October 21st 2025

# DESY Journal Club

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arXiv: 2506.17384

## Schur Connections: Chord Counting, Line Operators, and Indices

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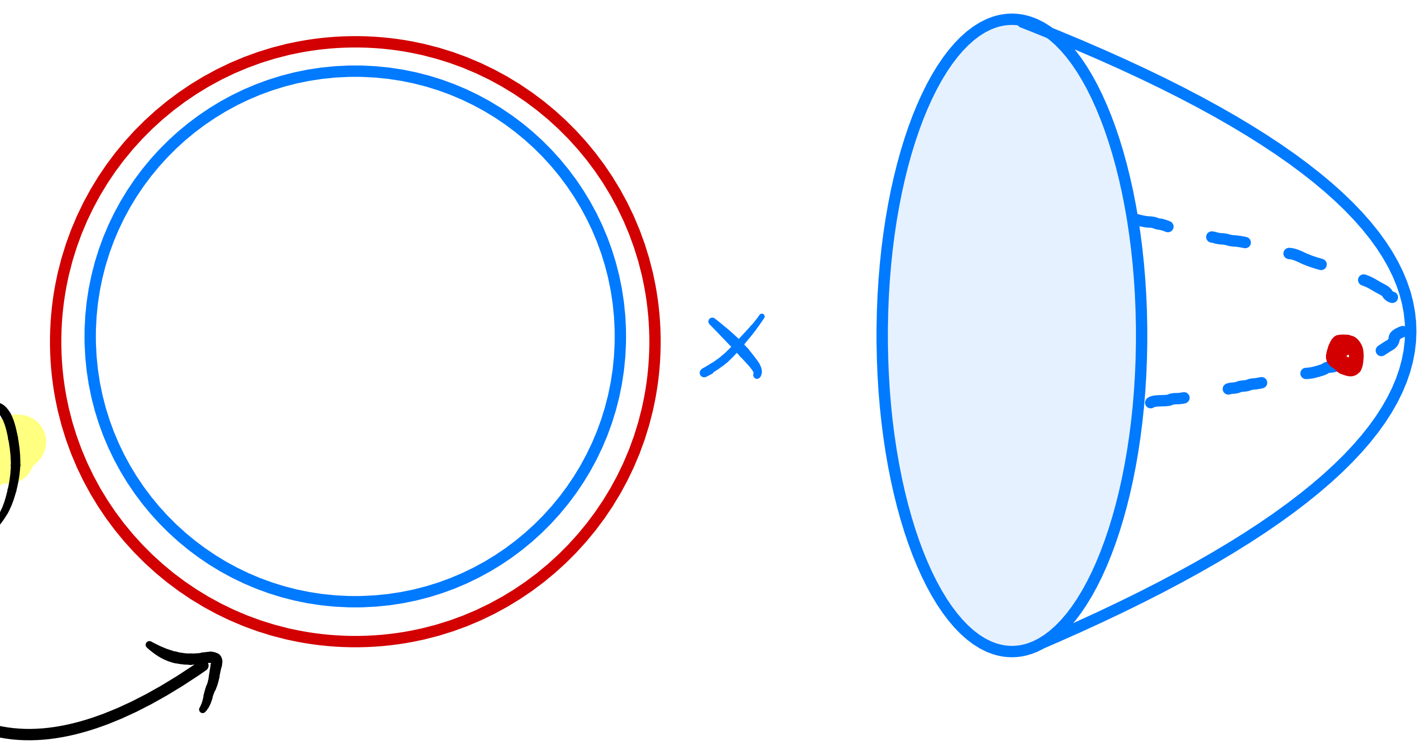
Recently, an intriguing correspondence was conjectured in [1] between Schur half-indices of pure 4d $SU(2)$ $\mathcal{N}=2$ supersymmetric Yang-Mills (SYM) theory with line operator insertions and partition functions of the double scaling limit of the Sachdev-Ye-Kitaev model (DSSYK). Motivated by this, we explore a generalization to $SU(N)$ $\mathcal{N}=2$ SYM theories. We begin by deriving the algebra of line operators, $\mathcal{A}_{\text{Schur}}$, representing it both in terms of the $\mathfrak{q}$-Weyl algebra and $\mathfrak{q}$-deformed harmonic oscillators, respectively. In the latter framework, the half-index admits a natural description as an expectation value in the Fock space of the oscillators. This $\mathfrak{q}$-oscillator perspective further suggests an interpretation in terms of generalized colored chord counting, and maps the half-index to a purely combinatorial quantity. Finally, we establish a connection with the quantum Toda chain, which is an integrable model whose commuting Hamiltonians can be identified with the Wilson lines of the $SU(N)$ SYM, and their eigenfunctions correspond to the function basis appearing in the half-index.

