = 4m^2 \sin^2 \frac{\phi}{2},$$

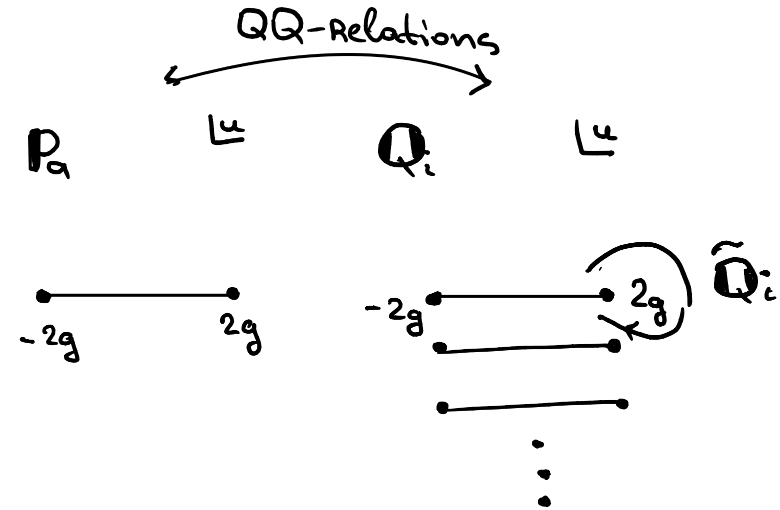


Regge trajectories from the Quantum Spectral Curve

Asymptotics at large u:

$$\mathbf{P}_a \sim 1$$

$$\mathbf{Q}_i \sim e^{\pm \varphi u}$$



$$\mathbf{P}_a = x(u)^{\mathrm{powP}} \sum_{n=0}^{\mathrm{cutP}} \frac{c_n}{x(u)^{2n}}$$

Solve perturbatively for c_n for small g.

Solve for the gluing condition between $\mathbf{Q}_i$ and $\tilde{\mathbf{Q}}_i$

In the end, the goal is to find $\Gamma_{\mathrm{cusp}}(\varphi, g)$

The workflow of the QSC

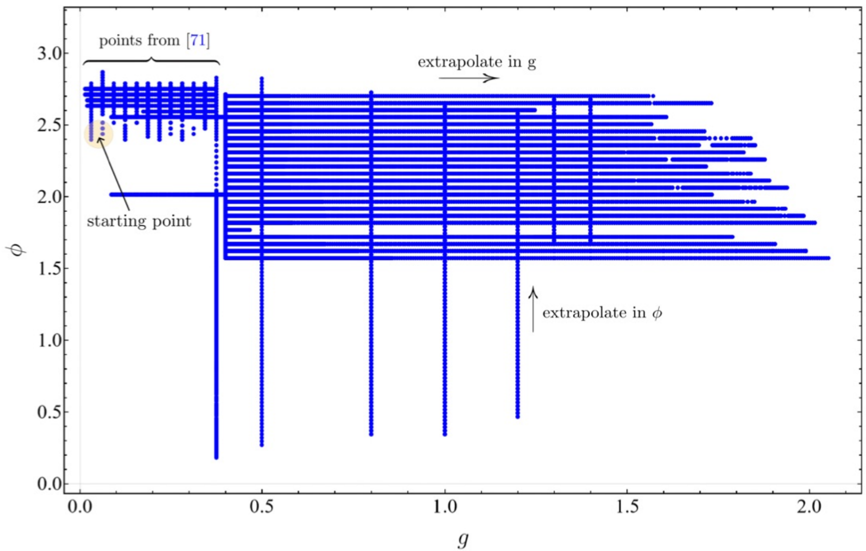
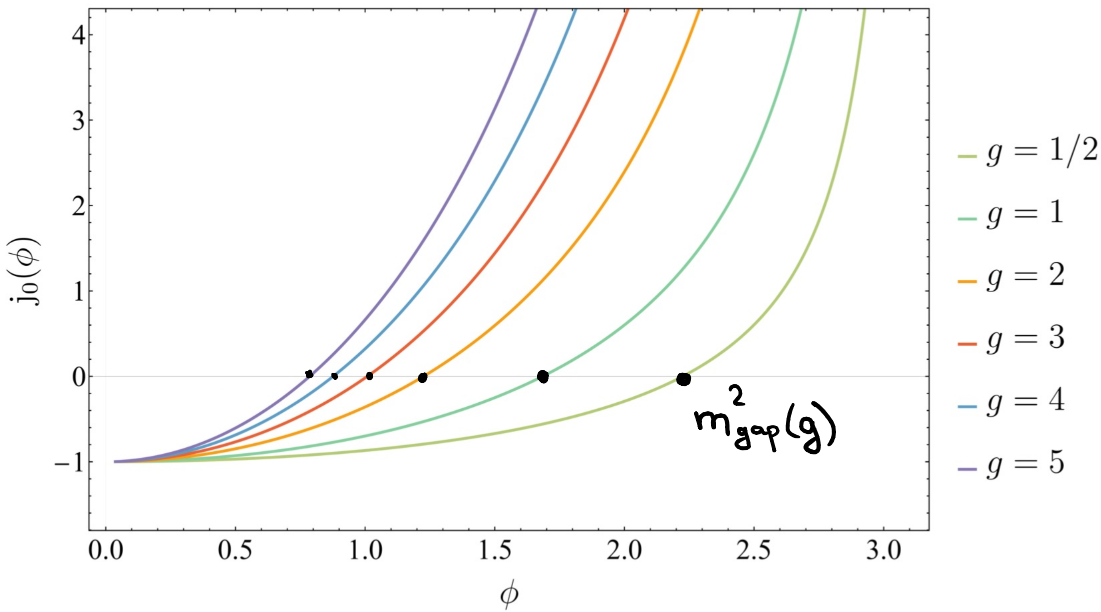
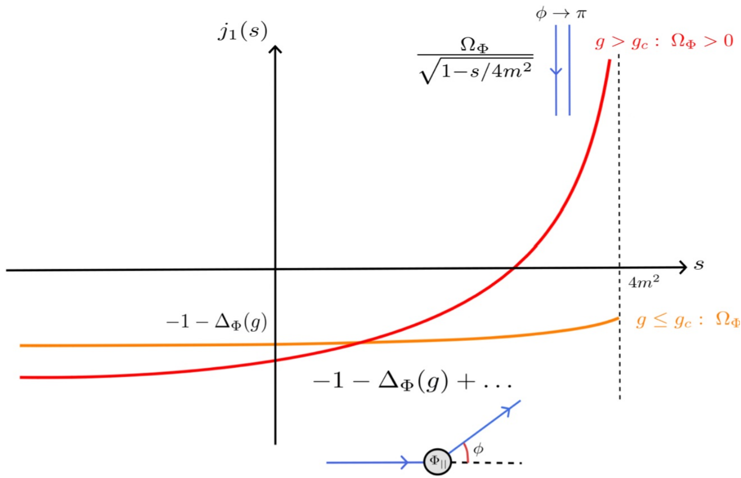


