Holographic thermal propagator from modularity

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- Submitted on 2 Sep 2025 -

[2509.02226]

Overview

Setting: Holographic 4d CFTs on the thermal geometry $S^1 \times \mathbb{R}^3$

Goal: Compute low T expansion of thermal two point function of scalars in momentum space

Strategy: Leverage connections with with SUSY gauge theories and Virasoro conformal blocks

Thermal propagator — Connection coefficients of Heun equation — NS instanton prepotential for 4d $\mathcal{N}=2$ SYM with gauge group SU(2) and $N_f = 4$ fundamental hypermultiplets — 4-pt semiclassical Virasoro conformal blocks in Liouville theory

How to make the most out of these connections?

Thermal propagator from gauge theory

Gauge theory: $a \leftrightarrow u,\ \vec{a} = (a_0, a_t, a_1, a_\infty) \leftrightarrow \vec{m},\ q \leftrightarrow t$

Propagator: $\omega, |\vec{k}|, \nu\ \ (= 2 - \Delta_\phi)$

$$G_R = \pi^{4a_1} e^{-\partial_{a_1} F} \frac{M(a, a_1; a_\infty)}{M(a, -a_1; a_\infty)}, \Bigg|_{q = \mathrm{e}^{-\pi}\ \text{(black brane)}}$$

Relation between parameters

$$\vec{a} = \left(0, \frac{i\omega}{4\pi}, \frac{\nu}{2}, \frac{\omega}{2\pi}\right) \qquad \frac{\omega^2 - 2k^2}{8\pi^2} - \frac{\nu^2}{4} = u = -a^2 + a_t^2 + a_0^2 - \frac{1}{4} + t\partial_t F,$$

where $\omega, |\vec{k}|$ are in unit of T

(at the end rescale $\omega \to \omega/\mathrm{T}$, $|\vec{k}| \to |\vec{k}|/\mathrm{T}$)

$\Rightarrow$ low T = high $\omega, |\vec{k}|$ = high a, a_t, a_∞

too many large parameters! So set $\omega = 0$

Computing the instanton prepotential

$$G_R = \pi^{4a_1} e^{-\partial_{a_1} F} \frac{M(a, a_1; a_\infty)}{M(a, -a_1; a_\infty)},$$

The prepotential F is an essential object, entering the propagator formula both directly and indirectly (to find a)

Typically, F computed via a *series expansion in q* (or t)

→ E.g., using the connection to conformal blocks

$$F(q, a, \vec{a}) \;=\; \ldots + H(q, a, \vec{a}) \quad \text{where} \quad H(q, a, \vec{a}) \equiv \lim_{\epsilon_1 \to 1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log \mathcal{H}(q, \Delta, \vec{\mu}).$$

conformal block

$$\boxed{\mathcal{H}(q, \Delta, \vec{\mu}) = 1 + \sum_{m,n=1}^{\infty} \frac{q^{mn} R_{m,n}(\vec{\mu})}{\Delta - \Delta_{m,n}} \mathcal{H}(q, \Delta_{m,n} + mn, \vec{\mu})}$$

Zamolodchikov q- recursion

However, since low T = high a, we would like to have an expansion $H(a, q, \nu) = \sum_{n=1}^{\infty} \frac{\alpha_{2n}(q, \nu)}{a^{2n}}$

Main task: find such expansion!

Prepotential as series in $1/a$

Main task: find an expansion $H(a,q,\nu)=\sum_{n=1}^{\infty}\frac{\alpha_{2n}(q,\nu)}{a^{2n}}$

They tackle the task in two steps

1) ***Expand Zamolodchikov q −series in a*** to find a (convoluted) representation of $\alpha_{2n}(q,\nu)$ as a series in q

$$\tilde{c}^l_\chi=\sum_{k=1}^{\chi}\sum_{\sum_{i=1}^k m_i n_i=\chi}\left(\frac{m_1\epsilon_1+n_1\epsilon_2}{2}\right)^{2l}\frac{\prod_{i=1}^k R_{m_i n_i}}{\prod_{i=1}^{k-1}\Delta_{m_i n_i}+m_i n_i-\Delta_{m_{i+1}n_{i+1}}}$$

$$\alpha_{2n}=-\lim_{\substack{\epsilon_1\to 1\\ \epsilon_2\to 0}}\left[\sum_{m=1}^{n}\frac{(\epsilon_1\epsilon_2)^{m+1}}{m}\sum_{\substack{\sum_{l=0}^{\infty}k_l=m\\ \sum_{l=0}^{\infty}k_l(l+1)=n}}\frac{m!}{\prod_{l=0}^{\infty}k_l!}\prod_{l=0}^{\infty}\left(\sum_{\chi=1}^{\infty}\tilde{c}^l_\chi q^\chi\right)^{k_l}\right]$$