# Schur Indices, Algebra, and Quantization

We consider: (half) Schur index $\mathcal{I}(q)$ with half-line insertions in 4d $\mathcal{N}=2$ $SU(N)$ SYM

$$\boxed{\mathcal{I} = \mathrm{tr}(-1)^F q^{j_2 - j_1 + R}}$$

"partition function" on $S^3 \times S^1$ or $HS^3 \times S^1$ (half-index)



+ line operators

## Localization:

$$\boxed{\mathbb{I}^{(N)}_{W_{R_1}^{k_1} W_{R_2}^{k_2} \dots} = \frac{(q;q)_\infty^{N-1}}{N!} \oint_{\mathbb{T}^{N-1}} \prod_{a=1}^{N-1} \frac{dz_a}{2\pi i z_a} \prod_{a<b}^{N} \left( (z_a z_b^{-1})^{\pm}; q \right)_\infty \prod_{i=1}^{N} \left( \chi_{R_i}(\underline{z}) \right)^{k_i} \Bigg|_{\prod_c z_c = 1}}$$

# Schur Indices, Algebra, and Quantization

Algebra of line operators $\sim$ quantization of K-th. Coulomb branch

(fancy word for $\mathrm{Fun}(\mathcal{M}_{vac}(\mathbb{R}^3 \times S^1))$)

$\downarrow$

as an algebra (i.e. $\{\cdot,\cdot\} \longrightarrow i\hbar[\cdot,\cdot]$)

but no Hilbert space !

$\longrightarrow$ Schur half-index = inner product on $\mathcal{A}_{Schur}$

GNS construction $\longrightarrow$ Hilbert space $\rightsquigarrow$ unitary representations

relation w/ complex CS b.c.

$\mathcal{M}_{vac}(\mathbb{R}^3 \times S^1) \cong \mathcal{M}_{flat}(C_{g,n})$

# q-Oscillators & Double Scaled SYK

Observation: Schur indices = vevs of q-oscillators:

$$\boxed{\begin{aligned} a_i a_j &= a_j a_i \\ a_i^\dagger a_j^\dagger &= a_j^\dagger a_i^\dagger \\ a_i a_j^\dagger - q\, a_j^\dagger a_i &= \delta_{ij} \end{aligned}}$$

$$\mathbb{I}^{(N)}_{W^{k_1}_{R_1}\dots} = \langle 0 | W^{k_1}_{R_1} \dots | 0 \rangle$$

$$\hookrightarrow W_{R_1} \sim \chi_{R_1}(\underline{v})\Big|_{\substack{v_i \to a_i^\dagger \\ v_i^{-1} \to a_i}} , \quad i,j = 1,\dots,N-1$$

<u>SU(2) case</u>: $W_1 \sim a + a^\dagger$

$$\rightsquigarrow \langle W_1^{2k} \rangle = \langle H^{2k} \rangle_J$$

<u>Recall</u>: $H_{SYK} = i^{p/2} \sum_{i_1 < \dots < i_p} J_I \psi_I$

↑ Gaussian ensemble

double scaled: $p, N \to \infty$, $\lambda = \frac{p^2}{N}$ finite

Hamiltonian moments of DSSYK!

# q - Oscillators & Double Scaled SYK

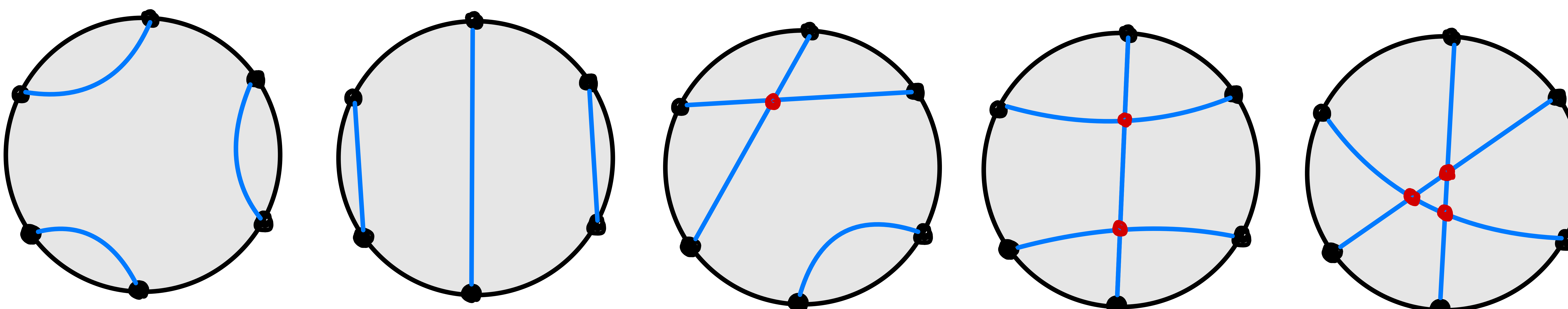
## Why?

