String Journal Club 28-10-2025

ABSTRACT: We use effective string theory (EST) to describe a toroidal 2d domain wall embedded in a 3d torus. In particular, we compute the free energy of the domain wall in an expansion in inverse powers of the area, up to the second non-universal order that involves the Wilson coefficient γ_3.

In order to test our predictions, we simulate the 3d Ising model with anti-periodic boundary conditions, using a two-step flat-histogram Monte Carlo method in an ensemble over the boundary coupling J that delivers high-precision free energy data. The predictions from EST reproduce the lattice results with only two adjustable parameters: the string tension, $1/\ell_s^2$, and γ_3. We find $\gamma_3/|\gamma_3^{\min}| = -0.82(15)$, which is compatible with previous estimates.

Effective string theory

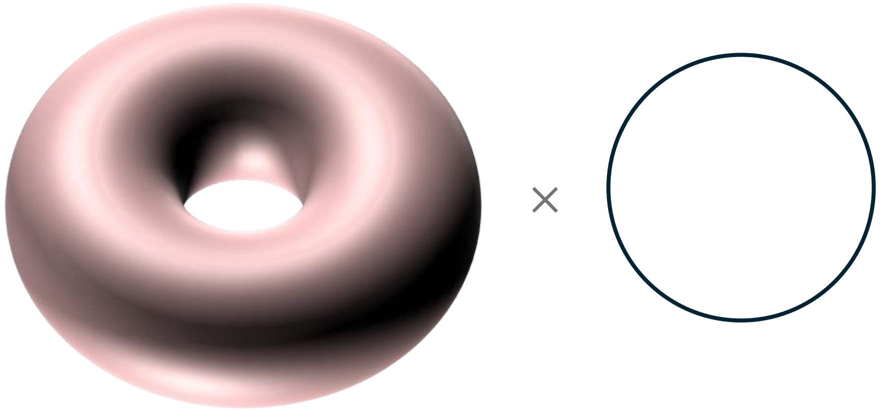
- In many contexts (spin systems, gauge theories…) some non-perturbative solutions are described by **flux tubes**
- At long distances from the tube, a good description is given by **effective string theory (EST)**
- **Deformation of Nambu-Goto action** (for a 3d theory)

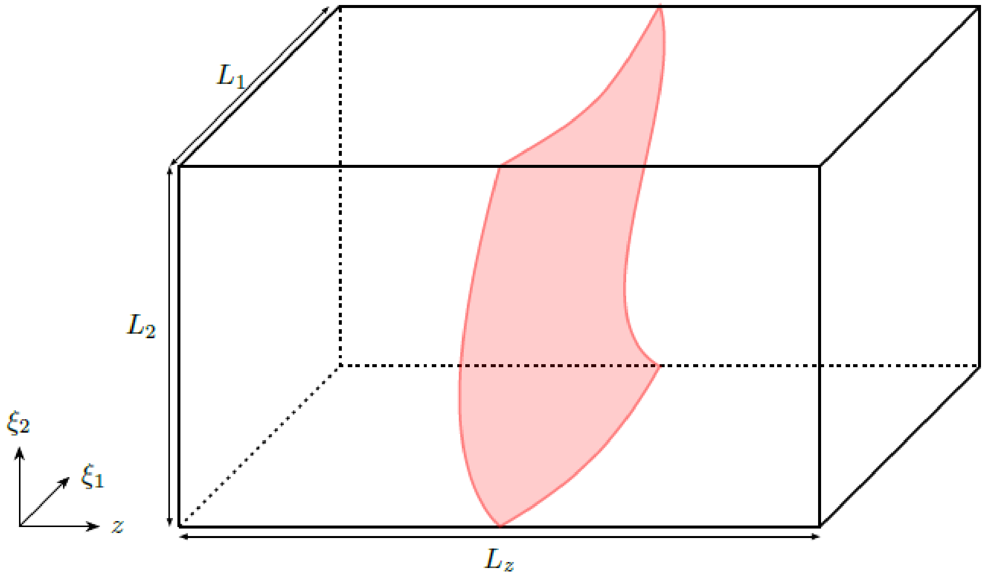
$$S = -\int d^2\sigma \sqrt{h} \left(\frac{1}{\ell_s^2} + 2\gamma_3 \ell_s^2 K^4 + \mathcal{O}\left(\ell_s^4\right) \right)$$

Theory dependent Wilson coefficient!