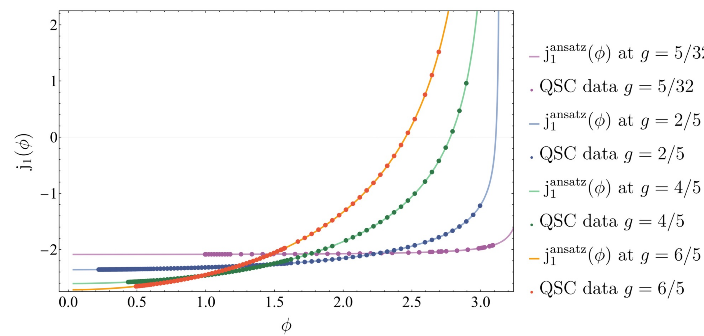
Figure 11: QSC data point for $\Gamma_{\text{cusp}}(g, \phi)$ as a function of (g, ϕ). Here we summarize our workflow: in the left corner, at small g, the QSC is initialized using the perturbative solution. Then we move in ϕ at fixed g or in g at fixed ϕ.

The results for the leading Regge trajectory

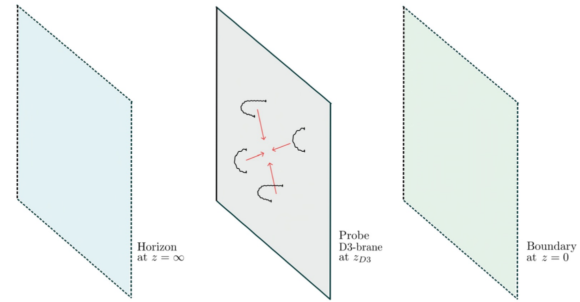


The subleading trajectory





Worldsheet bootstrap



$$\underbrace{f(s,t) = -\frac{g^2}{\pi^2 m^4}\frac{\Gamma\left(-\frac{gs}{m^2\pi}\right)\Gamma\left(-\frac{gt}{m^2\pi}\right)}{\Gamma\left(1-\frac{gs}{m^2\pi}-\frac{gt}{m^2\pi}\right)} + \ldots\,, \quad g \gg 1\,,}_{\text{flat-space tree-level}}$$

what is $1/g$ correction?

$$\underbrace{S = \frac{gs}{m^2\pi}, \quad T = \frac{gt}{m^2\pi}} \qquad \underbrace{\hat{f}(S,T) \equiv \frac{\pi^2 m^4}{g^2} f\left(\frac{m^2\pi}{g}S, \frac{m^2\pi}{g}T\right)}_{\text{dimensionless}}$$

$$\hat{f}(S,T) = \hat{f}^{(0)}(S,T) + \frac{1}{g}\hat{f}^{(1)}(S,T) + \cdots\,,$$

$$\hat{f}^{(1)}(S,T) = \int_0^1 \frac{x^{-S}(1-x)^{-T}}{x(1-x)}\underbrace{h(S,T;x)}dx,$$

linear combination of harmonic polylogs $H_w(x)$ up to weight 3 with letters $\{0,1\}$

$$\hat{f}^{(0)}(S,T) = -\frac{\Gamma(-S)\Gamma(-T)}{\Gamma(1-S-T)} \rightarrow \hat{f}^{(0)}(S,T) = \frac{1}{S+T}\int_0^1 \frac{x^{-S}(1-x)^{-T}}{x(1-x)}dx\,.$$

$$h(S,T;x) = \sum_w \frac{P_w(S,T)}{(S+T)^2}H_w(x),$$

The bootstrap problem

Amplitude: $A(s,t) = s^2 f(s,t)$ $\rightarrow$ improved Regge behavior

$$\lim_{|s|\to\infty} \frac{f(s,t)}{s^0} = 0$$

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EFT expansion:

$$f_{EFT}(s,t) = -\frac{1}{st} + a_{0,0} + a_{1,0}(s+t) + a_{2,0}(s^2+t^2) + a_{2,1}(st) + \dots$$

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fixed coeff.

1-loop exact

$$a_{0,0} = \frac{g^2}{6m^4}$$

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fixed coeff.

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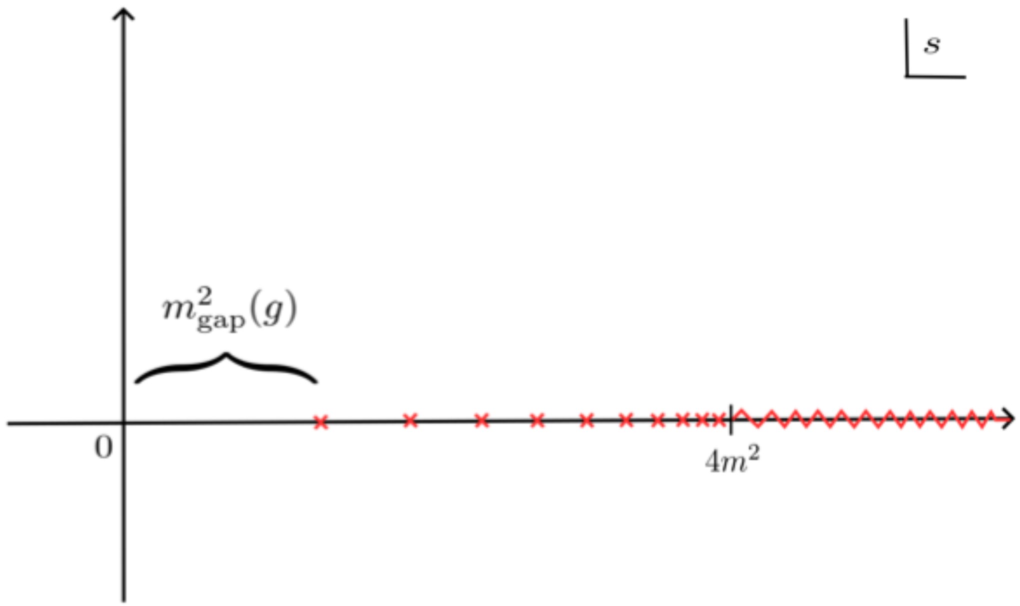
$$a_{0,0} = \frac{g^2}{6m^4}$$