- q - oscillators capture combinatorics of chord counting

$$\Vert$$

Wick contractions

Example: $\langle W_1^6 \rangle = q^3 + 3q^2 + 6q + 5$



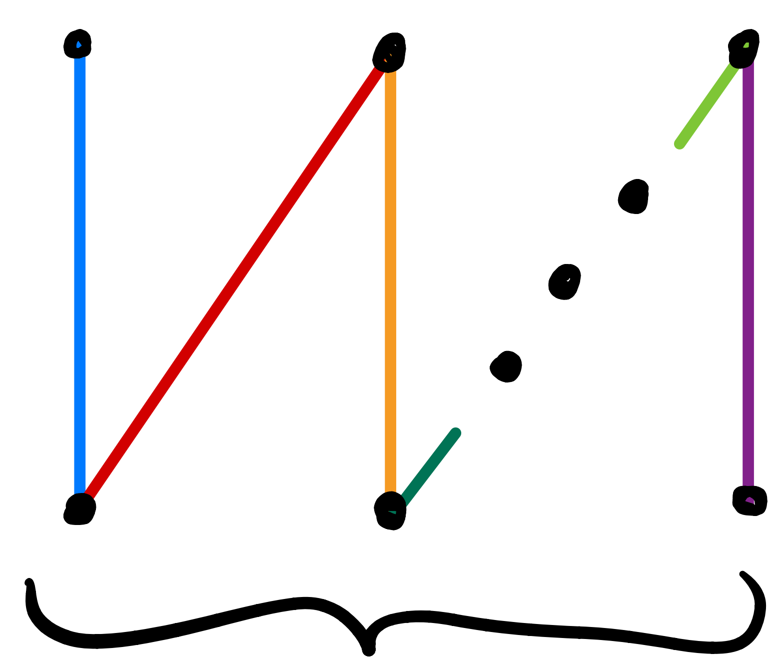
$$2 + 3 + 6q + 3q^2 + q^3$$

# Higher Rank & Generalized Chord Counting

Same idea, different operators:

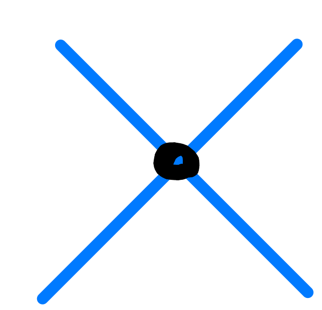
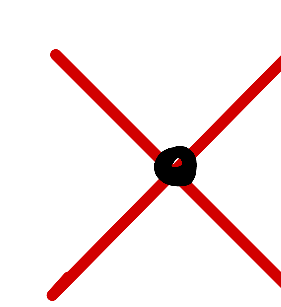
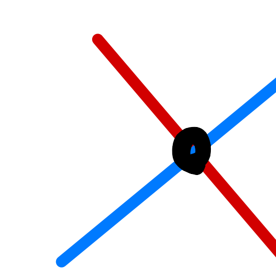
- SU(2) character: $\chi_1(v) = v + v^{-1} \longrightarrow a^\dagger + a$ (ordinary chords)
- SU(N) character: $\chi_{[1,0,\dots,0]} = v_1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} + \dots + \frac{v_{N-1}}{v_{N-2}} + v_{N-1}^{-1}$

$$\longrightarrow a_1^\dagger + a_2^\dagger a_1 + \dots + a_{N-1}^\dagger a_{N-2} + a_{N-1}$$

