Paper discussion: 2506.06431

Seeing through the confinement screen: DGLAP/BFKL mixing and light-ray matching in QCD

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DESY String Journal Club

July 2025

Outline

- Central problem in nutshell
- Lessons from previous work [2209.00008]
- DGLAP detectors and renormalization
- BFKL detector using soft theorem
- QCD phenomenology

Central problem

- Two point of views: pheno and formal
- Pheno problem: can we predict number of particles in a jet analytically?
- Formal problem: what is the space of detectors in a weakly coupled field theory?

Consider one-point event shape in collider physics:

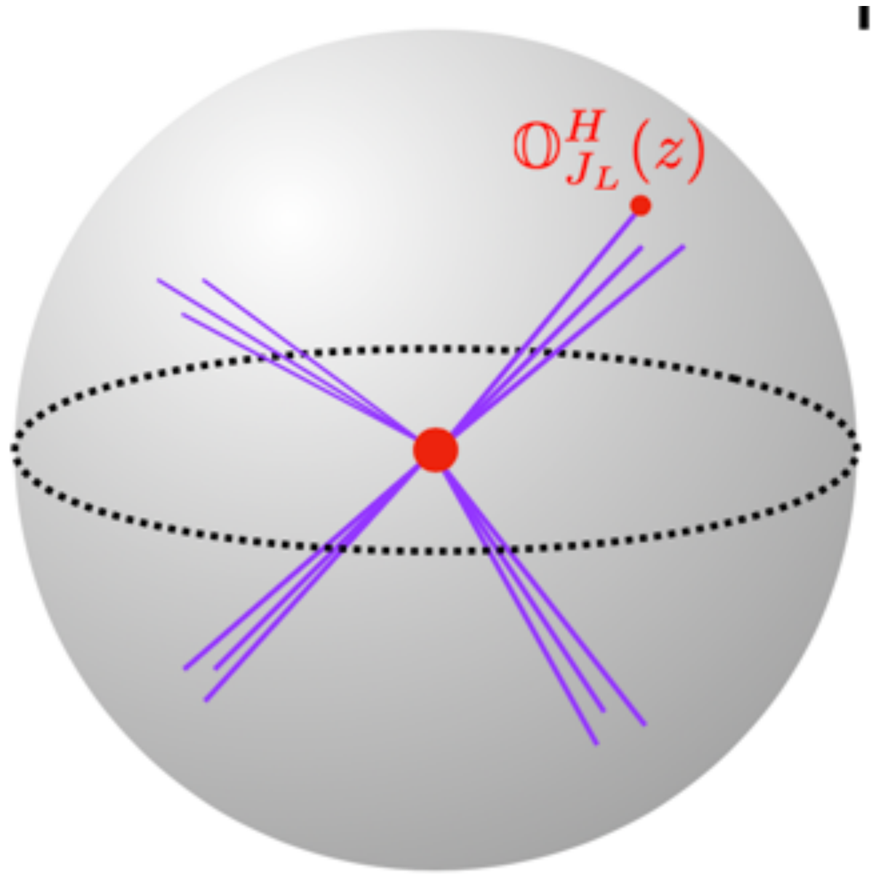
$$\langle \mathbb{N}_{J_L}(z)\rangle_Q \equiv -\int d^d x\, e^{-iq\cdot x}\langle\Omega|J_\mu(0)\mathbb{N}_{J_L}(z)J^\mu(x)|\Omega\rangle$$

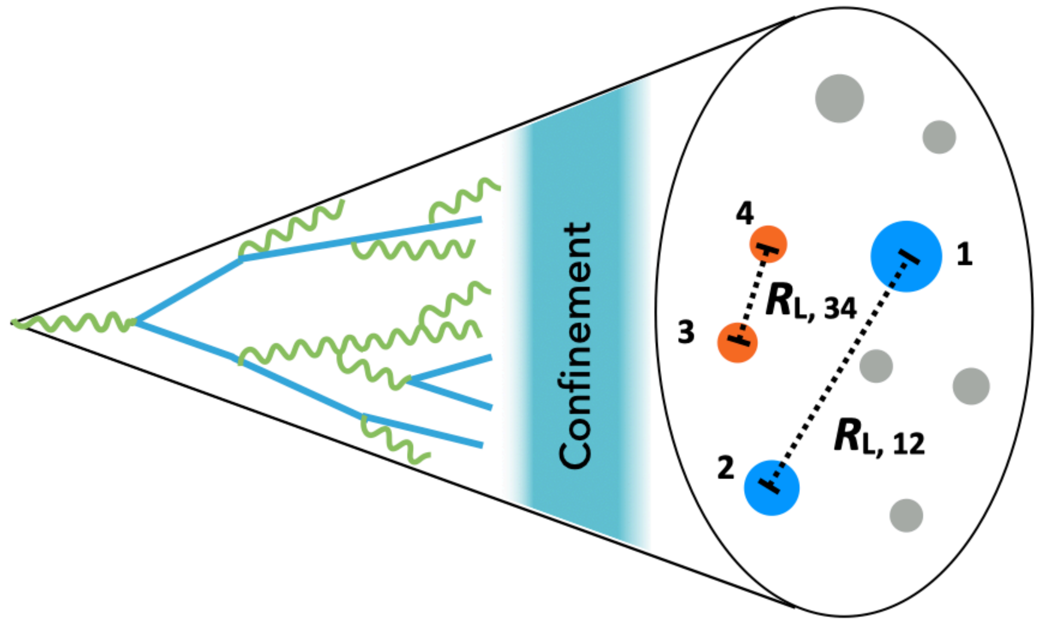
J_L is the Lorentz spin which is preserved by hadronization

$$\mathbb{N}(\vec{n}) = \sum_i C_i(J_L,\mu)\mathcal{D}_{J_L,i}(\vec{n},\mu)\Big|_{J_L=2-d}$$

Wilson Coefficients that describe parton-> hadron matching

More generally: $$\mathbb{N}_{J_L}(z) = \sum_i \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}2E} E^{2-d-J_L}\delta^{d-2}(\hat{p}-\hat{z})a_i^\dagger(\vec{p})a_i(\vec{p})$$





- $J_L = -2$: multiplicity
- $J_L = -3$: energy (only this one IRC safe)
- $J_L = -4$: energy squared

Central problem

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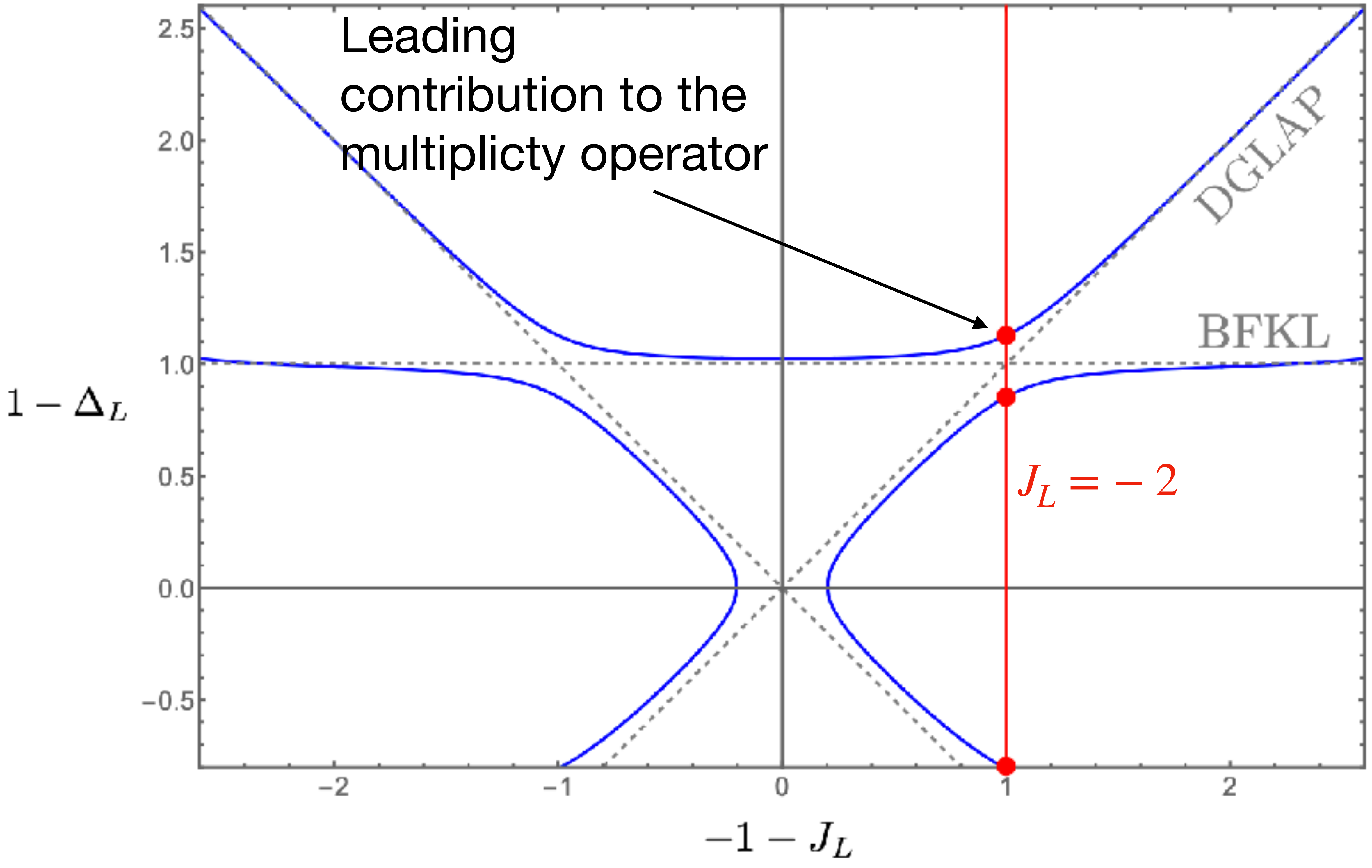

$$\mathbb{N}(\vec{n}) = \sum_i C_i(J_L,\mu)\mathcal{D}_{J_L,i}(\vec{n},\mu)\Big|_{J_L=2-d}$$

- Dependence of $\langle \mathbb{N}(\vec{n})\rangle_Q$ on Q can be understood by analyzing the matrix elements of $\mathscr{D}_{J_L,k}(\vec{n},\mu)$
- Above equation is an expansion in $(\Lambda_{\text{QCD}}/Q)^{\Delta_{L,i}}$ with $-\Delta_{L_i}$ being the Eigenvalue of RG equation of $\mathscr{D}_{J_L,k}(\vec{n},\mu)$
- What is the set of $J_L = -2$ operators that describe the expansion of $\mathbb{N}(\vec{n})$?