The system

- Consider a 3d theory with a 2d defect. The authors take **3d Ising**.
- The 2d defect is seen as an EST worldsheet.
- The authors consider a **toroidal worldsheet**.





area: $\mathcal{A} = \frac{L_1 L_2}{\ell_s^2}$

Free energy behaviour

- The authors want to compute the **free energy** of the toroidal defect.
- The free energy will organize in **inverse powers of the area of the torus $\mathcal{A}$**

$$F(\tau) = F_{\mathrm{U}}(\tau) - \frac{\gamma_3}{\mathcal{A}^3}\frac{2\pi^6}{225}(\tau-\bar{\tau})^4 E_4(\tau)\, E_4(-\bar{\tau}) + O(\mathcal{A}^{-4})$$

Universal part: can also be expanded in powers of $\mathcal{A}$

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Theory dependent part: depends on the Wilson coefficient

Strategy: **compute the free energy with Monte Carlo and then compare to evaluate γ_3!**

Universal behaviour

- Let's compute the universal part.

- Several results are already available: **GGRT spectrum** → We simply compute the partition function and then the free energy

$$\mathcal{E}_{k,k',p_z} = \sqrt{1 + \frac{4\pi u}{\mathcal{A}}\left(k + k' - \frac{1}{12}\right) + \left[\frac{2\pi u\,(k-k')}{\mathcal{A}}\right]^2 + \left(\frac{p_z}{\sigma L_2}\right)^2},$$

$$Z_{\mathrm{U}} = \mathrm{Tr}\left[e^{-L_1 H}\right] = \frac{L_z}{2\pi}\int dp_z \sum_{k,k'} p(k)p(k')e^{-\sigma L_1 L_2 \mathcal{E}_{k,k',p_z}},$$

$$= \left(\frac{\sigma \mathcal{A} L_z^2}{u\pi^2}\right)^{1/2} \sum_{k,k'} p(k)p(k')\mathcal{E}_{k,k'} K_1\left(\mathcal{A}\mathcal{E}_{k,k'}\right)$$

Universal behaviour

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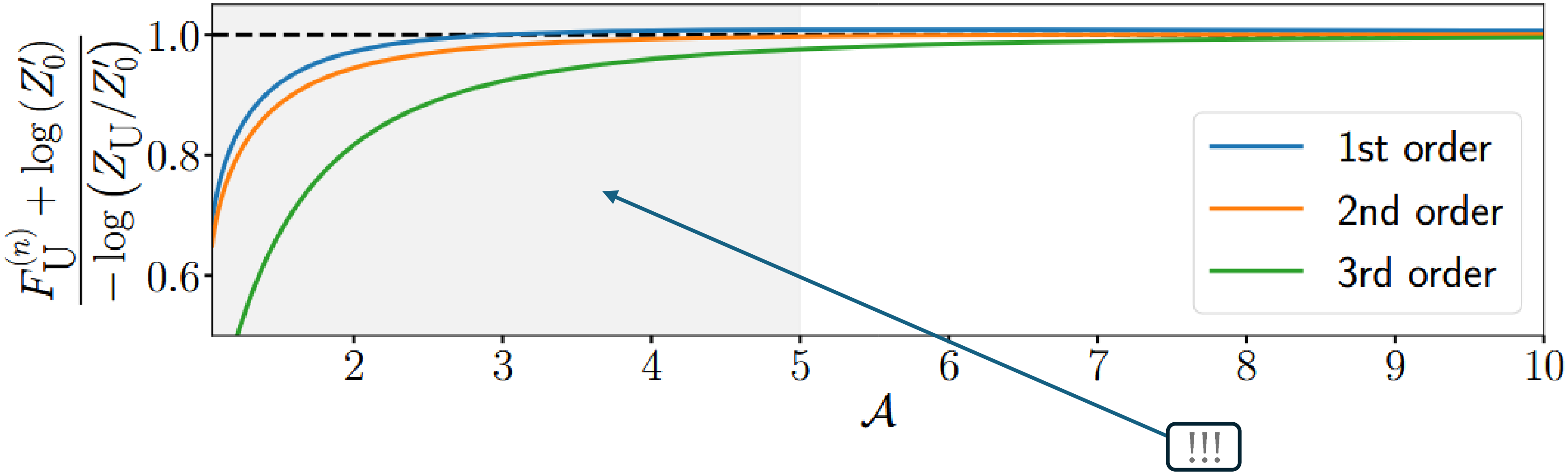
$$F_{\mathrm{U}} = -\log Z_{\mathrm{U}} = \mathcal{A} - \frac{1}{2}\log\left(\frac{\sigma i}{\pi(\tau-\bar{\tau})}L_z^2\right) + 2\log|\eta(\tau)| + \sum_{n=1}^{\infty}\frac{g_n(\tau,\bar{\tau})}{\mathcal{A}^n}$$



