Exploring Structure Constants in Planar $\mathcal{N} = 4$ SYM: From Small Spin to Strong Coupling

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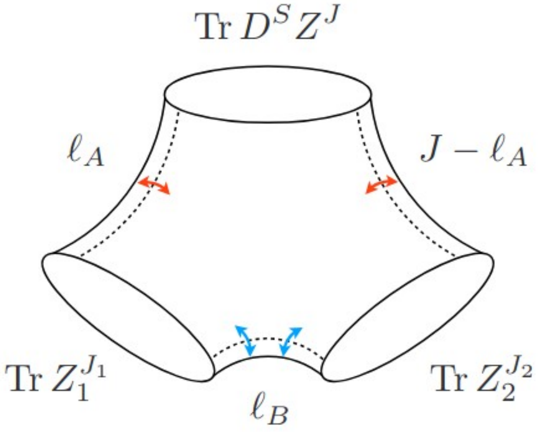
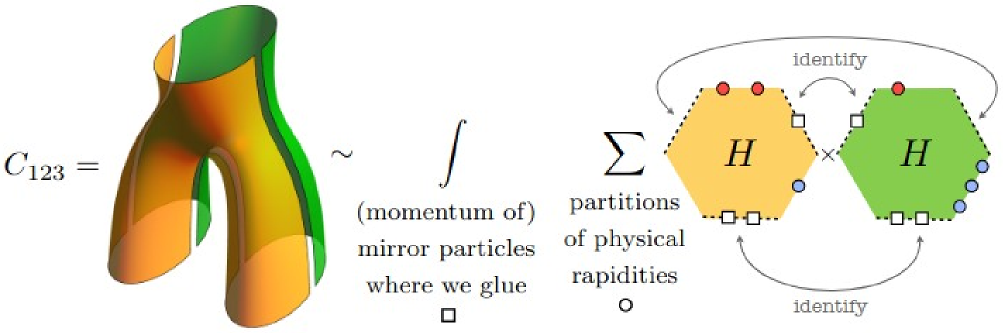
Summary: use integrability to compute correlation functions in planar N=4 SYM at small complex spin, then strong coupling, then finite spin



Compute using Hexagons (integrability):



Okay for weak coupling, large operators, integer spin...

Integrability dream: combine Hexagons with Quantum Spectral Curve (QSC) to compute correlation functions in any regime

This has been done… sort of...

Structure constants of short operators in planar $\mathcal{N}=4$ SYM theory

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$$C^{\circ\circ\bullet} = \mathcal{N} \times \sum\!\!\!\!\!\!\int_L \times \sum\!\!\!\!\!\!\int_R \times \sum\!\!\!\!\!\!\int_B e^{-\ell_L \mathcal{E}_L - \ell_R \mathcal{E}_R - \ell_B \mathcal{E}_B} \, |\mathcal{H}|^2 \qquad |\mathcal{H}|^2 = \prod_{i,j,k=1}^{N_{L,R,B}} \frac{\mathbb{W}^L_{a_i}(u_i)\mathbb{W}^R_{b_j}(v_j)\mathbb{W}^B_{c_k}(w_k)}{p^{\text{mir}}_{a_i b_j}(u_i,v_j)} \,,$$

$$\mathbb{W}^{R/L}_a(u) = e^{\frac{1}{2}L\mathcal{E}_a(u)} \frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^{+/-}_{a,0}(u)} \,, \qquad \mathbb{W}^B_a(u) = e^{-\frac{1}{2}L\mathcal{E}_a(u)} \mathbf{t}_{a,1}(u) \,.$$

Objects coming from Quantum Spectral Curve

Today’s paper: use small spin solution of QSC to compute structure constant

Quantum Spectral Curve: collection of functions P, Q, related by finite-diff eqns

Small spin limit: huge simplification, P ~ Q.