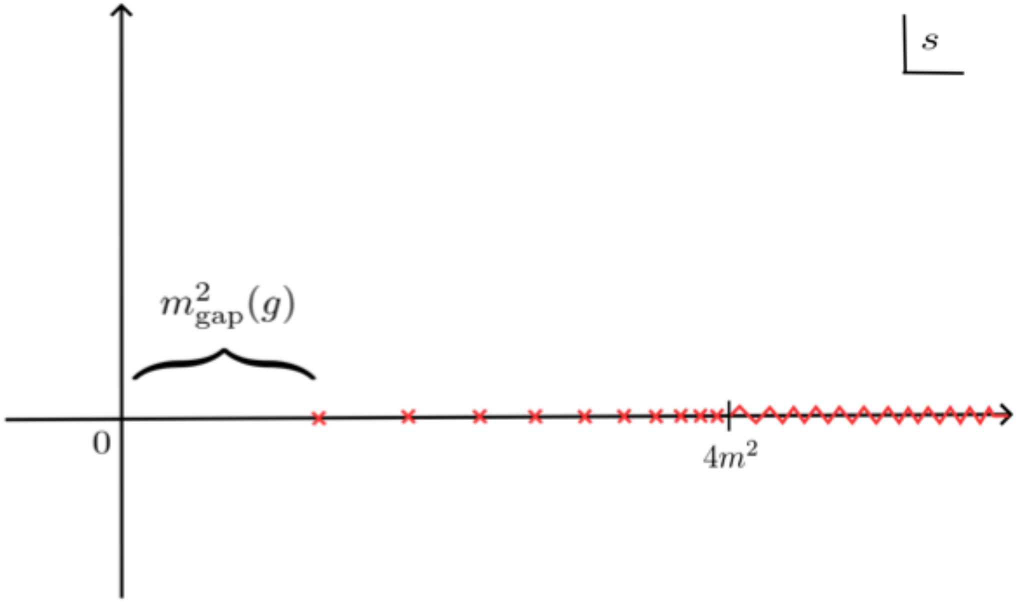
The bootstrap problem

Analytic structure:



The bootstrap problem

Analytic structure:



Unitarity

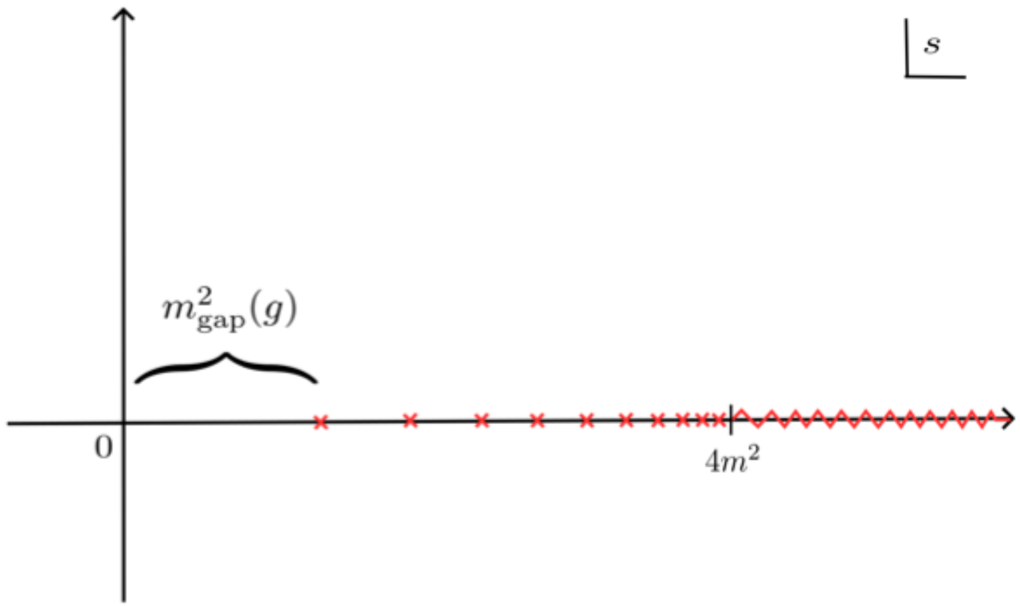
$SO(3)$

$$\text{Im} f = \sum_J \rho_J(s) P_J(\cos\theta)$$

$$\rho_J(s) \geq 0 , \quad s \geq 0$$

The bootstrap problem

Analytic structure:



Unitarity

$$SO(3) \xrightarrow[\text{dual conf. inv.}]{} \quad SO(4) \quad\quad SO(1,3)$$

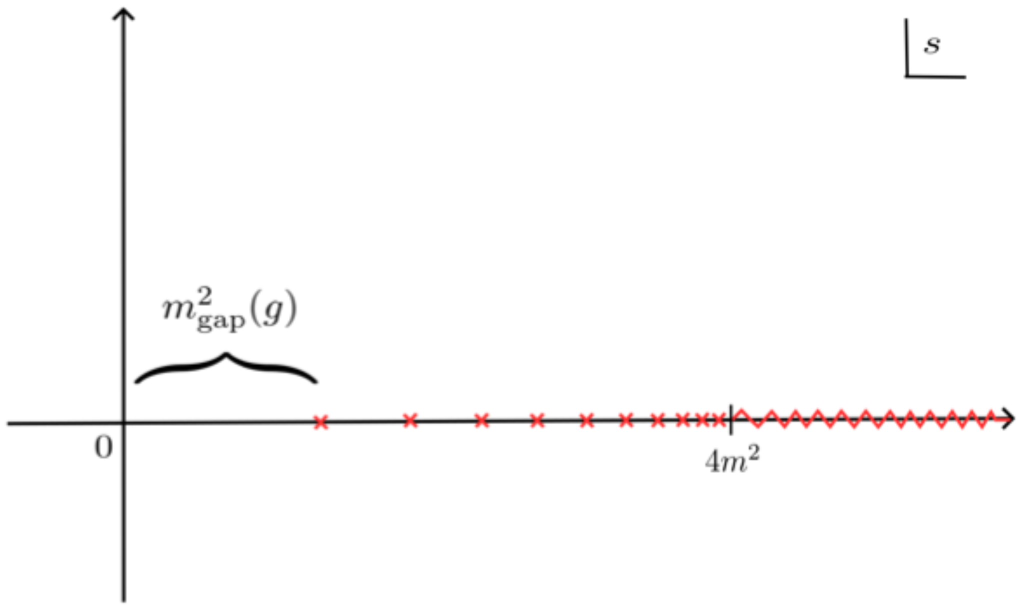
$0 \leq s \leq 4m^2$ $\quad\quad$ $4m^2 \leq s$

$$\mathrm{Im}\, f = \sum_J \rho_J(s) P_J(\cos\theta)$$

$$\rho_J(s) \geq 0\,, \quad s \geq 0$$

The bootstrap problem

Analytic structure :