Theory-dependent behaviour

- The really interesting part is the one **dependent on γ_3.**
- The leading correction is available **via path integration:**

$$F_{\mathrm{NU}}(iu) = -\frac{32\gamma_3\pi^6}{225\mathcal{A}^3}u^4E_4\,(iu)^2 + O(\mathcal{A}^{-4})$$

- It correctly reproduces the **long string limit:**

$$\Delta E_0 \equiv \lim_{L_1\to\infty}\frac{\Delta F_{\mathrm{NU}}(L_1,L_2)}{L_1} = -\frac{32\pi^6}{225}\frac{\sqrt{\sigma}\gamma_3}{(\sqrt{\sigma}L_2)^7} + \dots$$

Even more corrections?

- What if we could still compute the partition function and then expand?
- We need the spectrum → the **3d EST is integrable at low energy** [Dubovsky et al.]
- We can use **Thermodynamic Bethe Ansatz**

$$\mathcal{E}_{k,k',s,s'} = \mathcal{E}_{k,k'} \underbrace{- \gamma_3 \frac{2048\pi^6 u^4}{225\mathcal{A}^3} \frac{(240s+1)(240s'+1)}{\mathcal{E}_{k,k'}\left[\left(\mathcal{E}_{k,k'}+1\right)^2 - \frac{\pi^2 u^2}{\mathcal{A}^2}\left(k-k'\right)^2\right]^3}}_{-\Delta\mathcal{E}_{k,k',s,s'}},$$

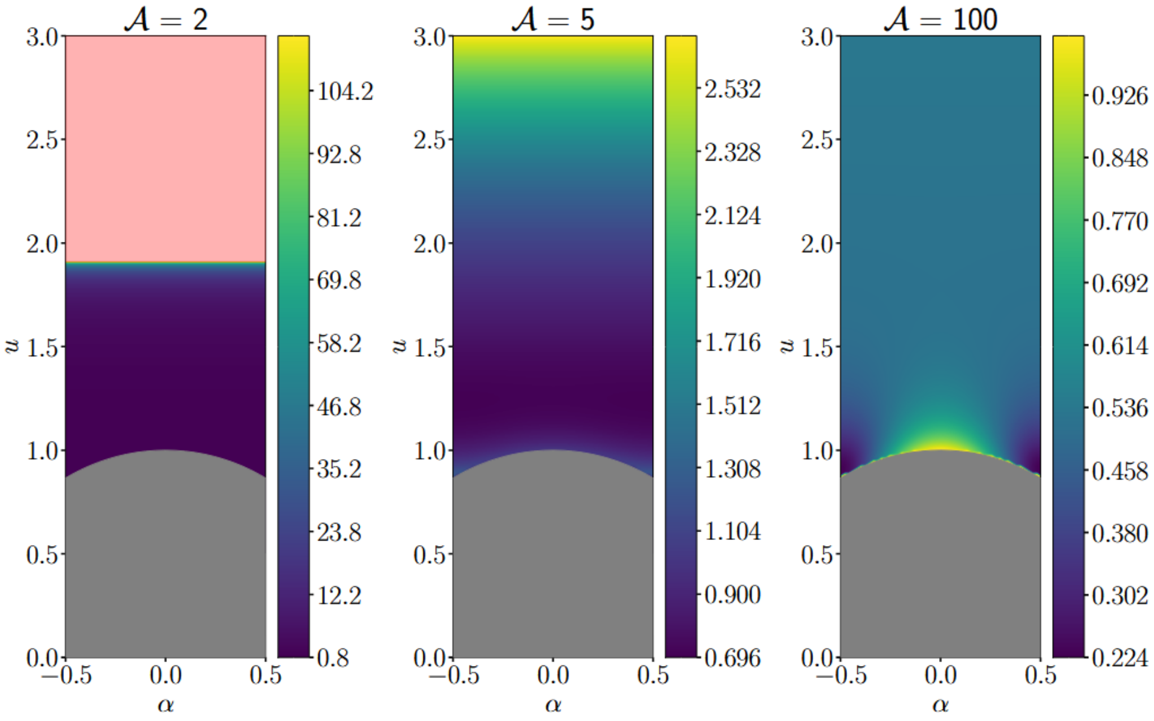
- TBA stops being valid at orders $\mathcal{O}(1/\mathcal{A}^5)$, where γ_3^2 **inelastic effects are induced.**