$$x(u) = \frac{u+\sqrt{u^2-4g^2}}{2g}$$

$$\mathbf{P}_1 = \epsilon x^{-J/2}$$

$$\mathbf{P}_2 = -\epsilon x^{+J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n}$$

$$\mathbf{P}_3 = \epsilon \left(x^{-J/2} - x^{+J/2} \right)$$

$$\mathbf{P}_4 = \epsilon \left(x^{J/2} - x^{-J/2} \right) \sinh_- - \epsilon \sum_{n=1}^{J/2} I_{2n-1} \left(x^{\frac{J}{2}-2n+1} + x^{-\frac{J}{2}+2n-1} \right) .$$

$$\epsilon \sim \sqrt{S}$$

$$I_n = I_n(\sqrt{\lambda})$$

t, T ~ det P

$$\Delta - L = S \frac{I_1(\sqrt{\lambda}) + I_3(\sqrt{\lambda})}{I_1(\sqrt{\lambda}) - I_3(\sqrt{\lambda})}$$

$$\frac{C_{123}}{C_{123}^{(0)}} = \mathcal{N}\mathcal{A}\mathcal{B}\,, \qquad \mathcal{A} = 1 + S\,F_J(-\ell_A) + \mathcal{O}\left(S^2\right)\,, \qquad \mathcal{B} = 1 + S\,F_J(\ell_B) + \mathcal{O}\left(S^2\right)\,,$$

where $F_J(\ell)$ is given by

$$F_J(\ell) = \sum_{a=1}^{\infty} \oint \frac{du}{2\pi} e^{-(\ell+\frac{1}{2}J)\tilde{E}_a(u)}\, \tilde{\mu}_a(u)\, t_a(u)\,,$$

$$\mathbf{t}_{a,1} = S\,t_a + \mathcal{O}\left(S^2\right).$$

$$\tilde{\mu}_a(u) = \frac{a}{g^2\left(x^{[+a]}x^{[-a]}\right)^2} \prod_{\sigma_1,\sigma_2=\pm} \left(1 - \frac{1}{x^{[\sigma_1 a]}x^{[\sigma_2 a]}}\right)^{-1}, \tag{2.5}$$

and

$$\tilde{E}_a(u) = \log\left(x^{[+a]}x^{[-a]}\right), \tag{2.6}$$

respectively, where $x^{[\pm a]} = x(u \pm ia/2)$, with $x(u)$ the Zhukovsky variable,

$$x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g}\,. \tag{2.7}$$

Weak coupling computation

ℓ_B	$F_J(\ell_B)$ for $J=2$
1	$3g^4(4\zeta_2\zeta_3+5\zeta_5)-48g^6(\zeta_3\zeta_4+\zeta_2\zeta_5+7\zeta_7)+4g^8(15\zeta_4\zeta_5+63\zeta_3\zeta_6-56\zeta_2\zeta_7+1470\zeta_9)$
2	$4g^6(-3\zeta_3\zeta_4+9\zeta_2\zeta_5+14\zeta_7)+84g^8(\zeta_3\zeta_6-4\zeta_2\zeta_7-20\zeta_9)$
3	$2g^8(-30\zeta_4\zeta_5+56\zeta_2\zeta_7+105\zeta_9)$

Table 1. Four-loop results for $F_J(\ell_B)$ for $J=2$ and bridge lengths $\ell_B=1,2,3$.

$$F_J(\ell)=f_J(\ell)+f_J(-J-\ell)\,,$$

Strong coupling

$$\begin{aligned}f_J(\ell)=&\frac{\gamma_J^{(1)}}{4}\left(\log\lambda-2\log(4\pi)-2\psi(\ell)\right)-\frac{1}{2J}+\frac{1}{2}(\gamma_{\mathrm{E}}-\log(8\pi))+\frac{3+4J}{8\sqrt{\lambda}}\\&+\frac{J+2\ell}{4J}\left(\log(8\sqrt{\lambda})+\gamma_{\mathrm{E}}\right)+\frac{J+2\ell}{\sqrt{\lambda}J}\left(\frac{2J^2-1}{8}+\frac{1-J^2-\ell(J+\ell)}{6}\zeta_2\right)\\&+\mathcal{O}\left(\frac{1}{\lambda}\right),\end{aligned}$$

$$\sqrt{\lambda}=4\pi g$$

Next: strong coupling, finite-spin ansatz

$$\boxed{\frac{C_{123}}{C_{123}^{(0)}} = \frac{\Gamma\left[AdS\right]}{\Gamma[Sphere]} \times \frac{\mathcal{D}_{J_1 J_2 J}(S)}{\lambda^{S/4}\Gamma\left(1+\frac{S}{2}\right)}\,,}$$

$$\Gamma[AdS] = \frac{\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+S}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta+S}{2}\right)}{\sqrt{\Gamma(\Delta+S)\Gamma(\Delta+S-1)}}\,,$$

$$\Gamma[Sphere] = \Gamma[AdS]_{\Delta\to J, S\to 0}\,.$$

local stringy corrections. The formula is constructed so that the Gamma functions in the prefactor align with those observed at small spin in the previous section (see eq. (2.48)). Furthermore, at leading order in the strong coupling limit, $\mathcal{D}\to 1$, and the second factor in eq. (3.2) is designed to reproduce the three-point coupling in flat-space string theory, see ref. [56] for a recent discussion.