Unitarity

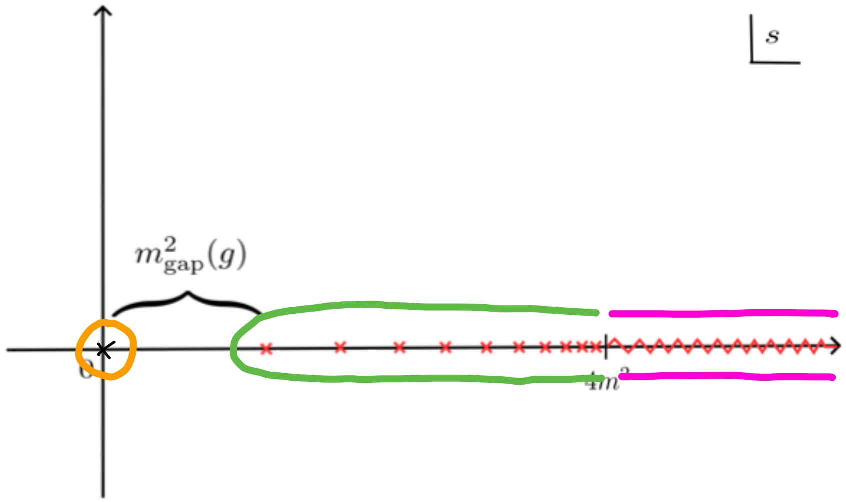
SO(3) $\xrightarrow{\text{dual conf. inv.}}$ SO(4) $\quad$ SO(1,3)

$0 \leq s \leq 4m^2 \qquad 4m^2 \leq s$

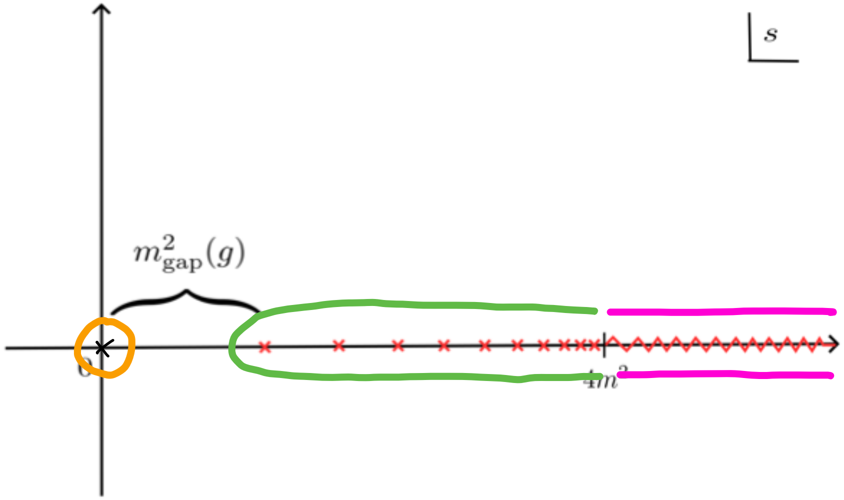
$$\mathrm{Im}\, f = \sum_J \rho_J(s) P_J(\cos\theta) = \sum_J c_J(s) \tilde{P}_J(\cos\tilde{\theta}) + \int d\nu\, c_\nu(s) \Omega_\nu(\cosh\tilde{\theta})$$

$\rho_J(s) \geq 0\,, \quad s \geq 0 \qquad c_J(s) \geq 0 \qquad c_\nu(s) \geq 0$

Sum rules



Sum rules



$$\boxed{a_{n,\ell}} = \left\langle \frac{1}{z^{2+n}} \left(\frac{z}{z_{5d}}\right)^{\ell} \frac{4^{\ell}\Gamma(J+\ell+2)}{(J+1)\Gamma(2\ell+2)\Gamma(J-\ell+1)} \right\rangle_{SO(4)} + \left\langle \frac{1}{z^{2+n}} \left(\frac{z}{\bar{z}_{5d}}\right)^{\ell} \frac{4^{\ell}\prod_{k=1}^{\ell}(k^2+\nu^2)}{\Gamma(2\ell+2)} \right\rangle_{SO(1,3)}$$

$$a_{n,\ell} = \left\langle \frac{1}{z^{2+n}} \left(\frac{z}{z_{5d}}\right)^{\ell} \frac{4^{\ell}\Gamma(J+\ell+2)}{(J+1)\Gamma(2\ell+2)\Gamma(J-\ell+1)} \right\rangle_{SO(4)}$$

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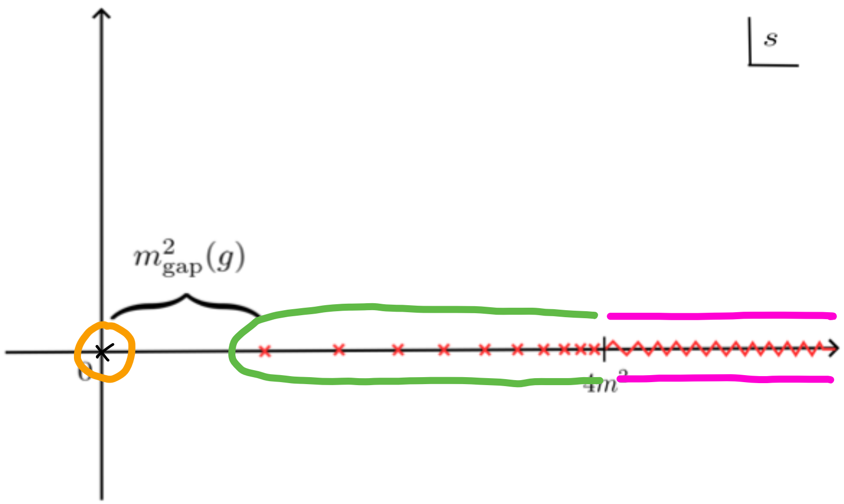