Even more corrections?

- We can produce a refined result for a square domain wall

$$F_{\gamma_3} = \frac{32\pi^6}{225\,\mathcal{A}^3} E_4(i)^2 \left(1 - \frac{13}{2\mathcal{A}} + \mathcal{O}(\mathcal{A}^{-2})\right)$$

- How does it compare with the universal sector?



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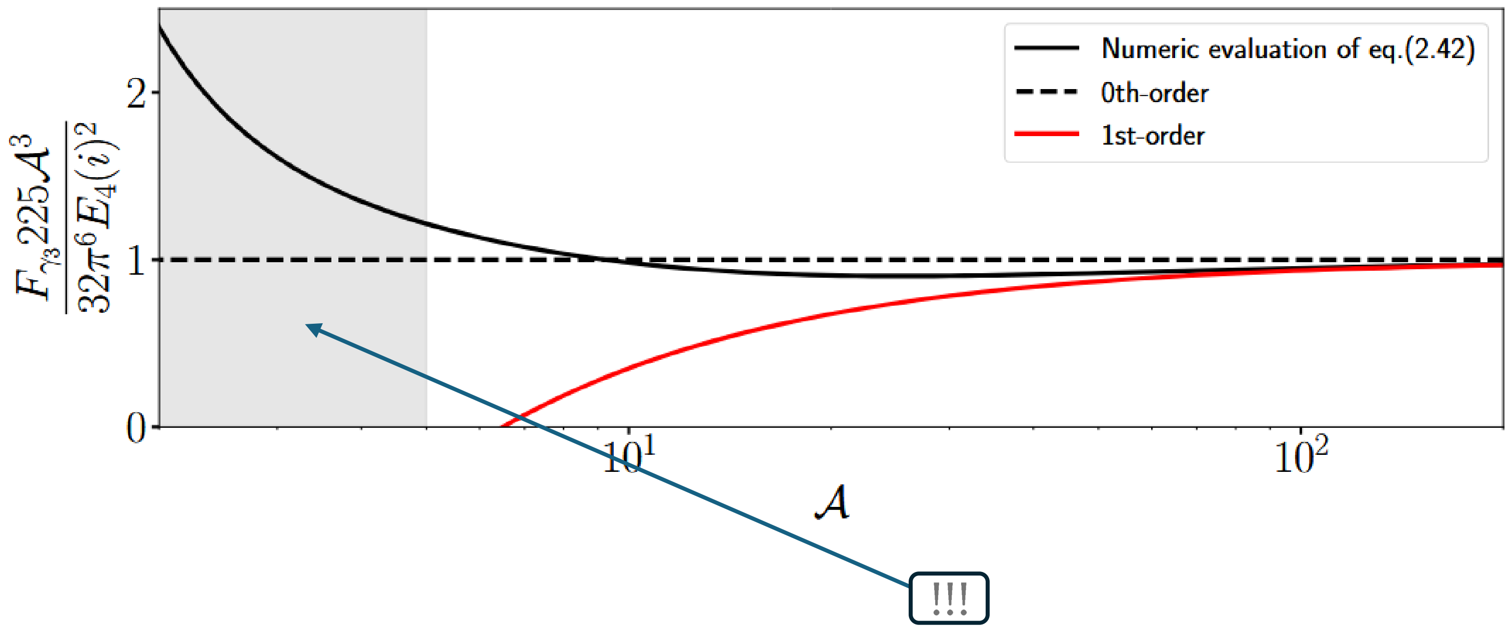
- When can it be compared with simulations?