This allows for the computation of first terms in the q series for a generic α_{2n}

e.g. $\alpha_4=-\frac{(\nu^2-1)^2}{8}q^2\left(2+3q^2(7-\nu^2)+8q^4(9-2\nu^2)+q^6(191-45\nu^2)+\dots\right)$

But no hope in resumming!

Prepotential as series in $1/a$

Main task: find an expansion $H(a,q,\nu) = \sum_{n=1}^{\infty} \frac{\alpha_{2n}(q,\nu)}{a^{2n}}$

2) *Use constraints coming from gauge theory* [Billò, Frau, ... ; 2013-2015]

→ The coefficients α_{2n}, paired with some known q-independent functions, are quasi-modular forms

Eisenstein series

$$\alpha_{2n}(q,\nu) + f_{2n}(\nu) = \sum_{2k+4l+6m=2n} c_{k,l,m}(\nu) {E_2}^k(q) {E_4}^l(q) {E_6}^m(q) \equiv \tilde{\alpha}_{2n}(q,\nu)$$

finite number of coefficients to be determined!

→ S-duality $\quad \mathcal{S} : \tau \to -\frac{1}{\tau}$ and also $\mathcal{S} : a \to a_D,\ a_D \to -a$ $\qquad q = e^{i\pi\tau}$

$$F_{\mathrm{NS}}(-1/\tau, a_D) = F_{\mathrm{NS}}(a) - a\frac{\partial F_{\mathrm{NS}}}{\partial a} \Rightarrow \frac{\partial \tilde{\alpha}_{2l}}{\partial E_2} = \frac{l-1}{12}(\nu^2 - 1)\tilde{\alpha}_{2(l-1)} + \frac{1}{6}\sum_{i=1}^{l-2} i(l-i-1)\tilde{\alpha}_{2i}\tilde{\alpha}_{2l-2i-2}$$

recursion relation for the coefficients $c_{k,l,m}$ with $k \neq 0$!

Summary of the strategy

$$G_R = \pi^{4a_1} e^{-\partial_{a_1} F} \frac{M(a, a_1; a_\infty)}{M(a, -a_1; a_\infty)}, \Bigg|_{q\,=\,\mathrm{e}^{-\pi}\ \text{(black brane)}}$$

1. Compute the coefficients $\alpha_{2n}(q,\nu)$ of the large a expansion of the prepotential (also of M, but that is simpler):
   - Use Zamolodchikov q-relation to determine the leading terms in their q series
   - Feed these into the recurrence relation + matching to the expansion in the Eisenstein basis to fully fix the $\alpha_{2n}(q,\nu)$
2. Invert the Matone relation and find a in terms of $|\vec{k}|$

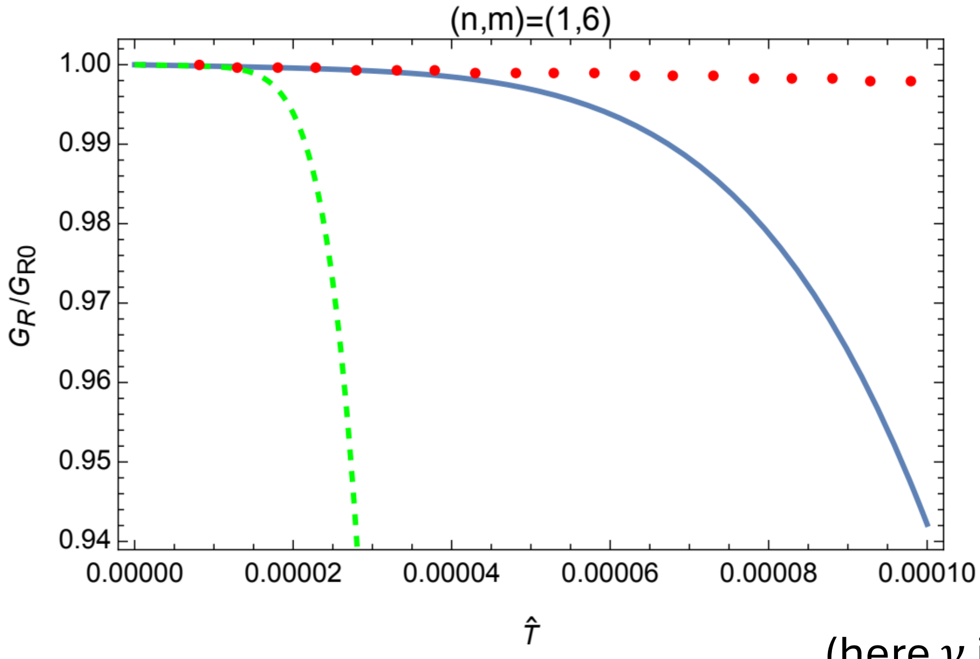
$$\frac{\omega^2 - 2k^2}{8\pi^2} - \frac{\nu^2}{4} = -a^2 + a_t^2 + a_0^2 - \frac{1}{4} + t\partial_t F,$$