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Detectors in free theory

- For a free massless scalar,

$$H \propto \int d^{d-1}\vec{p}\; a^\dagger(\vec{p})a(\vec{p}) = \int d^{d-2}\vec{n} \int_0^\infty dE\, E^{d-2}\, a^\dagger(E\vec{n})a(E\vec{n}) \equiv \mathscr{E}_2(\vec{n})$$

- One can generalize this to define an operator measuring E^{J-1} flux:

$$\mathcal{E}_J(\mathbf{n}) \propto \int_0^\infty dE\, E^{J+d-4} a^\dagger(E\mathbf{n})a(E\mathbf{n}).$$

- Define $n^\mu = z^\mu = (1,\vec{n})$. Under Lorentz boost along the direction $\vec{n}$, we have $z \to \lambda z$. The powers of λ under boost define the boost weight or the collinear spin or Lorentz spin J_L:

$$\mathcal{E}_J(\lambda z) = \lambda^{3-d-J}\mathcal{E}_J(z) \qquad (\lambda > 0)$$

- For $\mathscr{E}_J(\vec{n})$ we have mass dimension, $-\Delta_L = J - 1$, Lorentz spin, $J_L = 3 - d - J$

Detectors in interacting theory

- Instead of using creation-annihilation operators, define the detectors in terms of light-ray operators:

$$\mathcal{E}_J(z) = 2\mathbf{L}[\mathcal{O}_J](\infty, z) \qquad \mathcal{O}_J(x,z) = N_J :\phi(x)(z\cdot\partial)^J\phi(x): + (z\cdot\partial)(\cdots)$$

Contract Lorentz indicies with z^μ

Spin of the local operator

direction

$$\mathbf{L}[\mathcal{O}](x,n) = \int_{-\infty}^{\infty} d\alpha(-\alpha)^{-\Delta-J}\mathcal{O}\left(x - \frac{n}{\alpha}, n\right)$$

starting point

[Kravchuk, Simmons-Duffin, 2018]

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^2 \int_0^\infty dt\, \vec{n}_i T^{0i}(t, r\vec{n})$$

[Sveshnikov, Tkachov, 1996; Hofman, Maldacena, 2008;…]

- For free theory, $\Delta = J + 2(d-2)/2$, such that scaling dimension $\Delta_L = 1 - J$ and Lorentz spin $J_L = 1 - \Delta = 3 - d - J$

- In the interacting theory we can't have both Δ_L and J_L come out as expected:

$$E_2(\Omega) \equiv \int_\Omega d^{d-2}\mathbf{n}\,\mathcal{E}_2(\mathbf{n}) \xrightarrow{\text{Shrinking the } \Omega \text{ to } \vec{n} \text{ forces us to consider the OPE}} \widehat{E}_3(\Omega) \propto \theta^{d-2+\delta(3)}\mathcal{E}_3(\mathbf{n}) + \cdots$$

This happens because the dimension of $\mathcal{O}_J$ becomes anomalous

Disaster?

- Turn on ϕ^4 interactions and tune to Wilson-Fisher fixed point

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4 \qquad \beta(g) = -\varepsilon g + \frac{3}{16\pi^2}g^2 + O(\varepsilon^2) \qquad g_* = \frac{16\pi^2}{3}\varepsilon + O(\varepsilon^2)$$

- Mass dimension of $\mathcal{O}_J$: $\Delta(J) = 2\Delta_\phi + J + \gamma(J)$

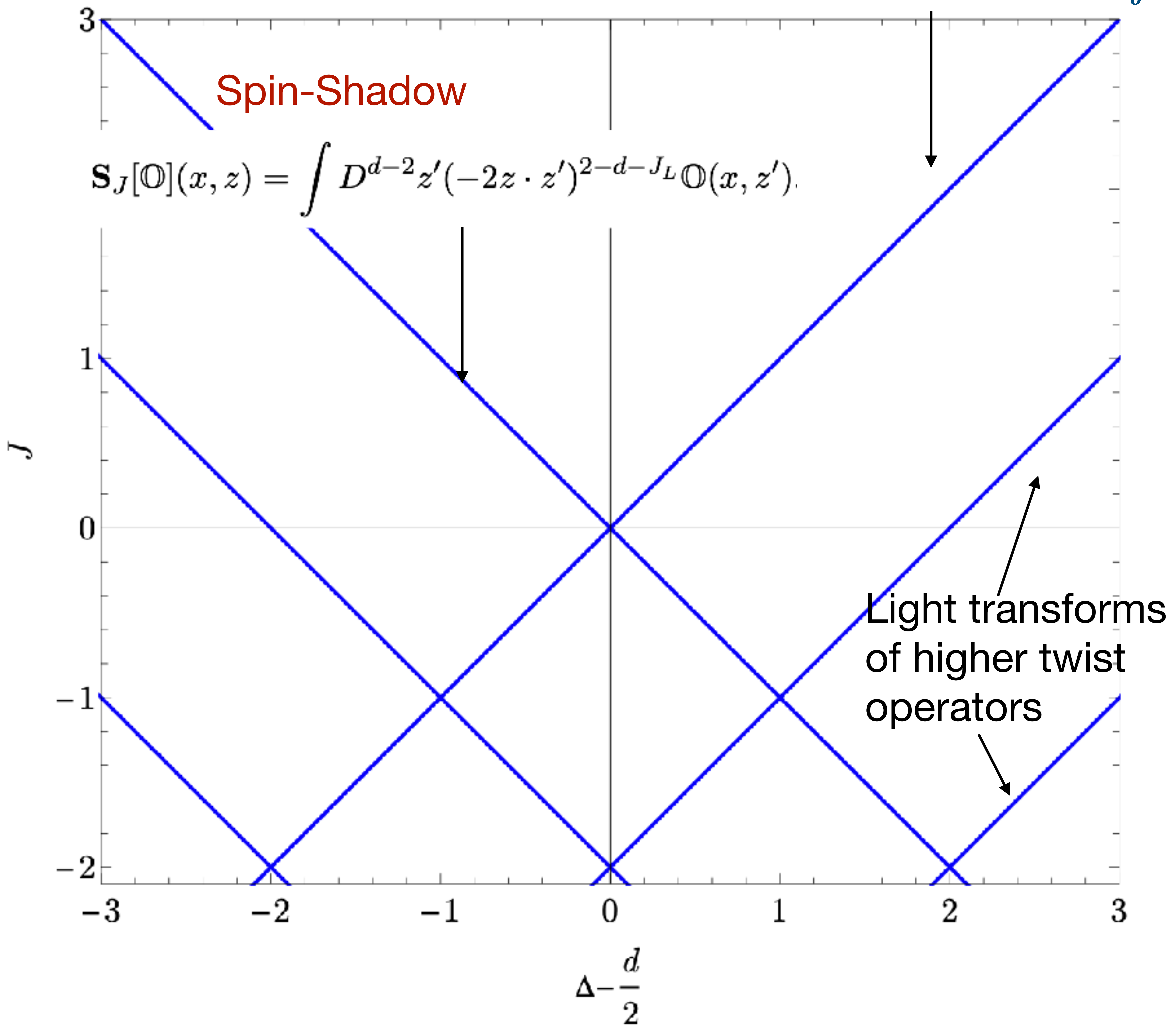
$$\Delta_\phi = 1 - \frac{1}{2}\epsilon + \frac{1}{108}\epsilon^2 + \frac{109}{11664}\epsilon^3 + \left(\frac{7217}{1259712} - \frac{2\zeta(3)}{243}\right)\epsilon^4 + O(\epsilon^5),$$

$$\gamma(J) = -\frac{1}{9J(J+1)}\epsilon^2 + \left(\frac{22J^2 - 32J - 27}{486J^2(J+1)^2} - \frac{2H(J)}{27J(J+1)}\right)\epsilon^3 + O(\epsilon^4).$$

- Notice that $\gamma(J)$ has poles for $J = 0, -1$. This means we have failed to appropriately renormalize the operators $\mathscr{E}_J(\vec{n})$ at these points. Predicts absurd scaling under boosts: $J_L = 1 - \Delta(J)$

- The reason for poles is because we have ignored other Regge trajectories

Regge trajectories in free theory



- It is important to know if there are other operators that could mix with $\mathbf{L}[\mathcal{O}_J]$
- The spin shadow defines another set of light-ray operators with the same $\Delta_L = 1 - J$ but $J_L \to 2 - d - J_L$
- For interacting theory, mixing at the Regge intercept described by a quadratic equation with roots:

$$\nu^2 = (2\Delta_\phi - d/2 + J + \gamma(J))^2, \quad \nu = \Delta - d/2$$

- Solution gives: $\Delta = d/2 \pm \sqrt{J^2 - \epsilon^2}$
- Naive expansion at a generic J:

$$\Delta = \frac{d}{2} + \left(J - \frac{\epsilon^2}{J}\right) + \ldots$$

Perturbation Theory in QM

Slides from Hao Chen's talk at EEC Workshop, Wuhan

Physicists are extremely good at doing perturbation theories.