Lattice setup: Monte Carlo procedure and free parameters

- Setup: usual 3d Ising Monte Carlo. **Antiperiodicity is enforced** on the transverse direction to **simulate the domain wall.**

- The authors sample:

  - **Different temperatures**: the farther from the critical one, the stronger the lattice effects

  - **Different sizes for the domain wall**: the γ_3 parameter can be determined more easily, but computation time largely increases

- **Asymptotic expansions, as highlighted, are not employed.**

Lattice setup: Monte Carlo procedure and free parameters

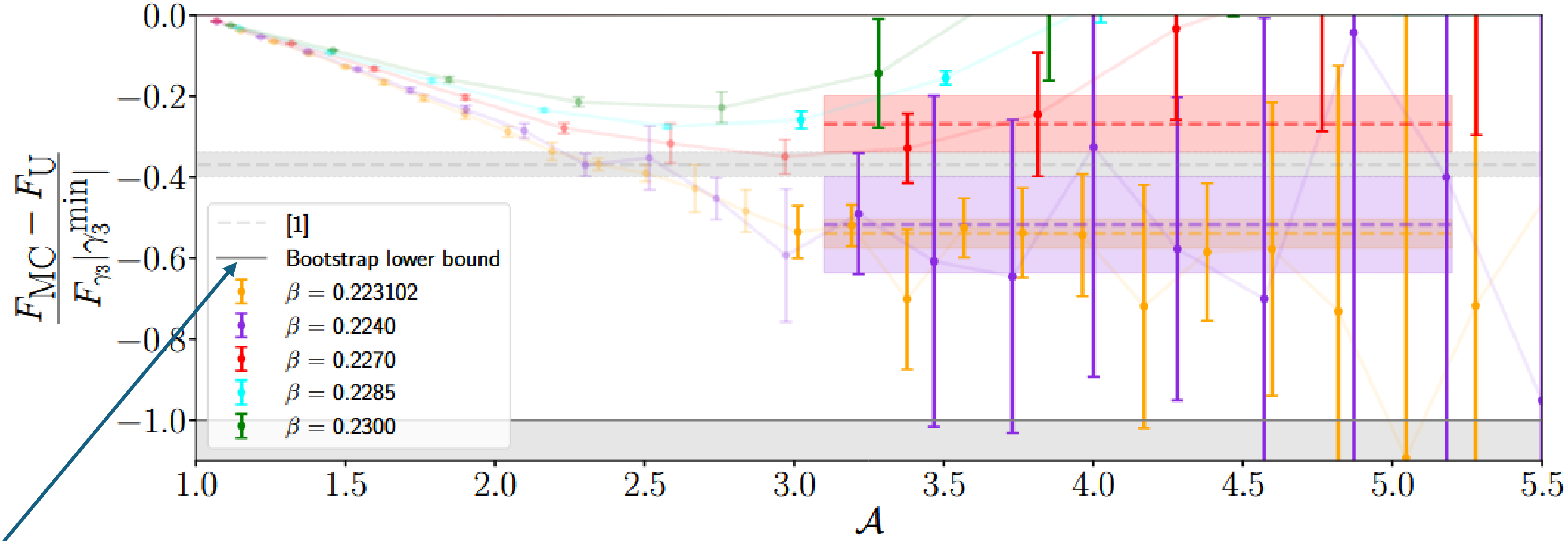


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Figure 10: γ_3^{MC} as defined in eq. (4.6), normalized by the bootstrap lower bound, shown as a function of the area $\mathcal{A}$ for several inverse temperatures. Horizontal lines indicate the γ_3 value extracted from plateau fits. For $\beta = 0.23$ and $\beta = 0.2285$ it increases with $\mathcal{A}$ and no plateau is observed. For $\beta = 0.227$ the behavior is unclear. For $\beta = 0.224$ and $\beta = 0.223102$, there are hints of a plateau, although it is not sharply defined due to large uncertainties. Overall, the true value will be somewhere between the orange data and the bootstrap bound.