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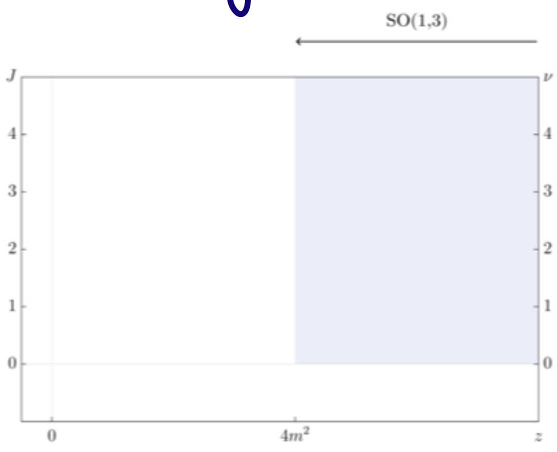
$$\langle \dots \rangle_{SO(4)} = \frac{1}{\pi}\int_{m^2_{\text{gap}}}^{4m^2} \frac{dz}{z} \sum_{J=0}^{\infty} c_J(z)(\dots)\,,$$

$$\langle \dots \rangle_{SO(1,3)} = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dz}{z} \int_0^{\infty} d\nu c_\nu(z)(\dots)\,,$$

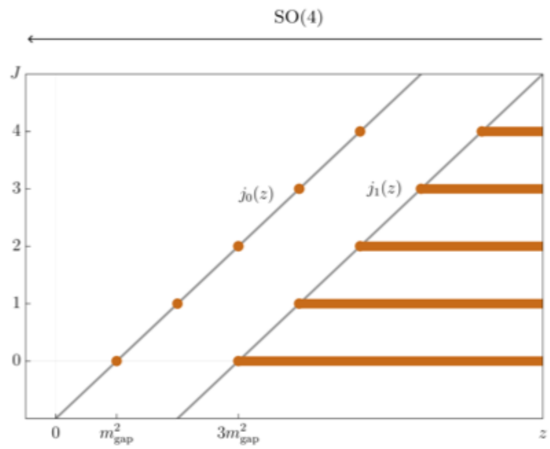
- normalize by $a_{0,0}$ & m_{gap} : bound $\bar{a}_{n,\ell} = \dfrac{a_{n,\ell}}{a_{0,0}} m_{gap}^{2n}$
- impose tree-level pole as extra constraint

Knowledge of g

$g = 0$



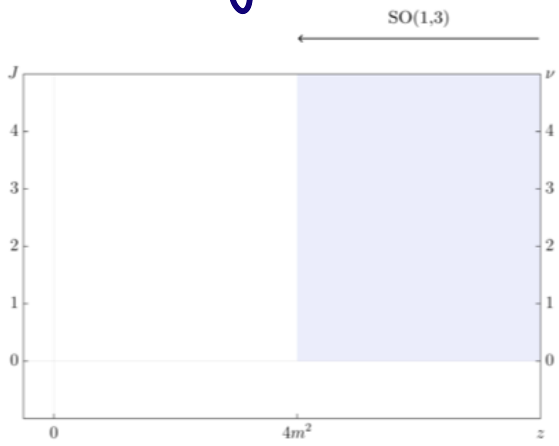
$g = \infty$

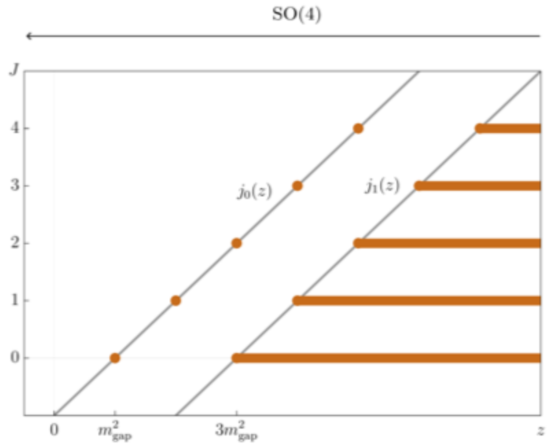


$0 < g < \infty \qquad m_{gap}^2 \sim \frac{\pi m^2}{g}$

Knowledge of g

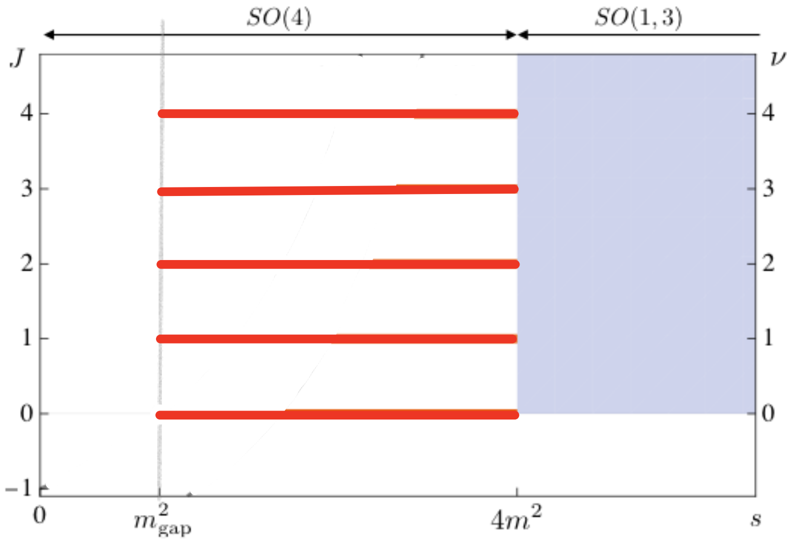
$g=0$ $\qquad$ $g=\infty$





$0<g<\infty \qquad m_{gap}^2 \sim \frac{\pi m^2}{g}$

without QSC data



Knowledge of g

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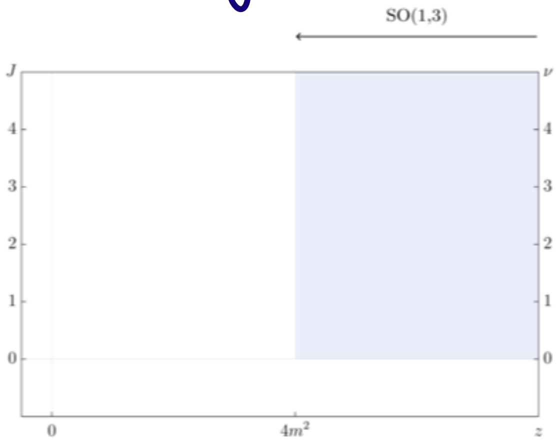