Hamiltonian: $H = H_0 + \lambda V$ ⟶ Goal: solving equation $H|\Psi_n\rangle = E_n|\Psi_n\rangle$

perturbation

Hilbert space: $H_0|n\rangle = E_n^{(0)}|n\rangle$ [assume no degeneracy]

Perturbative expansion:

$$E_n(\lambda) = E_n^{(0)} + \lambda\,\langle n|V|n\rangle + \lambda^2 \sum_{k\neq n} \frac{|\,\langle k|V|n\rangle\,|^2}{E_n^{(0)} - E_k^{(0)}} + \lambda^3 \left(-\,\langle n|V|n\rangle \sum_{k\neq n} \frac{|\,\langle k|V|n\rangle\,|^2}{(E_n^{(0)} - E_k^{(0)})^2} + \sum_{k\neq n}\sum_{m\neq n} \frac{\langle n|V|m\rangle\,\langle m|V|k\rangle\,\langle k|V|n\rangle}{(E_n^{(0)} - E_k^{(0)})(E_n^{(0)} - E_m^{(0)})} \right) + \cdots$$

At each order in the expansion, we find pole structures when energy levels are very close.

⟶ Numerically, this approximation is not good when the energy gap is $\mathcal{O}(\lambda)$ [resummation is needed]

Two-level system example

Slides from Hao Chen's talk at EEC Workshop, Wuhan

If the first excited state is close to the ground state, while all other states are far-separated,

→ the leading approximation for lowest two states is a two-level system

Example: $H = \frac{B}{2}\sigma_z + \lambda(3\sigma_x + \sigma_z)$

$|B|$ is the energy gap for "free" Hamiltonian $H_0 = \frac{B}{2}\sigma_z$

Perturbative expansion for the ground state energy

$$E_g = -\frac{B}{2} - \lambda - \frac{9\lambda^2}{B} + \frac{18\lambda^3}{B^2} + \frac{45\lambda^4}{B^3} - \frac{414\lambda^5}{B^4} + \cdots \qquad B > 0$$

Not easy to resum if one does not recognize the pattern of coefficients

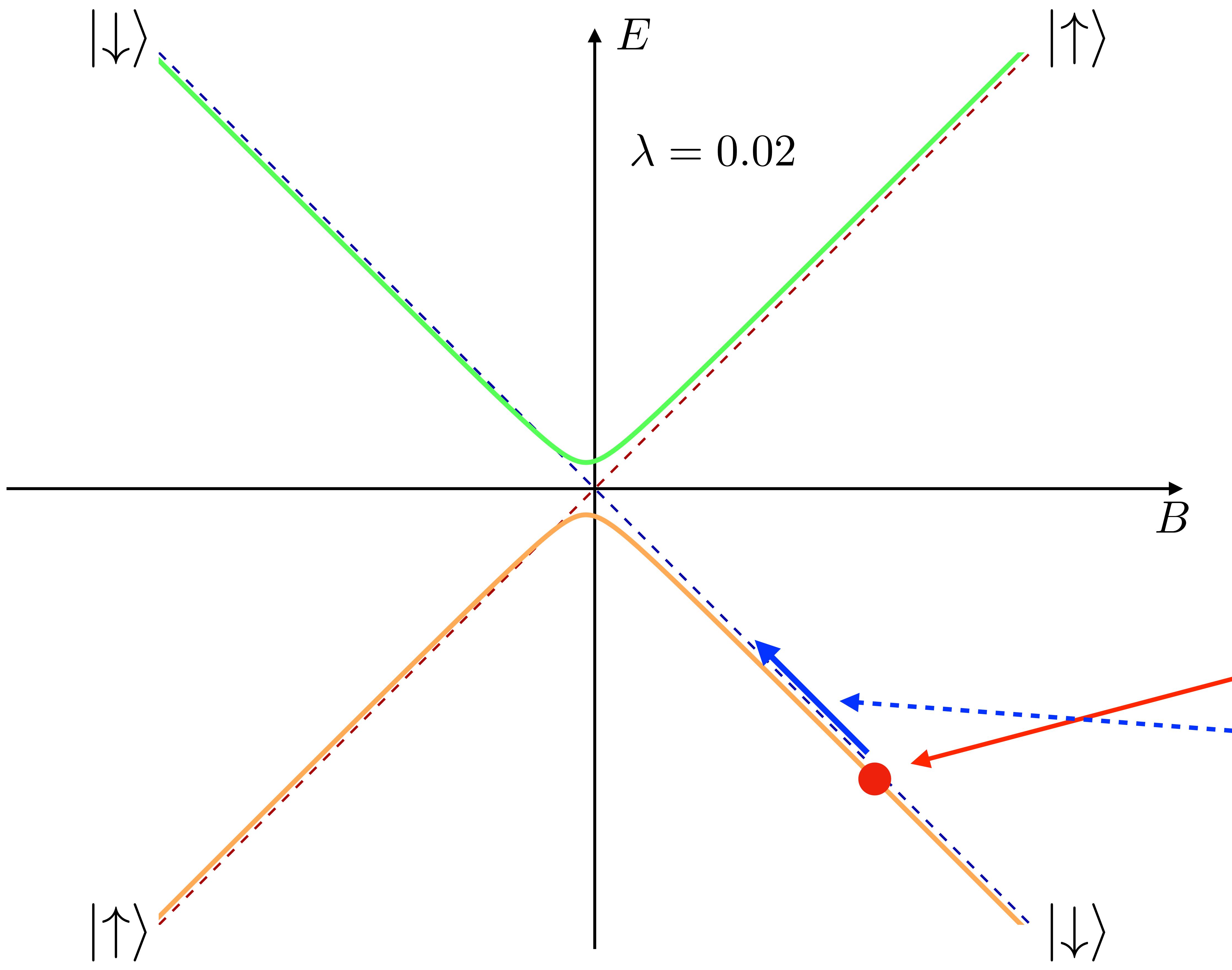
Hellmann-Feynman theorem $\frac{dE_g}{d\lambda} = \langle\psi_g|\frac{dH}{d\lambda}|\psi_g\rangle = \frac{a_1\lambda + a_2}{\sqrt{\lambda^2 + b_1\lambda + b_2}}$ $\xrightarrow{\text{solution}}$ $E_g = -\frac{1}{2}\sqrt{B^2 + 4B\lambda + 40\lambda^2}$

But everyone knows there is a straightforward way! [direct diagonalization]

$$\det(H - EI) = E^2 - (B^2/4 + B\lambda + 10\lambda^2) \longrightarrow E = \pm\frac{1}{2}\sqrt{B^2 + 4B\lambda + 40\lambda^2}$$

Avoided Level Crossing

Slides from Hao Chen's talk at EEC Workshop, Wuhan



Varying the external field B, we find avoided level crossing near $B \sim 0$.

The "free" Hamiltonian has degeneracy at $B = 0$, but is lifted by small perturbation.

Comparison btw two methods:

1. Perturbation + resummation

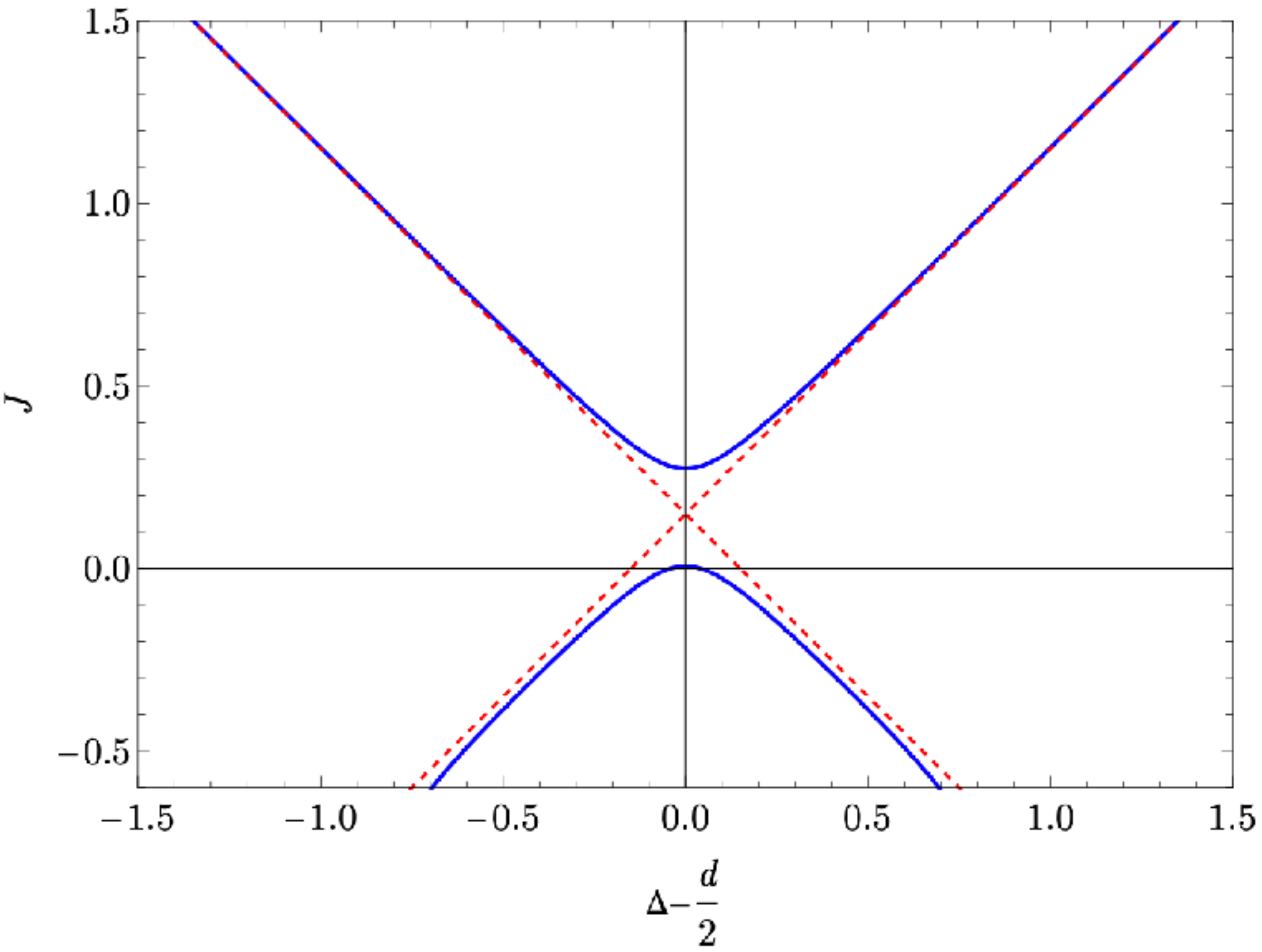
   [may not know the existence of the second level]

   Apply perturbation within the valid regime

   Resum the series near the intersection

2. The existence of the second level is known, the direct diagonalization is much simpler.

Regge trajectories in Wilson Fisher theory



- The poles at $J=0$ are in fact smooth once expanded properly:

$$\nu^2 = (2\Delta_\phi - d/2 + J + \gamma(J))^2$$
$$= J^2 - J\epsilon + \left(\frac{J}{27} + \frac{1}{4} - \frac{2}{9(J+1)}\right)\epsilon^2$$
$$+ \left(\frac{109J^3 + 164J^2 + 265J - 114}{2916(J+1)^2} - \frac{4H(J)}{27(J+1)}\right)\epsilon^3 + O(\epsilon^4),$$

- Poles at $J=0$ have cancelled.

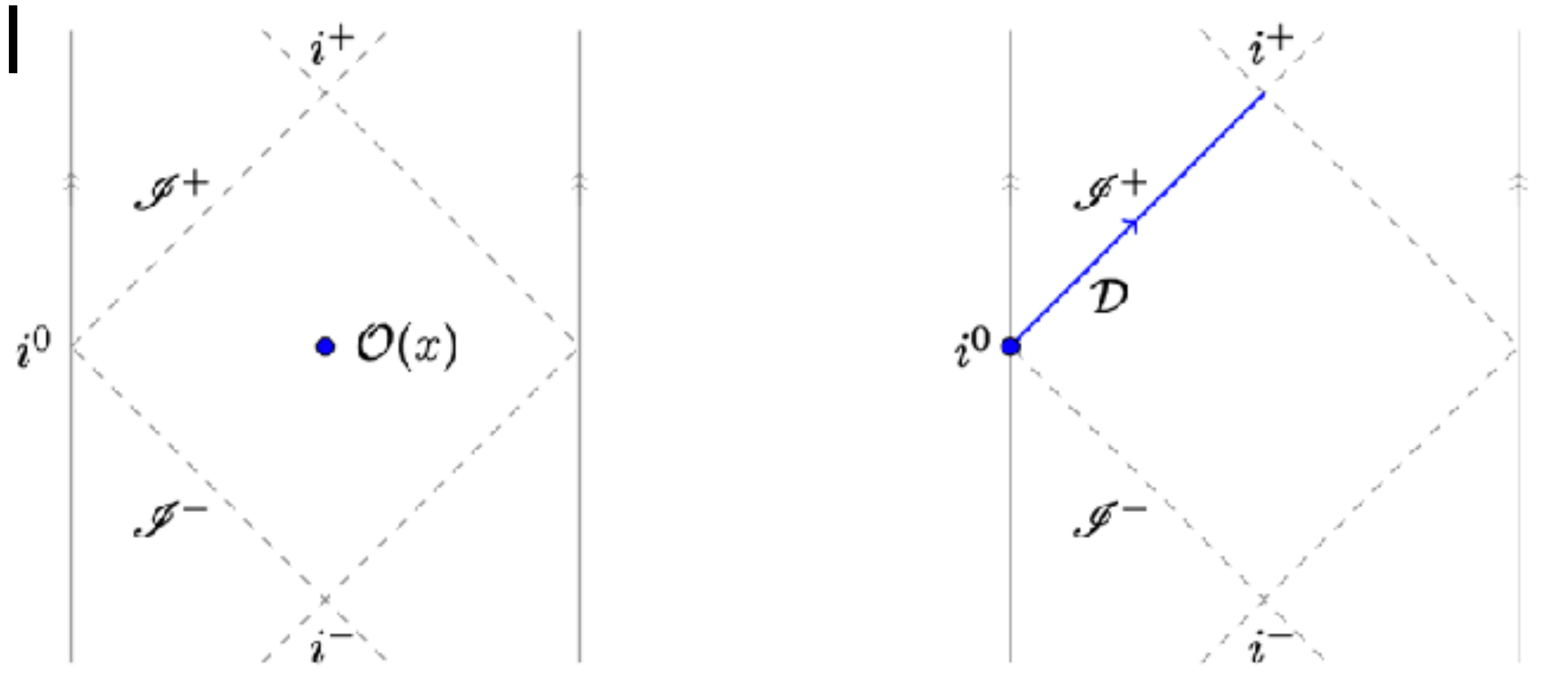
- One can obtain further Regge trajectories by combining light ray operators:

Clebsch-Gordon coefficient for the Lorentz group

$$\mathcal{H}_{J_L}(x,z) = \int D^{d-2}z_1 D^{d-2}z_2 \boxed{K_{J_L}(z_1,z_2;z)}\mathcal{H}(x,z_1,z_2). \qquad \mathcal{H}(x,z_1,z_2) \equiv :\mathbf{L}[\phi^2](x,z_1)\mathbf{L}[\phi^2](x,z_2):$$

- For $J = J_1 + J_2 - 1 > J_0$ (the Regge intercept, here $J=0$) the product requires regularization. For Wilson Fisher theory, these start at max $J = -1$.

Constructing detectors in perturbation theory



- Convenient to work in the *detector frame.* Define fields at future null infinity:

$$\phi(\alpha, z) = \lim_{L\to\infty} L^{\Delta_\phi}\phi(x + Lz)\,, \qquad \alpha = -2x\cdot z$$

- Detectors transform like primary operators at infinity: $[P^\mu, \mathscr{D}] = 0.$

- Dimension of the detector: $[D, \mathscr{D}(z)] = -\Delta_L \mathscr{D}(z)$

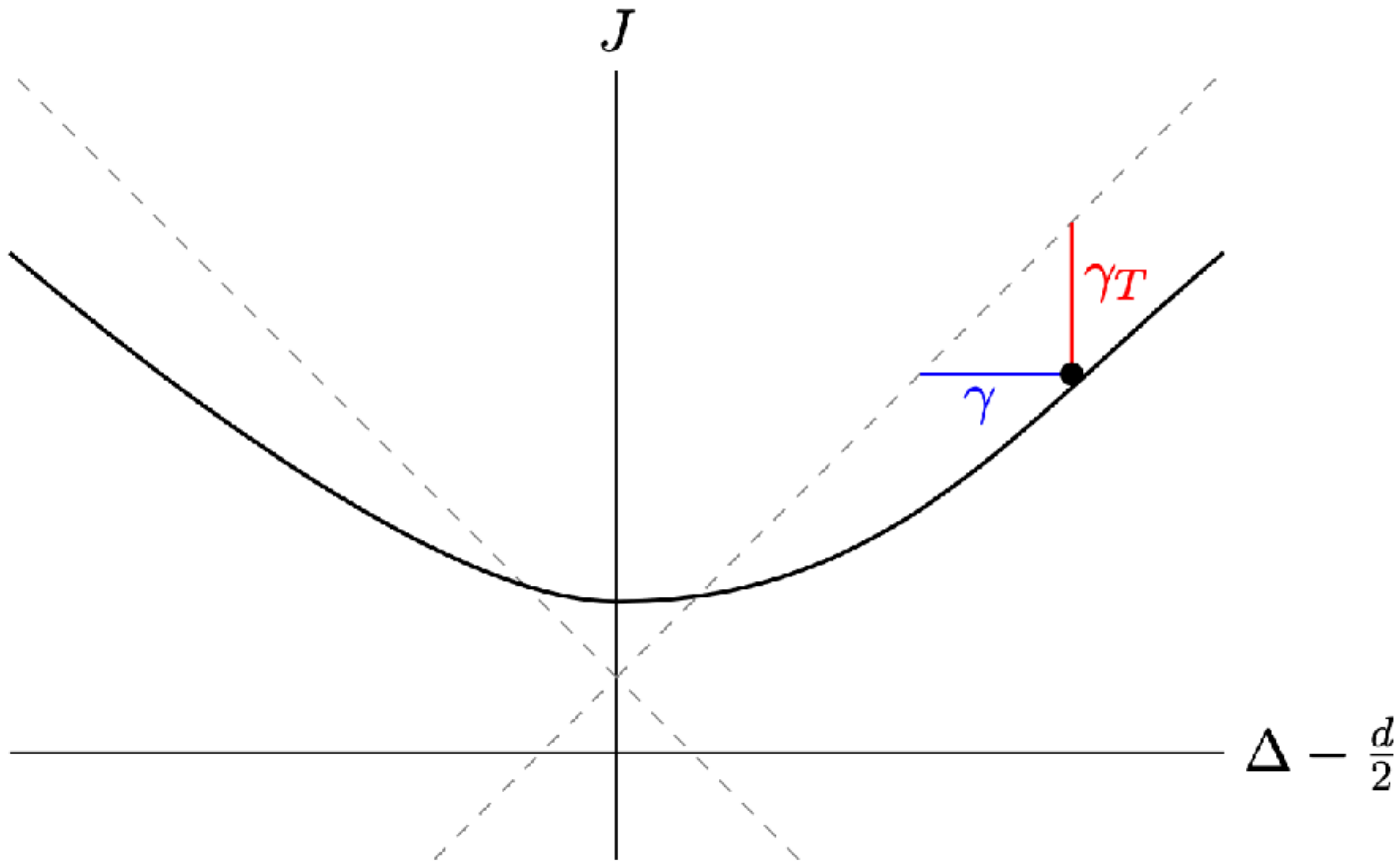
  QCD version: $\frac{d}{d\log Q}\log\langle \mathbb{N}_{J_L}(\vec{n})\rangle_Q = -\Delta_{L,i_{\min}}(J_L, \alpha_s(Q)) + \ldots$