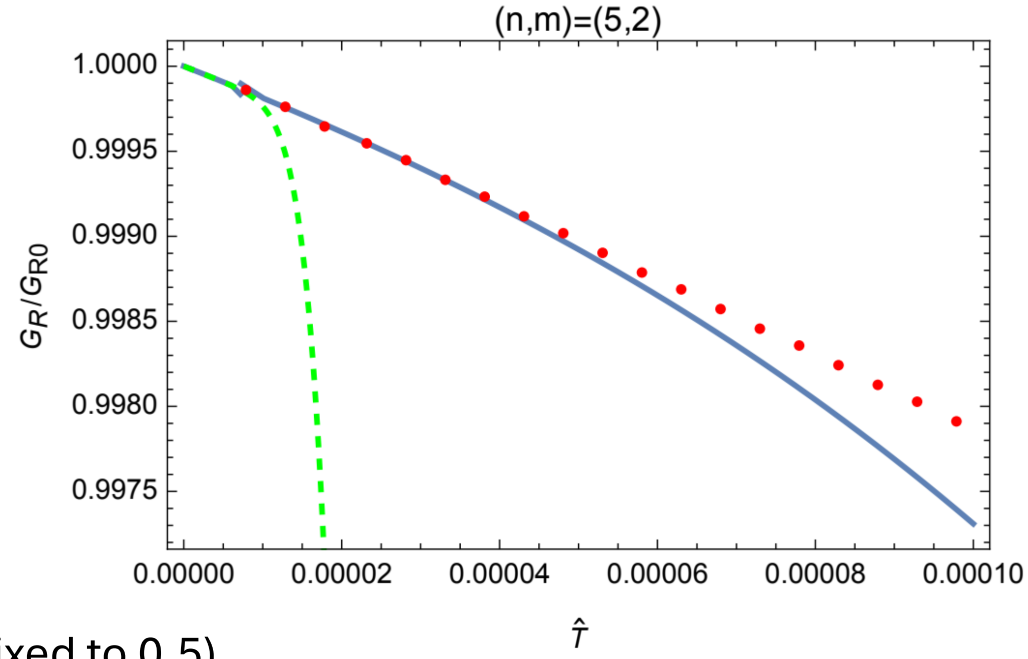
3. Reinstate the temperature as $|\vec{k}| \to |\vec{k}|/\mathrm{T}$ and finally expand everything in T

In this way they get an expansion for G_R up to T^{40}

Check of the result

- The expansion is tested against a numerical solution of the propagator, computing it from the Heun equation
- As the low T expansion obtained from the propagator seems asymptotic, they have also tested extracting some Padé approximants





(here ν is fixed to 0.5)

Conclusions

- The authors have derived a low T expansion for the momentum space two point function of scalars in 4d holographic CFTs on $S^1 \times \mathbb{R}^3$
- In order to obtain it, they have used tools coming from SUSY gauge theories (quasi-modularity and S-duality) and Virasoro blocks (q-recursion)
- The method is efficient, and can be used to push to high orders. The series appears to be asymptotic, confirming previous predictions [Čeplak, Liu, Parnachev, Valach; 2024]

Outlooks

- Study the case of $\omega \neq 0$
- Perform a resurgent analysis of the divergent series
- Interpret S-duality constraints as coming from crossing in the four-point block