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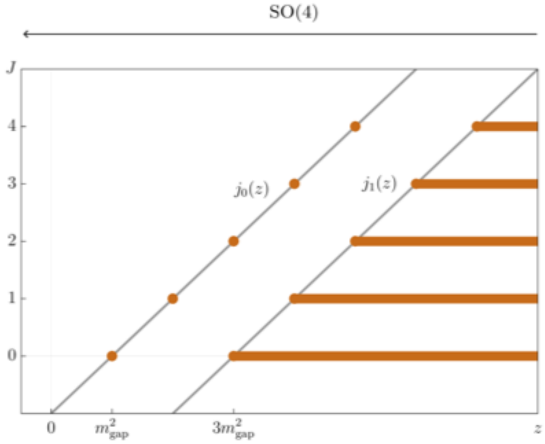
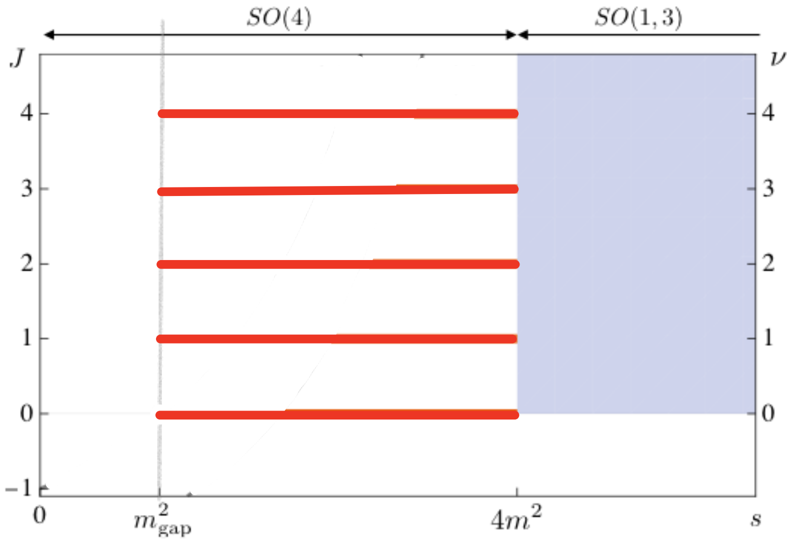
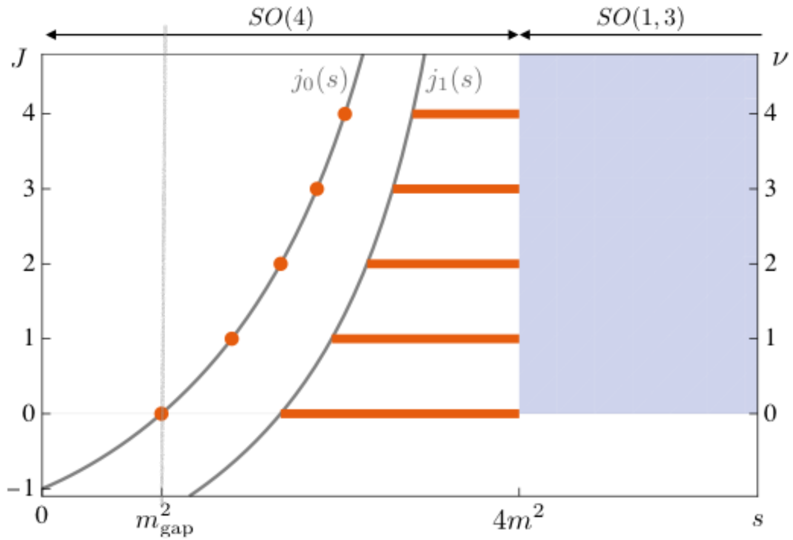