- Example of a primary detector: $\mathcal{D}_\psi(z) = \int d\alpha_1 \ldots d\alpha_n \boxed{\psi(\alpha_1, \ldots, \alpha_n)} : \phi(\alpha_1, z)\cdots\phi(\alpha_n, z) :$

  Translationally invariant kernel

  - The translation invariance condition and the detector spin J_L remains exact in perturbation theory
  - Interactions renormalize the detector dimension Δ_L: $\Delta_L = \Delta_{L,0}(J_L) + \gamma_L(J_L)$

Spacetime reciprocity



- Starting with the fact that it's the detector anomalous dimension Δ_L that gets renormalized, we can draw interesting conclusions. Use $(J_L, \Delta_L) = (1 - \Delta, 1 - J)$

  $$\Delta_L = \Delta_{L,0}(J_L) + \gamma_L(J_L)$$

  $$J = J_0 - \gamma_L(1 - \Delta)$$

- In the traditional frame we write $\Delta = \tau_0 + J + \gamma(J)\,,\ \Rightarrow J_0 = J + \gamma(J)$ This is renormalizing local operators in the bulk

- This yields $\gamma_T(J_0) = \gamma(J_0 - \gamma_T(J_0))$ with $\gamma_L(1 - \Delta) \equiv \gamma_T(J_0)$

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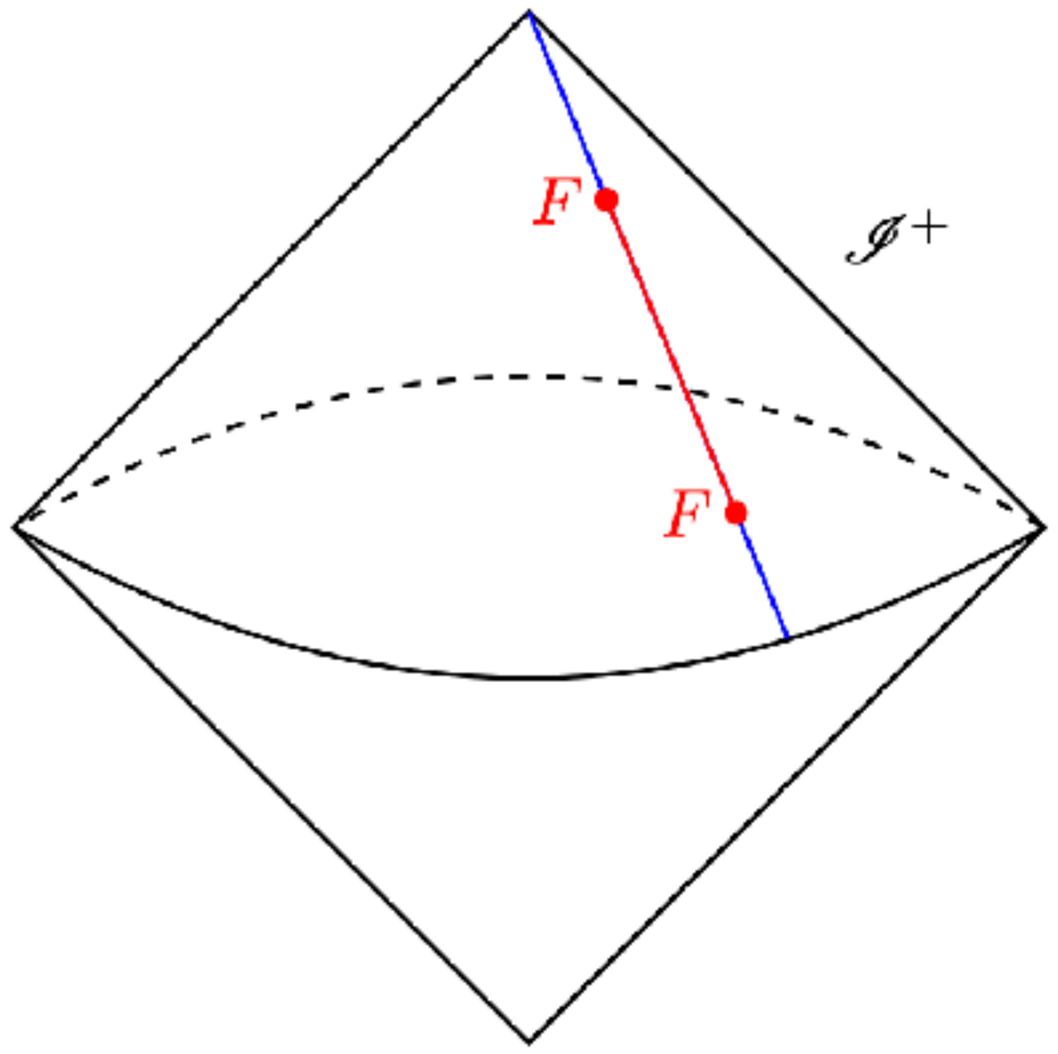
DGLAP detectors in QCD



- We now consider the E^{J-1}-flux detectors in QCD

$$\mathcal{D}^{\rm DGLAP}_{J_L,g}(z) = \sum_{\lambda,c} \int_0^\infty \frac{E^{-J_L} dE}{(2\pi)^{d-1} 2E} \left[a^\dagger_{\lambda,c}(p) a_{\lambda,c}(p) \right]\Big|_{p=Ez},$$

$$\mathcal{D}^{\rm DGLAP}_{J_L,q}(z) = \sum_{s,i} \int_0^\infty \frac{E^{-J_L} dE}{(2\pi)^{d-1} 2E} \left[b^\dagger_{s,i}(p) b_{s,i}(p) + d^\dagger_{s,i}(p) d_{s,i}(p) \right]\Big|_{p=Ez}.$$

- Turning on interactions will lead to IR divergences. Only the combination $\mathscr{D}_{J_L,g} + \mathscr{D}_{J_L,q}$ for $J_L = 1 - d$ for the energy flow operator $(J = 2)$ is IRC safe.

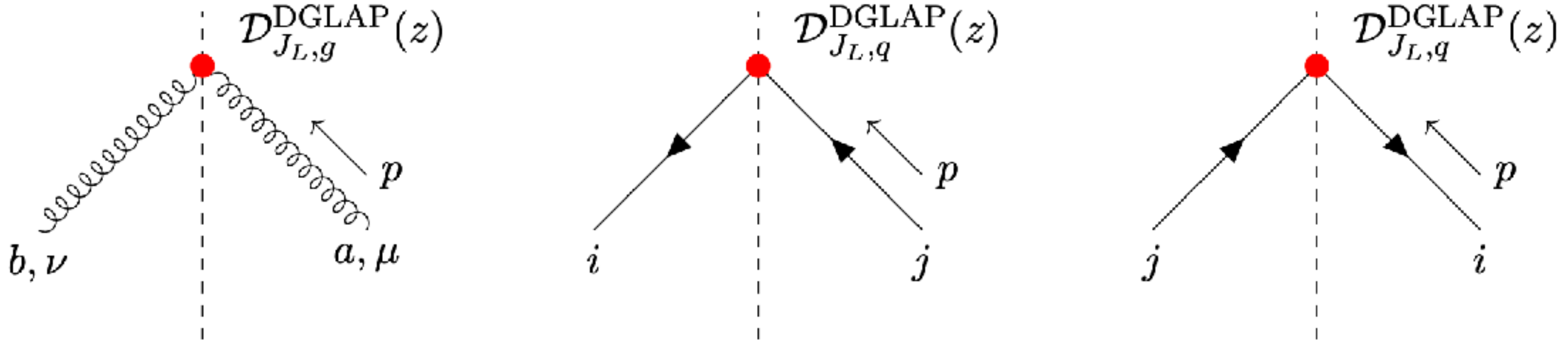
- Another way to write this:

$$\mathcal{D}^{\mathrm{DGLAP}(\bar z)}_{J_L,g}(z) \equiv \frac{1}{C_{J_L}} \int d\alpha_1 d\alpha_2 \ \left((\alpha_1 - \alpha_2 + i\epsilon)^{2\Delta_A + J_L} + (\alpha_2 - \alpha_1 + i\epsilon)^{2\Delta_A + J_L} \right)$$

$$\times : F^{(\bar z)\nu}_a(\alpha_1, z) W^{(\bar z)ab}_{\rm adj}(\alpha_1, \alpha_2) F^{(\bar z)}_{b\nu}(\alpha_2, z) :,$$

$$F^{(\bar z)}_\nu(\alpha, z) \equiv \lim_{L\to\infty} \frac{L^{\Delta_A}}{4} \bar z^\mu F_{\mu\nu}(Lz + \alpha \bar z/4)$$