So replace  with 

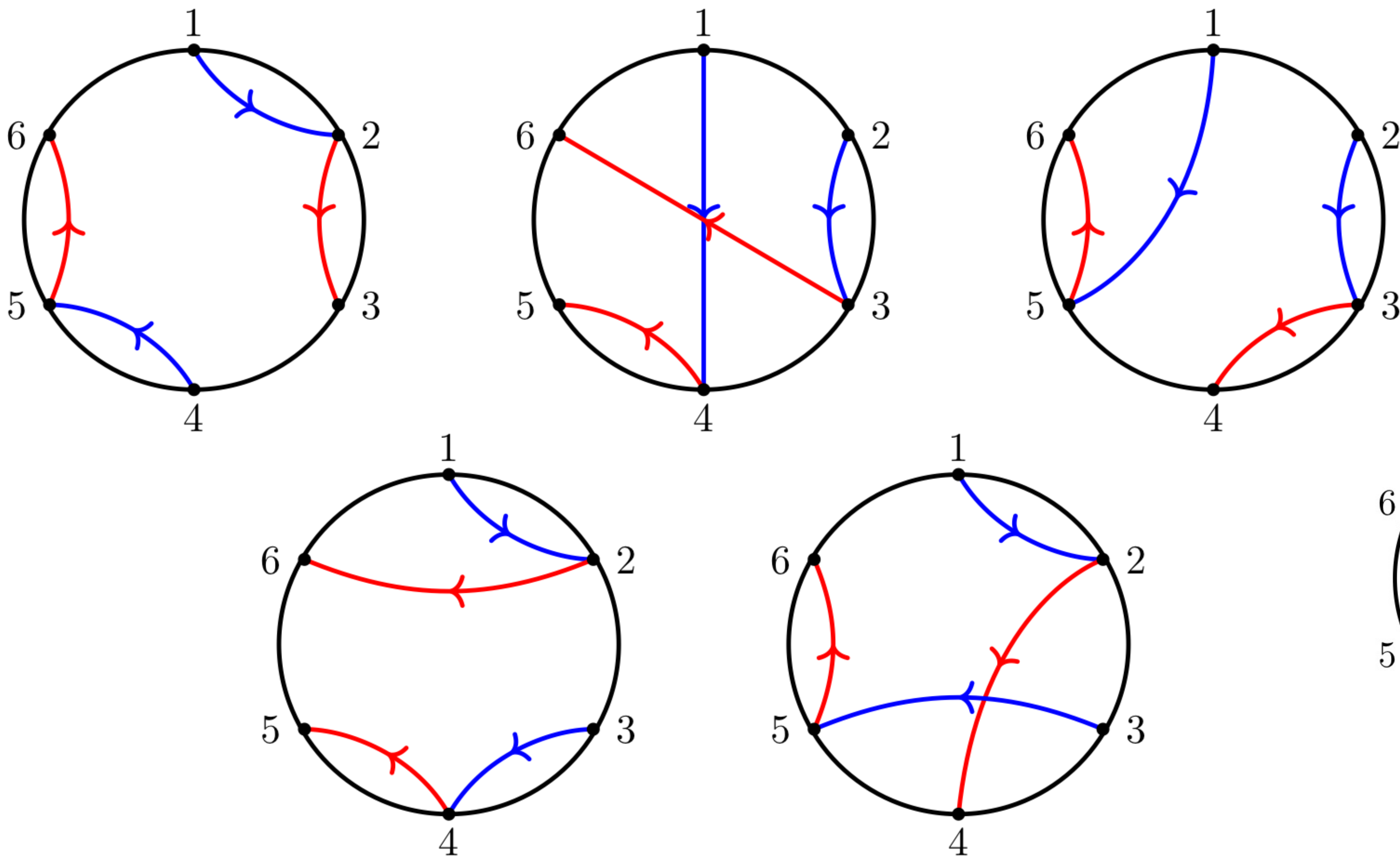
N−1 colors

Weights given by:

 =  $= q$ |  $= 1$

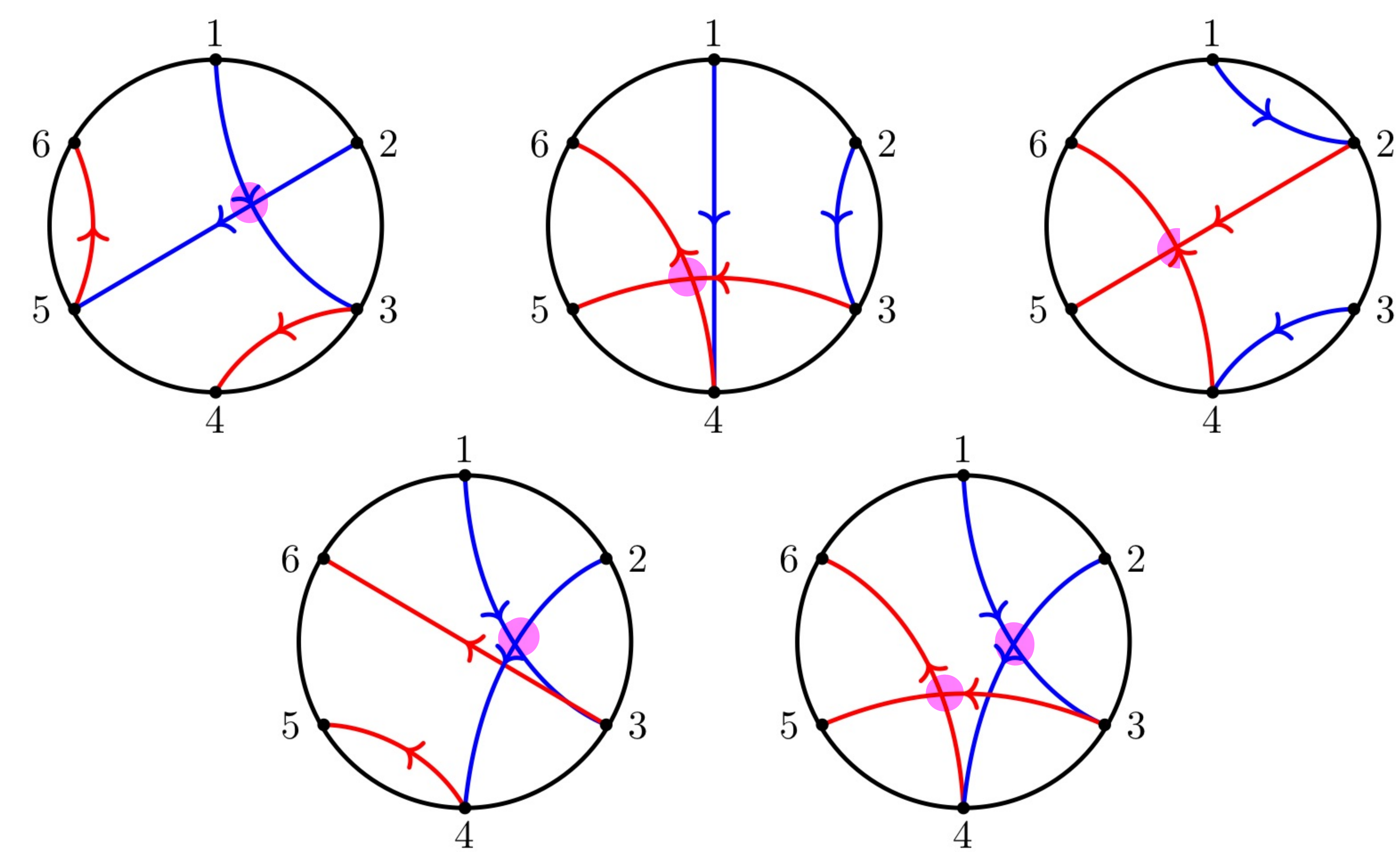
# Higher Rank & Generalized Chord Counting

Example: $\langle W^{6}_{[1,0]} \rangle$ for $SU(3)$



$\mathcal{O}(1)$ diagrams

$\mathcal{O}(q)$ diagrams



# Further generalizations to:

- other representations
- dyonic lines
- interfaces

} involves insertions of $q^{\hat{n}}$ ~ "matter chords"

$\hat{n}$: number operator

Same principle $\longrightarrow$ read off vertex rules from oscillators

This gets quite cumbersome quickly...

**Q**: What is the generalization of DSSYK generating gen. chords?

# Relativistic (open) Toda & Proof

**Note:** Wilson lines commute $\rightsquigarrow$ $N-1 = \mathrm{rank}(su(N))$

lin. indep. self. adj. ops.

$\longrightarrow$ integrability ?