$0 < g < \infty \qquad m_{gap}^2 \sim \frac{\pi m^2}{g}$

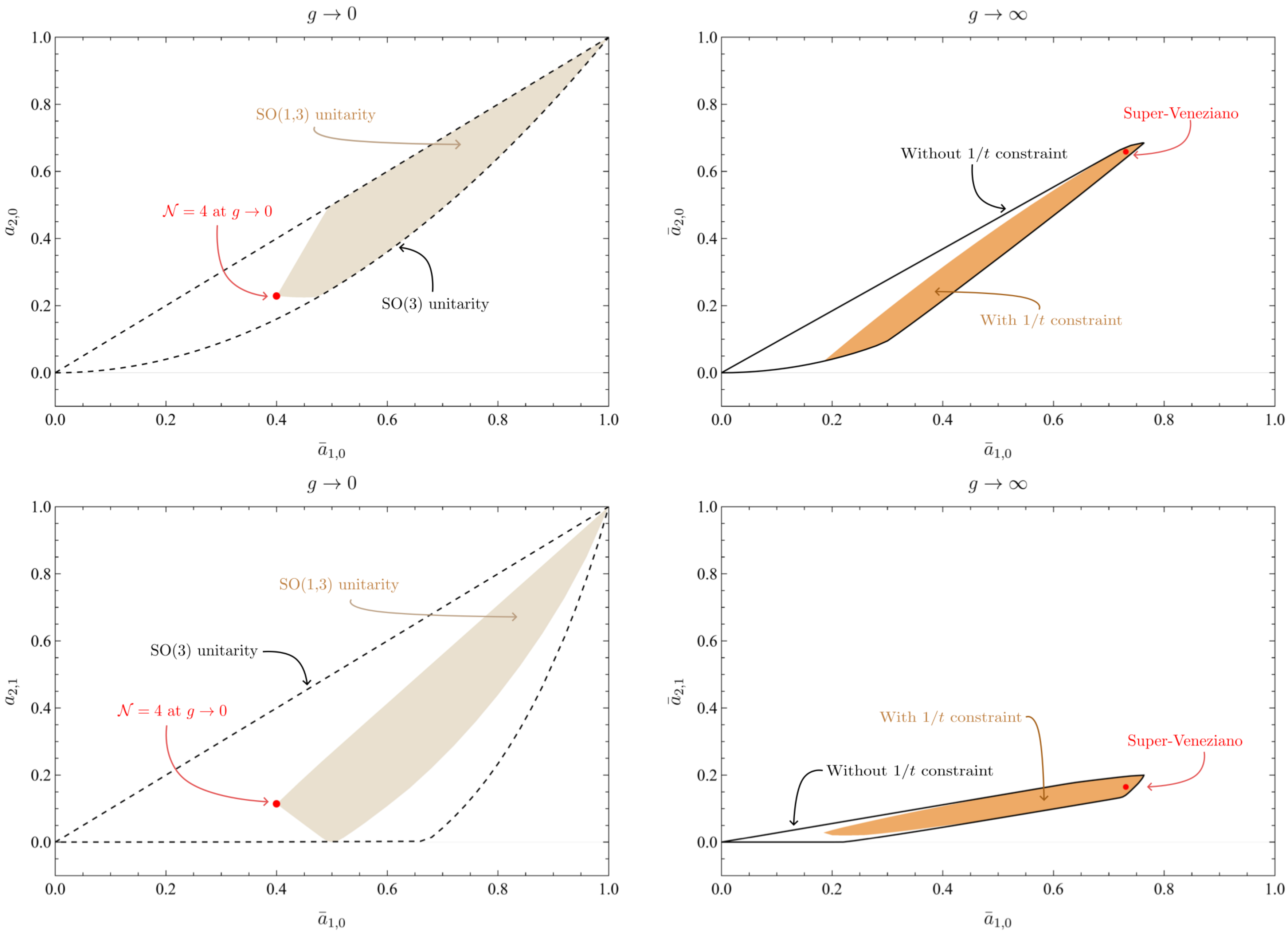
without QSC data



with QSC data

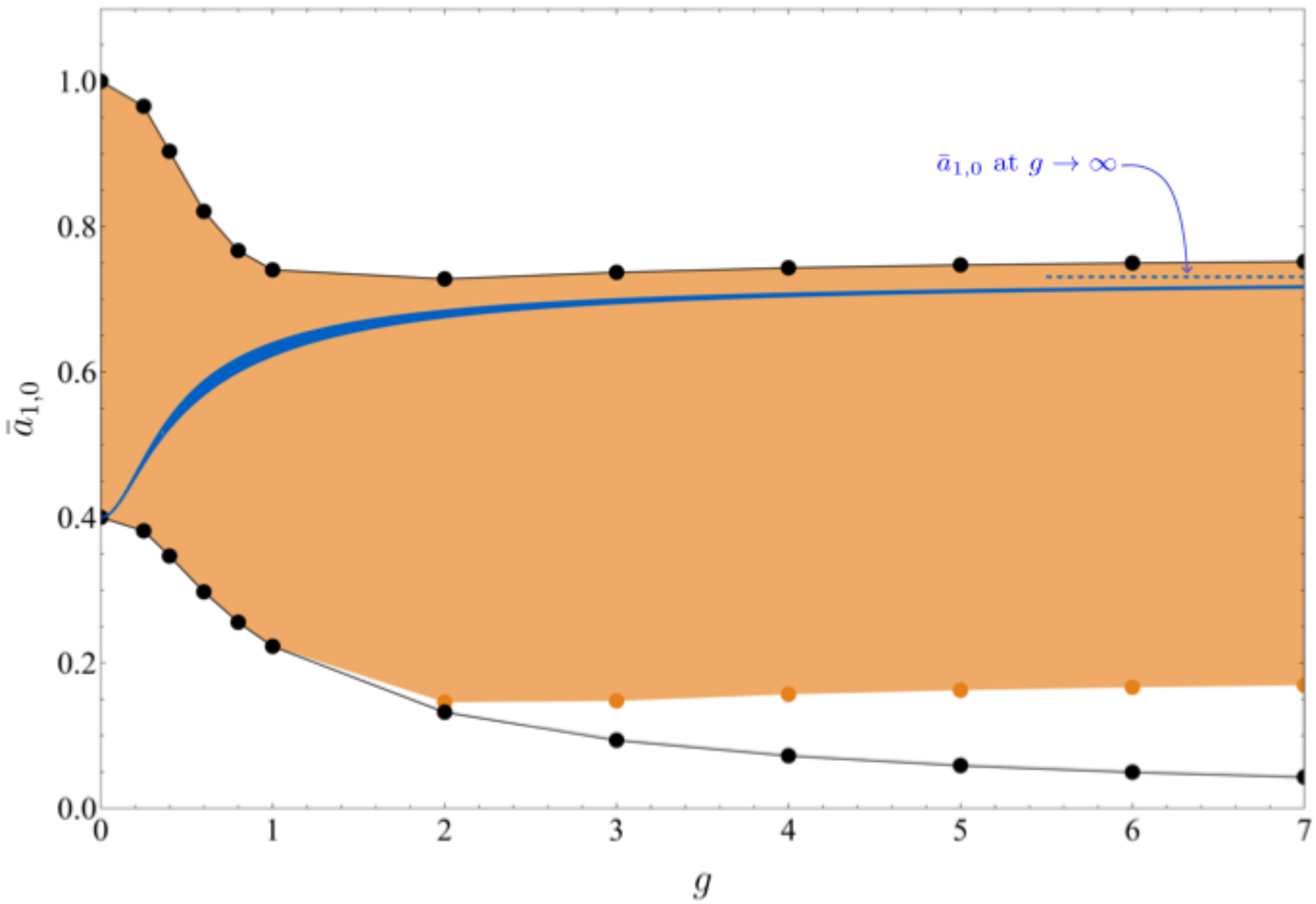


Bounds



$g \to 0$
$g \to \infty$
SO(1,3) unitarity
SO(3) unitarity
$\mathcal{N} = 4$ at $g \to 0$
Without $1/t$ constraint
With $1/t$ constraint
Super-Veneziano
$a_{2,0}$
$\bar{a}_{2,0}$
$a_{2,1}$
$\bar{a}_{2,1}$
$\bar{a}_{1,0}$

Bounds



Padé Model

Interpolates between strong and weak coupling

$$g \ll 0 : \quad A(g) = A_{0,0} + g^2 A_{0,1} + g^4 A_{0,2}$$

$$g \sim \infty : \quad A(g) = A_{\infty,0} + \frac{A_{\infty,1}}{g} + \dots$$

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Map cplx. g-plane to unit disk in z-plane

$$A_{[2,2]}(g) = \left. \frac{b_0 + b_1 z + b_2 z^2}{1 + c_1 z + c_2 z^2} \right|_{z = z(g)}$$