Tree-level matrix elements



(a) Gluon case (b) Quark case (c) Anti-quark case

$$\langle 0|A^b_\nu(-q)\mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)A^a_\mu(p)|0\rangle = (2\pi)^d\delta^{(d)}(p-q)\left[\delta^{ab}\Pi_{\mu\nu}(z)V_{J_L}(z;p)\right] \qquad V_{J_L}(z;p) = \pi\int_0^\infty d\beta\,\beta^{-J_L-1}\delta^{(d)}(p-\beta z)$$

$$\langle 0|\psi_{i,\alpha}(-q)\mathcal{D}^{\text{DGLAP}}_{J_L,q}(z)\overline{\psi}_{j,\beta}(p)|0\rangle = (2\pi)^d\delta^{(d)}(p-q)\left[\delta_{ij}\not{z}_{\alpha\beta}V_{J_L}(z;p)\right]$$

$$\langle 0|\overline{\psi}_{j,\beta}(-q)\mathcal{D}^{\text{DGLAP}}_{J_L,q}(z)\psi_{i,\alpha}(p)|0\rangle = (2\pi)^d\delta^{(d)}(p-q)\left[\delta_{ij}\not{z}_{\alpha\beta}V_{J_L}(z;p)\right]$$

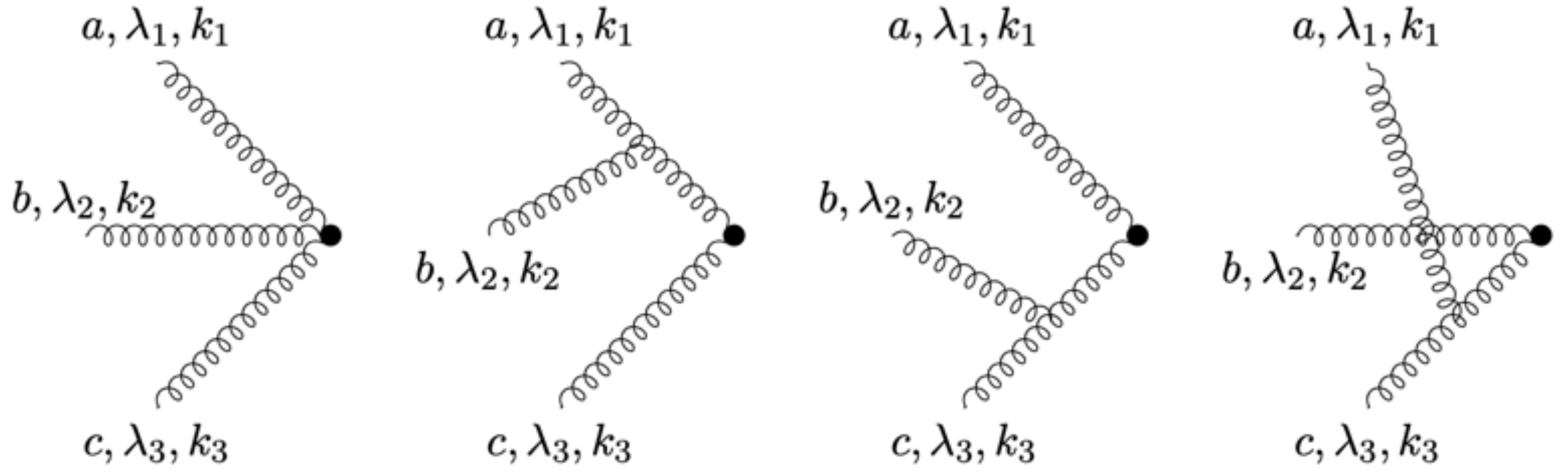
- These rules and a lot of algebra gives the tree-level matrix elements:

$$\langle\mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)\rangle^{\text{tree}}_{\mathcal{O}(p)} = \frac{d-2}{2^{d+1}\pi^{d-2}}(N_c^2-1)(2z\cdot p)^{J_L}(p^2)^{1-J_L}\,,$$

$$\langle\mathcal{D}^{\text{DGLAP}}_{J_L,q}(z)\rangle^{\text{tree}}_{J(p)} = \frac{d-2}{2^{d-3}\pi^{d-2}}N_c(2z\cdot p)^{J_L}(p^2)^{-J_L}\,,$$

One-loop computation

- At one-loop one has real emission and virtual contributions
- The one-loop calcualtion yields an ϵ pole which defines the DGLAP anomalous dimension:



$$\langle \mathcal{D}^{\mathrm{DGLAP}}_{J_L,g}(z)\rangle^{\text{1-loop}}_{[\mathcal{O}]_R(p)} = \frac{g^2(N_c^2-1)}{256\pi^4\epsilon}\frac{(2z\cdot p)^{J_L}}{(p^2)^{J_L-1}}\left[4C_A\left(\psi(-J_L)+\gamma_E-\frac{1}{(J_L+2)(J_L+1)}-\frac{1}{J_L(J_L-1)}\right)-\beta_0\right]+O(\epsilon^0)$$

$$[\vec{\mathcal{D}}^{\mathrm{DGLAP}}_{J_L}]_R(z;\mu) = \left[\mathcal{Z}^{\mathrm{DGLAP}}_{J_L}(\alpha_s(\mu))\right]^{-1}\vec{\mathcal{D}}^{\mathrm{DGLAP}}_{J_L}(z) \qquad \mu\frac{d}{d\mu}[\vec{\mathcal{D}}^{\mathrm{DGLAP}}_{J_L}]_R(z;\mu) = \gamma^{\mathrm{DGLAP}}_{J_L}(\alpha_s(\mu))[\vec{\mathcal{D}}^{\mathrm{DGLAP}}_{J_L}]_R(z;\mu)$$

- Renormalizaing this way does not remove the J_L poles
- These poles as before signal recombination of the DGLAP trajectory with another trajectory

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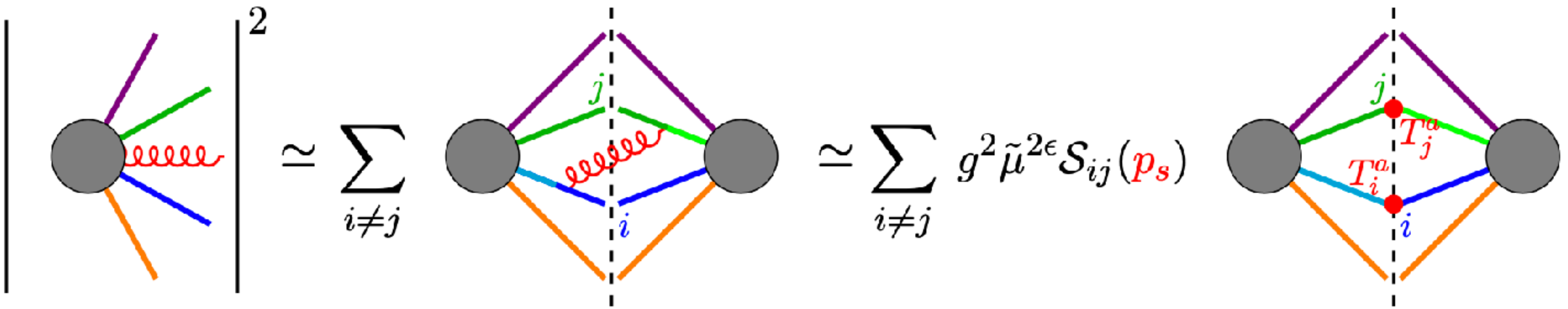
The origin of the $J_L = -2$ pole



- The poles at $J_L = -2 + \mathbb{N}$ arise from the soft limit $E \to 0$ in the loop computation of the DGLAP detector

$$\langle \mathcal{D}^{\mathrm{DGLAP}}_{J_L,g}(z)\rangle^{\mathcal{F}_{n+1}}_{\mathcal{O}(p)}$$

$n+1$ particle contribution

$$= \frac{1}{n!}\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\int \left[\prod_{i=1}^{n}\frac{d^{d-1}\vec{k}_i}{(2\pi)^{d-1}2E_i}\right](2\pi)^d\delta^{(d)}(p - Ez - \sum_{i=1}^{n}k_i)|\mathcal{F}_{n+1}(k_1,\ldots,k_n,Ez;p)|^2$$

$$= \frac{1}{J_L+2}\frac{g^2\tilde{\mu}^{2\epsilon}}{2^d\pi^{d-1}}\frac{1}{n!}\int \mathrm{dLIPS}_n\sum_{i\neq j}\frac{z_i\cdot z_j}{(z\cdot z_i)(z\cdot z_j)}\langle \mathcal{F}_n(k_1,\ldots,k_n;p)|T_i^aT_j^a|\mathcal{F}_n(k_1,\ldots,k_n;p)\rangle + \cdots$$