Lattice setup: Monte Carlo procedure and free parameters

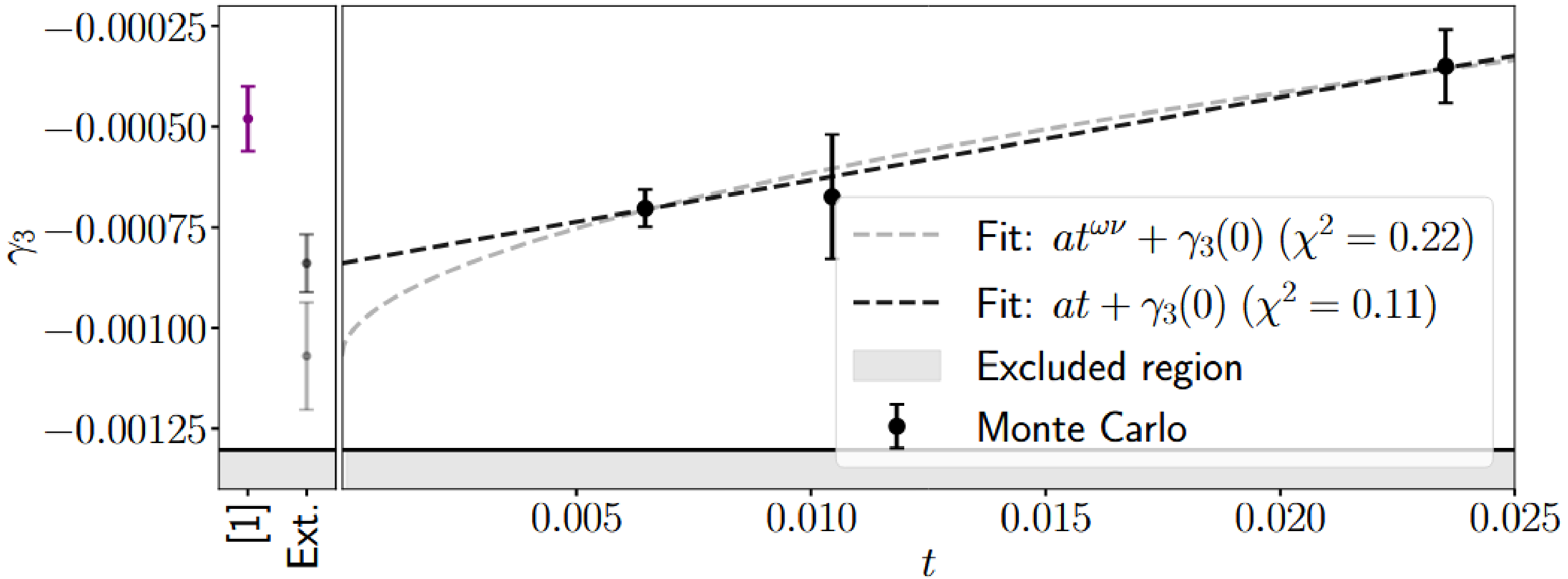

Figure 11: Extracted values of γ_3 versus reduced temperature. The black line is the bootstrap lower bound [15]. On the left subplot the ticks are: **left)** is the state-of-the-art [1] (purple dot); **right)**: linear extrapolation.

$$\boxed{\gamma_3 = -0.82(15)|\gamma_3^{\text{min}}| = -0.00106(18),}$$

Limits and possible new physics insights

- When the area is small, deviations from theory are large: breakdown of EST.
- However**, EST is a good model when the area is large enough.**
- It is possible to run **Monte Carlo simulations for 3d interfaces** and obtain good results.
- In some limits it is possible to witness **interactions between bulk modes and the interface.**