Padé Model

Interpolates between strong and weak coupling

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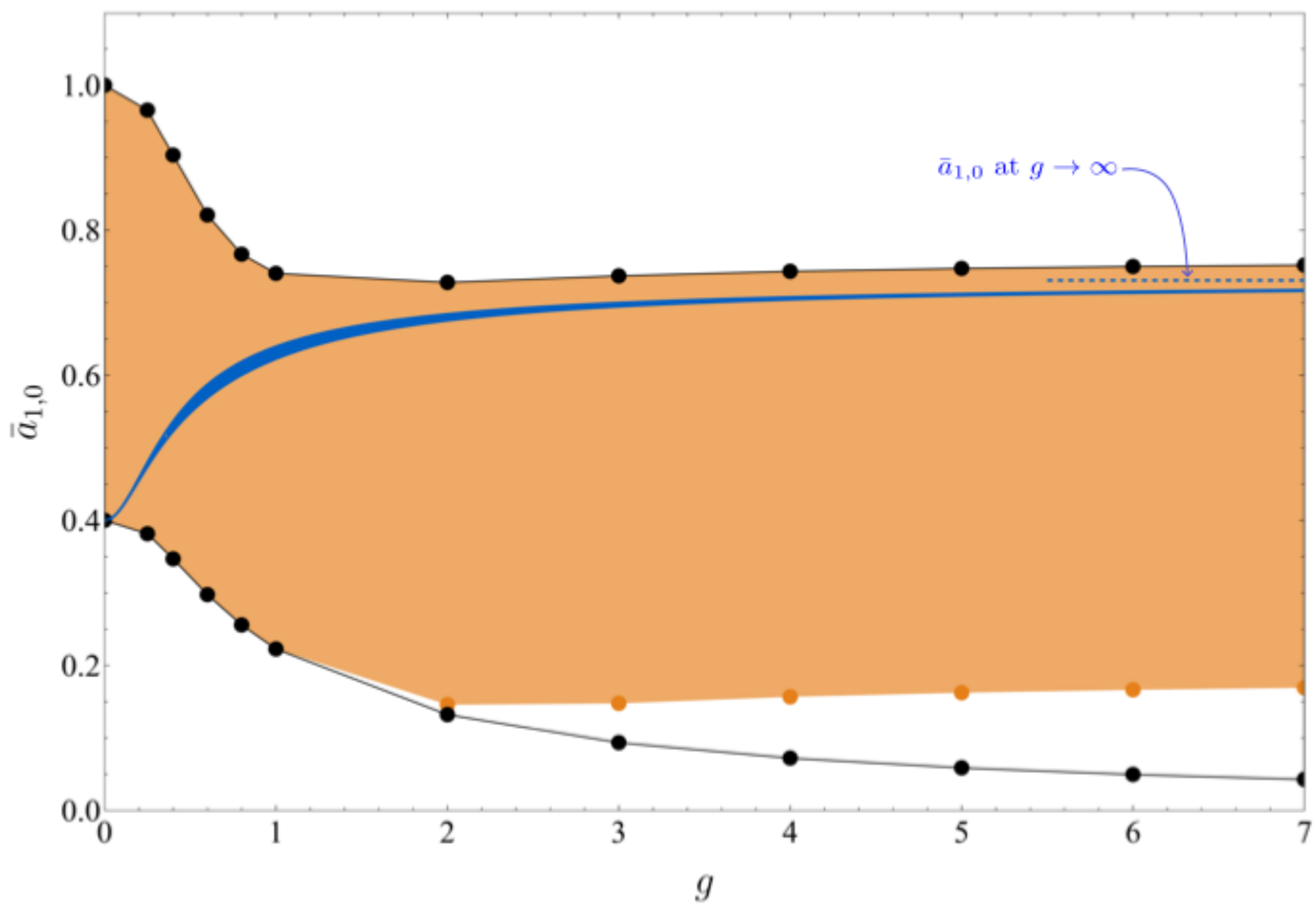
$$g \sim \infty: \quad A(g) = A_{\infty,0} + \frac{A_{\infty,1}}{g} + \ldots$$

Map cplx. g-plane to unit disk in z-plane

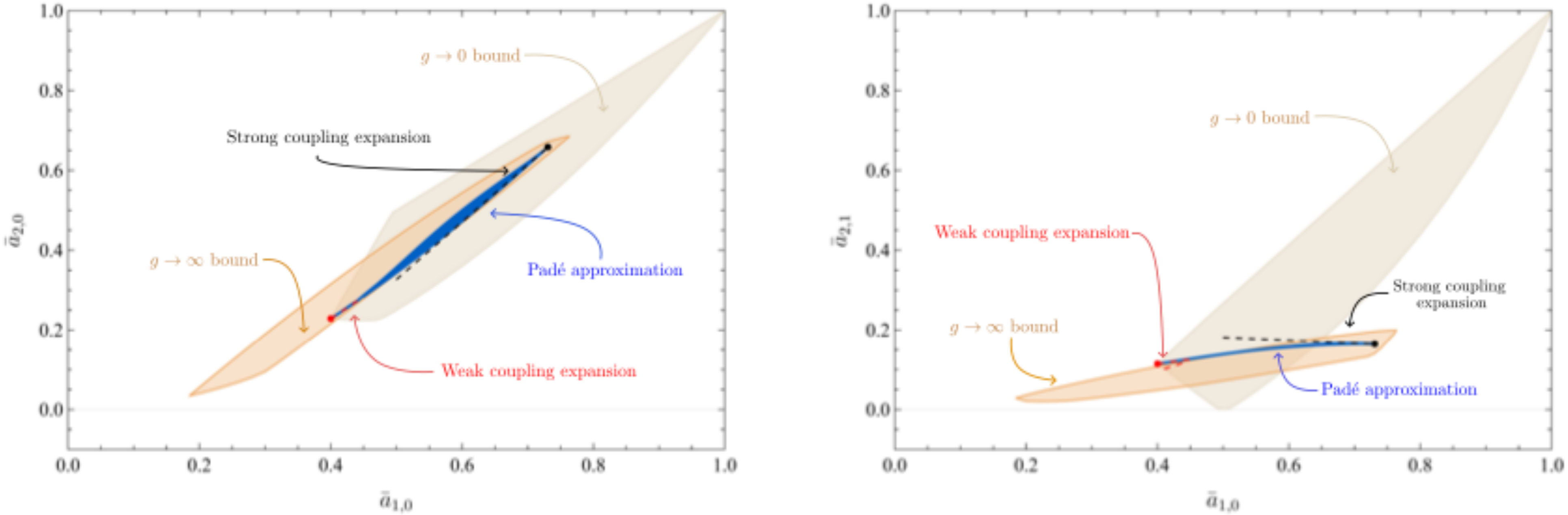
$$A_{[2,2]}(g) = \left. \frac{b_0 + b_1 z + b_2 z^2}{1 + c_1 z + c_2 z^2} \right|_{z = z(g)}$$

- some subtleties with log term
- results checked against bootstrap

Bounds



Bounds



$g \to 0$ bound
Strong coupling expansion
$g \to \infty$ bound
Padé approximation
Weak coupling expansion
$\bar{a}_{2,0}$
$\bar{a}_{1,0}$
Weak coupling expansion
$g \to 0$ bound
Strong coupling expansion
$g \to \infty$ bound
Padé approximation
$\bar{a}_{2,1}$
$\bar{a}_{1,0}$

The end.