Cross-section level soft factor $\mathcal{S}_{ij}$

n particle form factor

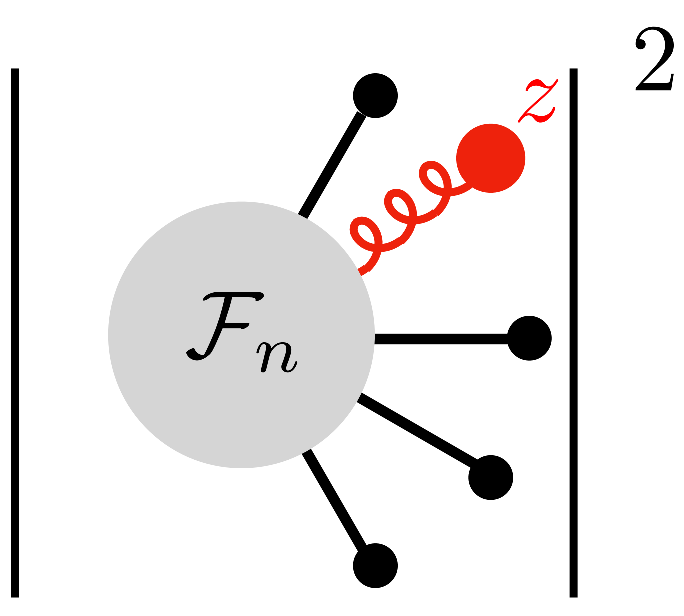
$$\mathcal{S}_{ij}(p_s) = \frac{1}{E^2}\frac{z_i\cdot z_j}{(z\cdot z_i)(z\cdot z_j)}$$

BFKL Detector

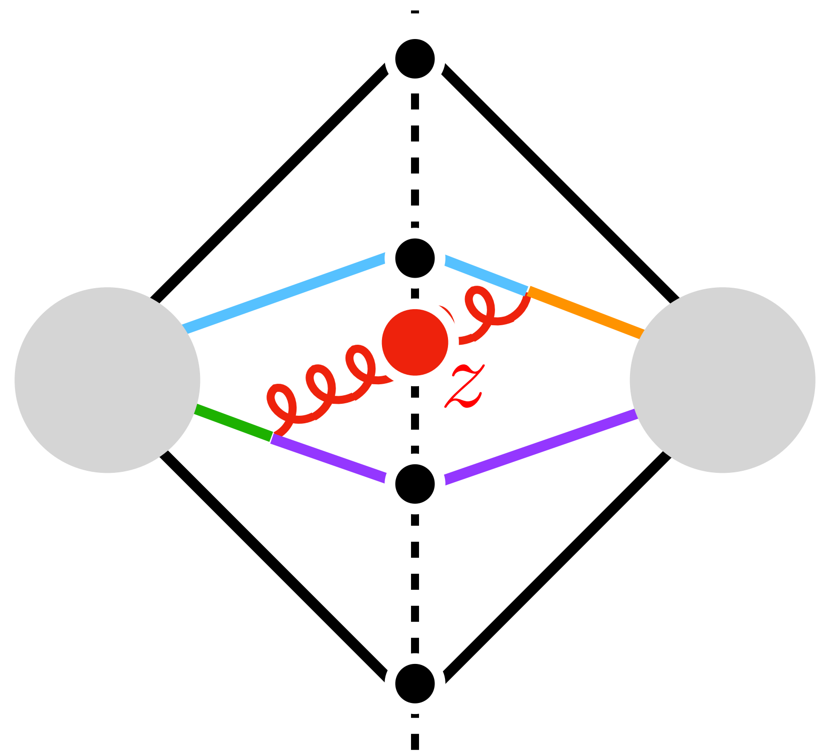
Slides from Hao Chen's talk at EEC Workshop, Wuhan

Apply DGLAP measurement and extract its leading J_L pole from soft theorem

- full phase space
  $$\int \frac{d^{d-1}\vec{p}_i}{(2\pi)^{d-1}2E_i}$$
- **DGLAP detector**
  $$\int \frac{E^{-J_L}dE}{(2\pi)^{d-1}2E}\int d^dp\,\delta(p-Ez)$$
  [constrained P.S.]

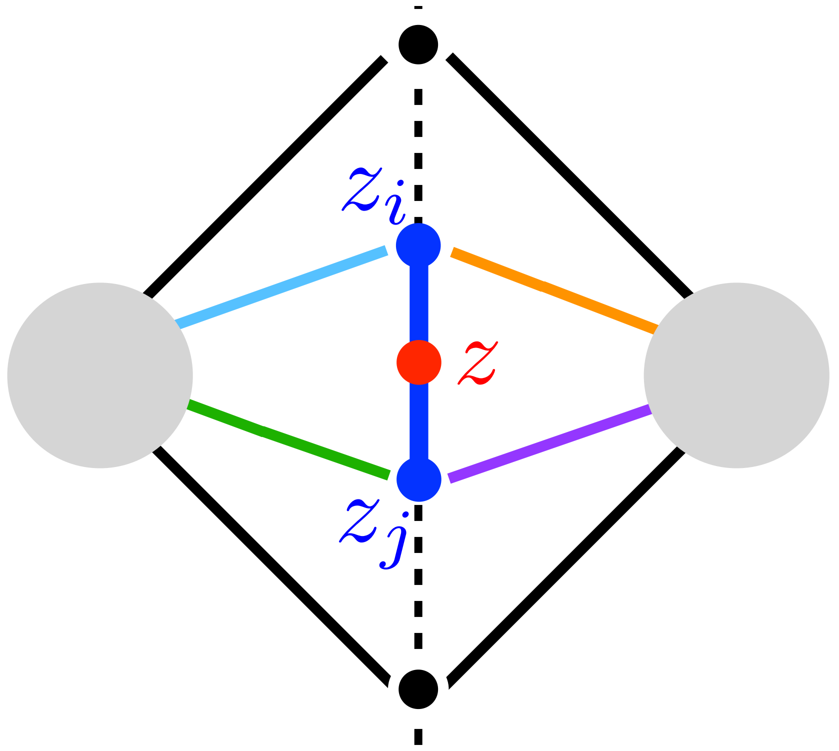


$\longrightarrow \sum_{i\neq j}$

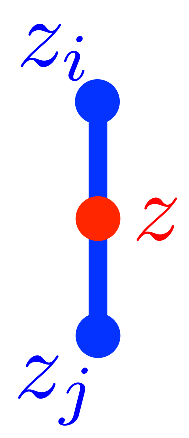


Soft Thm $\longrightarrow$ **+ integrate** E

$\sim \frac{g^2}{J_L+2}\sum_{i\neq j}$



New "measurement" function — BFKL detector



$\longrightarrow$ detector at $J_L = -2$

$$\int d^{d-2}z_i d^{d-2}z_j \frac{z_i\cdot z_j}{(z\cdot z_i)(z\cdot z_j)}\,\mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

$\longrightarrow$

BFKL detector

$$\mathcal{D}^{\mathrm{BFKL}}_{J_L}(z) = \frac{\Gamma(J_L+d-2)}{\Gamma(\frac{J_L+d-2}{2})^2}\int d^{d-2}z_i d^{d-2}z_j \left(\frac{2z_i\cdot z_j}{(2z\cdot z_i)(2z\cdot z_j)}\right)^{-J_L/2}\mathcal{N}^c(z_i)\mathcal{N}^c(z_j)$$

color-interference number detector

$$\mathcal{N}^c(z_i) \leftrightarrow \mathbf{T}_i^c \int \frac{E_i^{d-2}dE_i}{(2\pi)^{d-1}2E_i}\int d^dp_i\,\delta(p_i - E_i z_i)$$

Structure of the BFKL-DGLAP mixing

- One aims to define renormalized DGLAP and BFKL detectors whose loop matrix elements have no ϵ poles and no J_L poles near $J_L \sim -2$

- At generic J_L we have

$$\langle \mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)\rangle^{\text{1-loop}} = \frac{\alpha_s}{4\pi}\frac{\widehat{\gamma}^{(0)}_{gg}(J_L)}{\epsilon}\langle \mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)\rangle^{\text{tree}} + \mathcal{O}(\epsilon^0)\,,$$
$$\langle \mathcal{D}^{\text{BFKL}}_{J_L,g}(z)\rangle^{\text{1-loop}} = \frac{\alpha_s}{4\pi}\frac{\gamma_{\text{BFKL}}(J_L)}{\epsilon}\langle \mathcal{D}^{\text{BFKL}}_{J_L,g}(z)\rangle^{\text{tree}} + O(\epsilon^0).$$

