A strong–weak duality for the 1d long-range Ising model

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Abstract

We investigate the one-dimensional Ising model with long-range interactions decaying as $1/r^{1+s}$. In the critical regime, for $1/2 \leq s \leq 1$, this system realizes a family of nontrivial one-dimensional conformal field theories (CFTs), whose data vary continuously with s. For $s > 1$ the model has instead no phase transition at finite temperature, as in the short-range case. In the standard field-theoretic description, involving a generalized free field with quartic interactions, the critical model is weakly coupled near $s = 1/2$ but strongly coupled in the vicinity of the short-range crossover at $s = 1$. We introduce a dual formulation that becomes weakly coupled as $s \to 1$. Precisely at $s = 1$, the dual description becomes an exactly solvable conformal boundary condition of the two-dimensional free scalar. We present a detailed study of the dual model and demonstrate its effectiveness by computing perturbatively the CFT data near $s = 1$, up to next-to-next-to-leading order in $1 - s$, by two independent approaches: (i) standard renormalization of our dual field-theoretic description and (ii) the analytic conformal bootstrap. The two methods yield complete agreement.

Motivation

- IR dualities: different UV models, same IR (CFT) limit.
- Examples: Seiberg duality, particle/vortex duality, etc.
- Often they are strong-weak dualities.
- Long-range Ising (LRI) in $d \geq 2$ is dual to short-range Ising.

$$S_{\mathrm{LRI}}[\varphi] = S_{\mathrm{GFF}}[\varphi] + \int \mathrm{d}^d x \left(\frac{\lambda_2}{2} \varphi^2 + \frac{\lambda_4}{4} \varphi^4 \right) ,$$

$$\widetilde{S}_{\mathrm{LRI}}[\sigma, \chi] = S_{\mathrm{SRI}}[\sigma] + S_{\mathrm{GFF}}[\chi] + g \int \mathrm{d}^d x \, \sigma \, \chi \, .$$

  This does not work in $d = 1$.
- Goal: strong-weak duality for 1d long-range Ising model.

Known facts

- On the lattice the 1d long-range Ising model is described by

$$\beta\,\mathcal{H}_{\mathrm{LRI}} = \frac{\mathcal{J}}{2}\sum_{i\neq j}\frac{(\sigma_i-\sigma_j)^2}{|i-j|^{1+s}}\,,\quad \mathcal{J}>0\,,\quad \sigma_i=\pm 1\,.$$

- In the continuum: GFF with $\Delta_\varphi = \frac{1-s}{2}$ + quartic interaction,

$$S_{\mathrm{LRI}} = \frac{c_s}{4}\int \mathrm{d}x_1 \mathrm{d}x_2 \frac{(\varphi(x_1)-\varphi(x_2))^2}{|x_1-x_2|^{1+s}} + \int \mathrm{d}x \left(\frac{\lambda_2}{2}\varphi^2+\frac{\lambda_4}{4}\varphi^4\right)\,.$$

$$\begin{aligned} s &\leq 1/2 : \text{GFF fixed point.}\\ s &\gtrsim 1/2 : \text{weakly coupled interacting fixed point.}\\ s &\lesssim 1 : \text{strong coupling?}\\ s &> 1 : \text{no fixed point.}\end{aligned}$$

- If $s \lesssim 1$, LRI = weakly coupled Coulomb gas of domain walls,

$$Z_{\mathrm{AYK}} = \sum_{n=0}^{+\infty} g^{2n}\int_{\mathrm{I}_{2n}(a)}\Big(\prod_{i=1}^{2n}\frac{\mathrm{d}x_i}{a}\Big)\, e^{2\mathcal{J}\sum_{i<j}(-1)^{i-j}(1-s)^{-1}((|x_i-x_j|/a)^{1-s}-1)}\,.$$

New results for the 1d long-range Ising (LRI)

Main result: a new QFT that is dual to the 1d LRI in the IR.

- This description is weakly coupled at $s \lesssim 1$ and exactly solvable at $s = 1$ ($\sim$ Kondo model): strong-weak IR duality.
- They show that this QFT reproduces the AYK model upon perturbative expansion.
- They find RG flow fixed points, study CFT data at $s \lesssim 1$ in perturbation theory and through analytic conformal bootstrap.