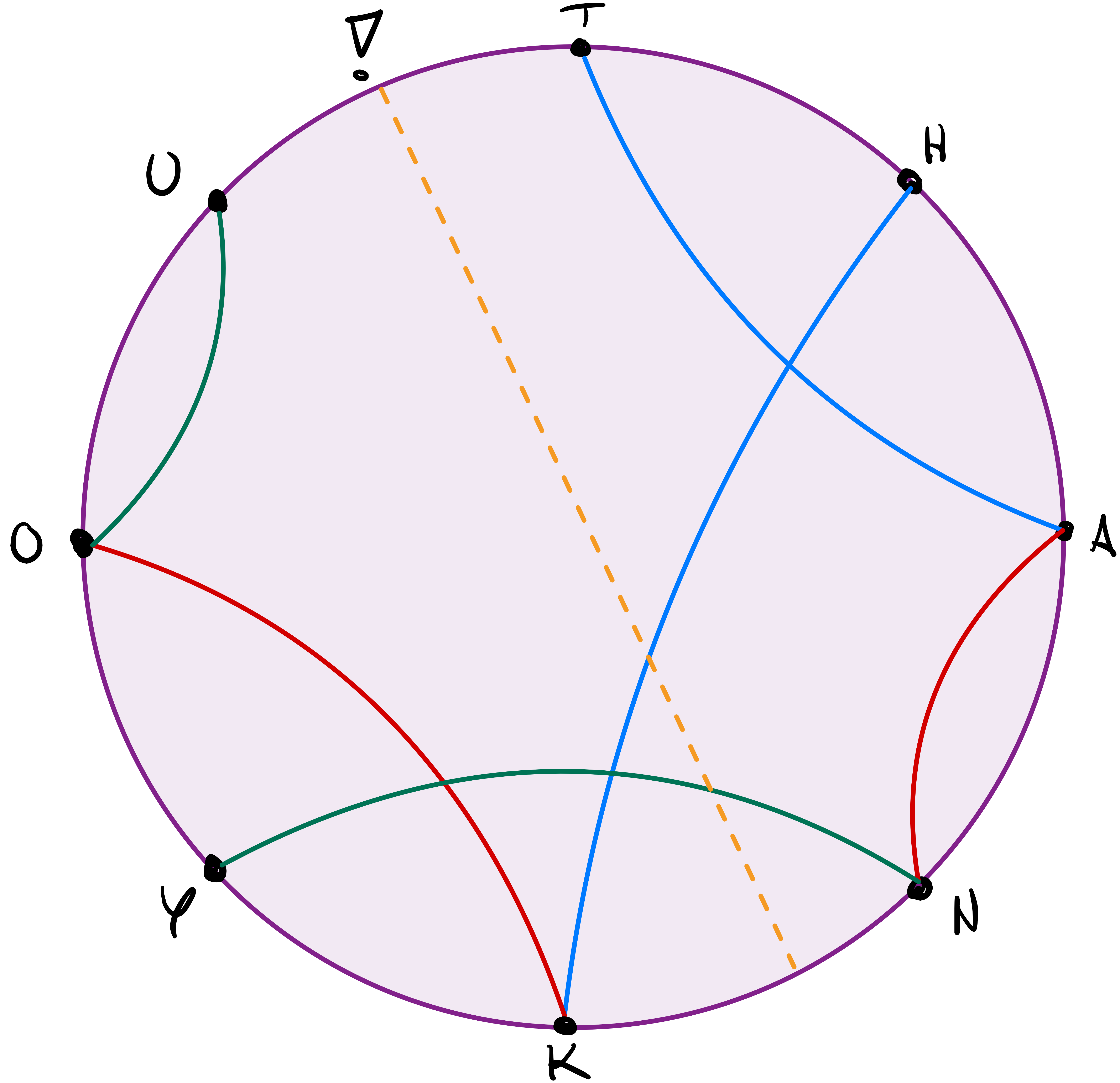
**Idea:** Simultaneously diagonalize $W_{[1,0,\dots,0]}, W_{[0,1,0,\dots,0]}, \dots$

$\longrightarrow$ closed form expression for $\langle W \rangle$ !

- $W$ = difference operators on $\mathcal{H}_{osc}$ $\rightsquigarrow$ $W_{[\underbrace{0,\dots,0}_{i-1},1,0,\dots,0]} = H_i^{q\text{-Toda}}$

q-Toda eigenfcts.

$$\boxed{\text{Insert } \quad \mathbb{1} = \oint_{\mathbb{T}^{N-1}} \prod_{a=1}^{N} \frac{dz_a}{2\pi i z_a} \Delta(\underline{z}; q) |\underline{z}\rangle\langle\underline{z}| \quad \text{into } \langle W \rangle \longrightarrow \text{localization formula !}}$$



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