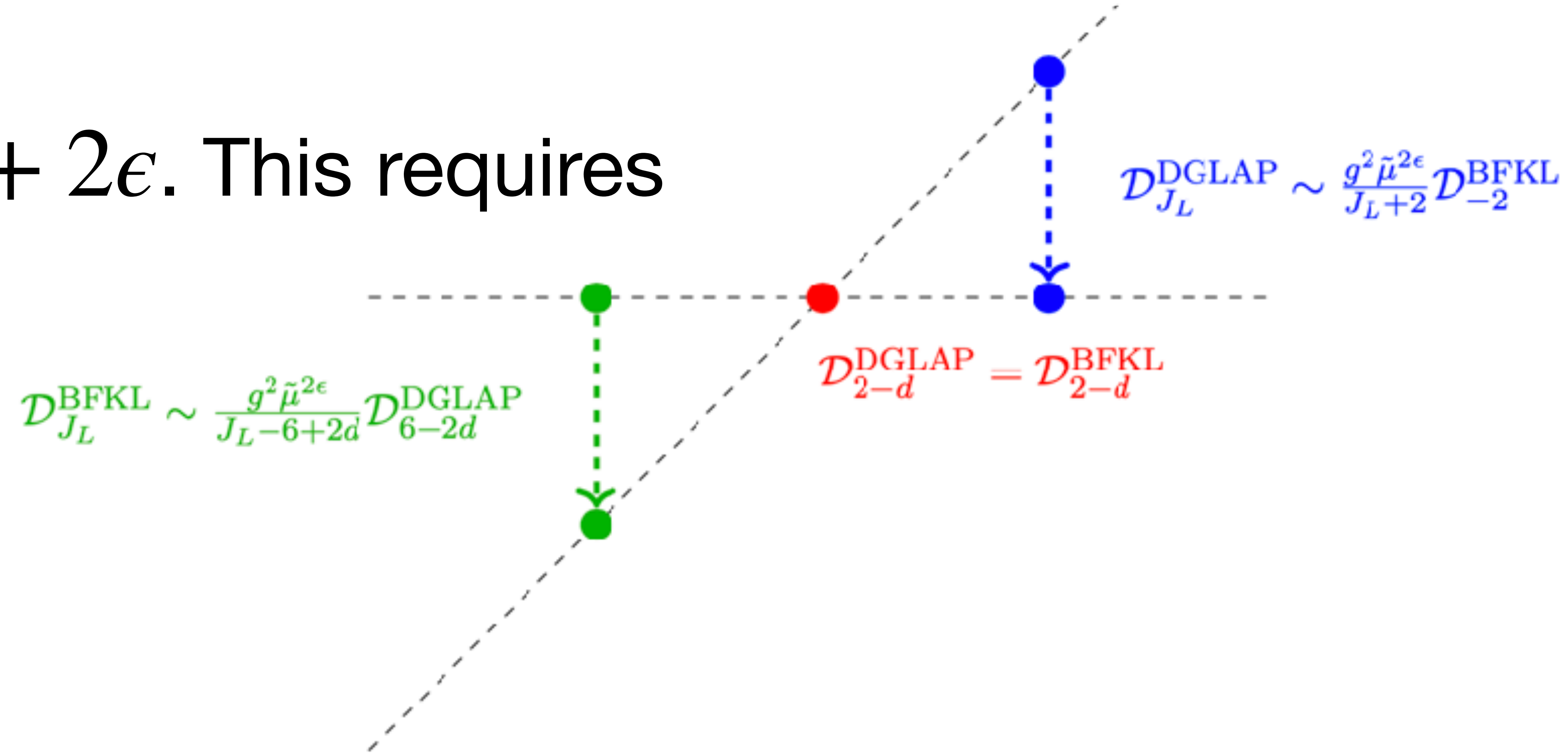
- For generic ϵ the J_L poles are given by

$$\langle \mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)\rangle^{\text{1-loop}} = \frac{\alpha_s\mu^{2\epsilon}}{4\pi}\frac{\mathcal{R}_1(\epsilon)}{J_L+2}\langle \mathcal{D}^{\text{BFKL}}_{J_L,g}(z)\rangle^{\text{tree}} + O((J_L+2)^0)\,,$$
$$\langle \mathcal{D}^{\text{BFKL}}_{J_L,g}(z)\rangle^{\text{1-loop}} = \frac{\alpha_s\mu^{2\epsilon}}{4\pi}\frac{\mathcal{R}_2(\epsilon)}{J_L+2-4\epsilon}\langle \mathcal{D}^{\text{DGLAP}}_{J_L,g}(z)\rangle^{\text{tree}} + O((J_L+2-4\epsilon)^0).$$

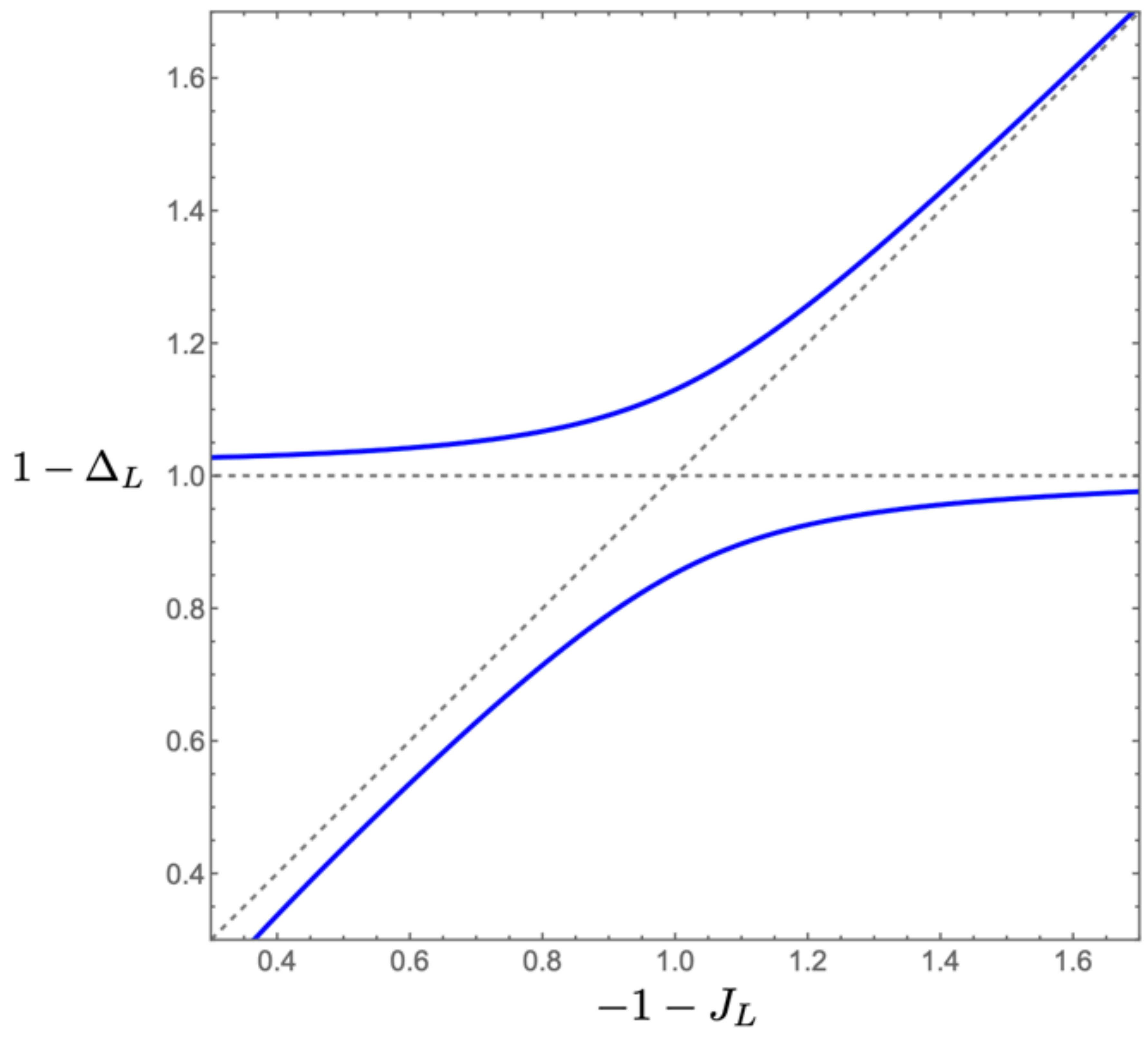
- These two detectors become identical at $J_L = -2 + 2\epsilon$. This requires working with a non-degenerate basis.

- Renormalization yields the detector anom. dim:

$$(\Delta_L(J_L = -2))_{\pm} = \pm\sqrt{\frac{2C_A}{\pi}\alpha_s} + \frac{11C_A}{12\pi}\alpha_s + O(\alpha_s^{3/2})$$



Renormalized Regge trajectories in pure YM



This technology allowed them to predict the leading and subleading poles of the γ_T and the leading poles of γ_S

$$\gamma^T(J,\alpha_s) = \alpha_s\left(-\frac{2C_A}{\pi(J-1)} + \frac{11C_A}{6\pi} + \ldots\right) + \alpha_s^2\left(\frac{4C_A^2}{\pi^2(J-1)^3} - \frac{11C_A^2}{3\pi^2(J-1)^2} + \ldots\right) + \alpha_s^3\left(-\frac{16C_A^3}{\pi^3(J-1)^5} + \frac{22C_A^3}{\pi^3(J-1)^4} + \ldots\right) + \alpha_s^4\left(\frac{80C_A^4}{\pi^4(J-1)^7} - \frac{440C_A^4}{3\pi^4(J-1)^6} + \ldots\right) + \ldots . \quad (4.57)$$

$$\gamma^S(J,\alpha_s) = \alpha_s\left(-\frac{2C_A}{\pi(J-1)} + \ldots\right) + \alpha_s^2\left(\frac{0}{(J-1)^2} + \ldots\right) + \alpha_s^3\left(\frac{0}{(J-1)^3} + \ldots\right) + \alpha_s^4\left(-\frac{4C_A^4\zeta(3)}{\pi^4(J-1)^4} + \ldots\right) + \ldots ,$$