The dual model

Claim: the 1d LRI CFT can be identified with the IR fixed point of

$$Z_{dual} = \left\langle \mathrm{tr}\,\mathrm{Pexp}\left\{ \int_{-L/2}^{L/2} \mathrm{d}x \, [g\,\mathcal{O}_g(x) + h\,\mathcal{O}_h(x)] \right\} \right\rangle_0 ,$$

where $\langle \# \rangle_0 \equiv \int \mathcal{D}\phi \; \# \; e^{-S_{GFF}(\phi)}$,

$$\phi \sim \phi + 2\pi n / b_0 \,, \quad \Delta_\phi = \frac{s-1}{2} < 0 \,, \quad b_0 = \mu^{(s-1)/2} \,.$$

$$\mathcal{O}_g(x) \equiv \hat{\sigma}_+ V_+(x) + \hat{\sigma}_- V_-(x) \,, \quad \mathcal{O}_h(x) \equiv \frac{i\hat{\sigma}_3}{\sqrt{2}} \partial\phi(x) \,, \quad V_\pm = e^{\pm i b_0 \phi(x)} .$$

Here $\hat{\sigma}_i$ are the Pauli matrices ($Z_{dual} \sim$ GFF + spin impurity).

Evidence

- Same symmetries as LRI model:

$$\mathbb{Z}_2 : \phi(x) \mapsto -\phi(x)\,, A \mapsto \hat{\sigma}_1 A \hat{\sigma}_1\,,$$
$$\text{parity}: \quad \phi(x) \mapsto -\phi(-x)\,, \quad A \mapsto A^T\,.$$

- Z_{dual} at $g \ll 1$ matches exactly $Z_{\rm AYK}$ with $\mathcal{J} = b^2$.
- At $s = 1$, Z_{dual} = bosonized Kondo model (2d bounday CFT), that can be solved exactly. Agrees with Yuval and Anderson.
- Fixed point at $s \sim 1$ becomes complex for $s > 1$ (no 2nd order phase transition at $s > 1$).
- Spectrum of (protected) operators matches with the one of quartic model.

RG flow

They analyze the RG flow perturbatively

$$\beta_g = -2gh(\sqrt{2} - h) + g^3 \,, \quad \beta_h = -\frac{1-s}{2} h - g^2(\sqrt{2} - 2h) \,.$$

- They find weakly coupled fixed points

$$g_*^2 \sim h_* \sim 1 - s \ll 1 \,.$$

  Conclusion: strong-weak IR duality at $s \lesssim 1$.
- They compute scaling dimensions and OPE coefficients, e.g.

$$\Delta_\sigma = \frac{1-s}{2} \,, \quad \Delta_\chi = 1 - \frac{1-s}{2} \,, \quad \chi \equiv \frac{i}{\sqrt{2}} \partial\phi$$

- Spectrum matches with the one of quartic model.

Analytic bootstrap

Assumptions:

- 1d LRI flows to unitary CFT with $\mathbb{Z}_2$ and parity symmetry.
- CFT data admit an asymptotic expansion in non negative powers of $\sqrt{1-s}$ and reduce to exact solution at $s=1$.
- Spectrum contains $\mathbb{Z}_2$-odd parity-even primaries σ, χ.

Strategy:

- $\langle \underbrace{\sigma\sigma}_{OPE} \overbrace{\chi\chi} \rangle_{Z_{dual}} = \langle \underbrace{\sigma \overbrace{\sigma\chi} \chi} \rangle_{Z_{dual}} \rightarrow$ crossing equations.
- Solve the crossing equation using analytic functionals.

Results:

- The crossing equations admit a unique perturbative solution.
- The OPE data match the RG predictions exactly.
- Confirms that the 1d LRI with $s<1$ flows to a unitary CFT.

Conclusions and outlook

Main results:

- A new QFT that describes the 1d LRI CFT, weakly coupled at $s \lesssim 1$.
- Detailed study of 1d LRI CFT at $s \lesssim 1$ using perturbative RG flow and analytic conformal bootstrap.

Future directions:

- Reformulate it as boundary theory of a massive scalar in AdS_2.
- Numerical bootstrap to go beyond perturbation theory at $s \gtrsim 1/2$ and $s \lesssim 1$.
- Other 1d long-range models ($O(N)$, tricritical Ising, Potts,...).