$$\log\mathcal{D} = \mathcal{D}_1\, S + \frac{\mathcal{D}_2}{\sqrt{\lambda}}\, S^2 + \frac{\mathcal{D}_3}{\lambda}\, S^3 + \mathcal{O}\left(\frac{S^4}{\lambda^{3/2}}\right) \qquad \mathcal{D}_n = \mathcal{D}_n^{(0)} + \frac{\mathcal{D}_n^{(1)}}{\sqrt{\lambda}} + \frac{\mathcal{D}_n^{(2)}}{\lambda} + \ldots\,,$$

Remainder of paper: testing + fixing this ansatz

4 Classical limit

In this section, we examine our ansatz in the classical limit, where $S, \sqrt{\lambda} \to \infty$ with $\mathcal{S} = S/\sqrt{\lambda}$ held fixed. For reasons that will become clear later, we also adopt a similar scaling for the R-charges, introducing $\mathcal{J} = J/\sqrt{\lambda} = \mathcal{O}(1)$, and likewise for $\mathcal{J}_{1,2} = J_{1,2}/\sqrt{\lambda}$. In this regime, scaling dimensions correspond to the energies of classical spinning strings,

classical integrability [77]. Structure constants have also been computed in this limit [30] by solving a minimal surface problem in $AdS_3 \times S^3$,

$$\log C_{123} = \sqrt{\lambda}\,\text{Area} + \mathcal{O}(1)\,, \tag{4.1}$$

where "Area" refers to the area of a classical string worldsheet ending on the three operators at the AdS boundary. In what follows, we recall the expression for this area and explain how to evaluate it in the limit $\mathcal{S}, \mathcal{J}, \mathcal{J}_1, \mathcal{J}_2 \to 0$.

5.3 Two-loop prediction

By combining our results, we can formulate a two-loop prediction for the structure constants of three operators with arbitrary lengths at strong coupling. This extends to any length the short-string data obtained for $J_1 = J_2 = J = 2$ in ref. [76] and expressed in our notation

$$
\begin{aligned}
&\log \mathcal{D}_{J_1 J_2 J} \\
&\quad = \frac{1}{\sqrt{\lambda}} \left[\frac{5}{8} S - \frac{7 - 4\zeta_3}{16} S^2 \right] \\
&\quad\quad + \frac{1}{\lambda} \left[\frac{(19 - 8J^2) + 8(1 + J^2 - \vec{J}^2)\zeta_3}{32} S - \frac{49 - 8\zeta_3}{64} S^2 + \frac{25 - 12\zeta_3 - 12\zeta_5}{64} S^3 \right] \\
&\quad\quad + \mathcal{O}\left(\frac{1}{\lambda^{3/2}} \right) ,
\end{aligned}
\tag{5.28}
$$

with $\vec{J}^2 = J_1^2 + J_2^2$. The terms linear in S follow directly from eq. (5.27), while the terms proportional to $S^{n+1}/(\sqrt{\lambda})^n$ originate from the classical string analysis (see eq. (4.66)).

6 Conclusion

In this paper, we examined structure constants of single-trace operators as functions of the coupling constant and spin. Our starting point was the hexagon representation, whose study at small spin proved remarkably simple thanks to the exact solution of the QSC equations. This approach led to concise all-loop expressions for structure constants of operators of any length, up to a normalization factor that remains challenging to study and is currently accessible only in the large-length limit.

Moreover, much like scaling dimensions, the small-spin limit offers valuable insight into structure constants at strong coupling. In particular, it indicates that, after factoring out a specific ratio of Gamma functions, the structure constants simplify and take a polynomial form in spin and R-charges at each order in the strong coupling expansion. We showed that this structure is fully consistent with existing data for short strings and smoothly interpolates between the small-spin expansion and the classical limit. Building on this observation, we extended recent two-loop results for the shortest operators to those of arbitrary length.