TASI Lectures on Conformal Field Theory in Lorentzian Signature

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These notes are from TASI 2019. They cover the analytic continuation between Euclidean and Lorentzian signature in QFT and CFT, a Lorentzian proof of the unitarity bounds, null-integrated operators and the average null energy condition (ANEC), Hofman-Maldacena bounds, and a brief introduction to analyticity in spin and light-ray operators. These notes are in progress and have not polished for publication.

Contents

1. Introduction

Both Lorentzian and Euclidean CFTs appear across theoretical physics. Lorentzian CFT_d's describe quantum critical points with $d-1$ spatial directions and 1 time direction. Their symmetries include the Lorentz group $\mathrm{SO}(d-1,1)$. Euclidean CFT_d's describe equilibrium statistical systems at criticality. Their symmetries include the rotation group $\mathrm{SO}(d)$.

Unitary Lorentzian CFTs are related to reflection-positive Euclidean CFTs by Wick rotation. This is the Osterwalder-Schrader reconstruction theorem. (We describe the relationship in more detail below.) Thus, in principle, everything about a Lorentzian CFT is encoded in the usual CFT data (operator dimensions and OPE coefficients) that can be studied in Euclidean signature. However, many observables, and many constraints on CFT data are deeply hidden in the Euclidean correlators. Lorentzian dynamics provides a clearer lens to understand these observables and constraints.

Some examples that we'll study in this course are unitarity bounds, the average null energy condition, and analyticity in spin.

2. From Euclidean to Lorentzian signature

2.1. *Euclidean correlators and time ordering*

Euclidean and Lorentzian correlation functions are related by analytic continuation. Many aspects of this relationship can be understood by modeling a QFT as a quantum-mechanical system with a Hermitian Hamiltonian H whose spectrum is bounded from below, but unbounded above. We assume the system has a vacuum state $|\Omega\rangle$ with zero energy $H|\Omega\rangle = 0$, and all other states have $H > 0$. Let us consider correlators as a function of time. We later generalize to include spatial dependence.

The operator that evolves by Euclidean time τ is $e^{-\tau H}$. Given a local operator $\mathcal{O}(0)$, we define Euclidean Heisenberg picture operators by

$$\mathcal{O}_E(\tau) = e^{\tau H}\mathcal{O}(0)e^{-\tau H}. \tag{1}$$

A correlation function in the vacuum state is given by

$$\begin{aligned}
&\langle\Omega|\mathcal{O}_{1E}(\tau_1)\cdots\mathcal{O}_{nE}(\tau_n)|\Omega\rangle \\
&= \langle\Omega|\mathcal{O}_1(0)e^{-(\tau_1-\tau_2)H}\mathcal{O}_2(0)\cdots\mathcal{O}_{n-1}(0)e^{-(\tau_{n-1}-\tau_n)}\mathcal{O}_n(0)|\Omega\rangle \\
&= \sum_{\psi_1,\ldots,\psi_{n-1}} \langle\Omega|\mathcal{O}_1(0)|\psi_1\rangle\langle\psi_1|\mathcal{O}_2(0)|\psi_2\rangle\cdots\langle\psi_{n-1}|\mathcal{O}_n(0)|\Omega\rangle \\
&\qquad\qquad \times e^{-(\tau_1-\tau_2)E_1}\cdots e^{-(\tau_{n-1}-\tau_n)E_{n-1}},
\end{aligned} \tag{2}$$

where we inserted complete sets of energy eigenstates $\psi_1,\ldots,\psi_{n-1}$. Our first observation is:

Claim 1. *In Euclidean signature, only time-ordered correlation functions of local operators make sense.*

Indeed, suppose the $\mathcal{O}_i$ are time-ordered

$$\tau_1 > \tau_2 > \cdots > \tau_n \qquad \text{(time-ordered)}. \tag{3}$$

The time evolution operators $e^{-(\tau_1-\tau_2)H}$, $e^{-(\tau_2-\tau_3)H}$, etc., cause high-energy states to be exponentially damped. The amplitude for a local operator to create high-energy states grows like a power of the energy.

Exercise 2.1. *Consider a Euclidean two-point function in a 1d CFT*

$$\langle\Omega|\mathcal{O}_E(\tau)\mathcal{O}_E(0)|\Omega\rangle = \sum_\psi |\langle\Omega|\mathcal{O}(0)|\psi\rangle|^2 e^{-\tau E_\psi} = \frac{1}{\tau^{2\Delta}}, \tag{4}$$

where Δ *is the dimension of* $\mathcal{O}$*. Show that the density in energy space* $\rho(E) = \sum_\psi |\langle\Omega|\mathcal{O}(0)|\psi\rangle|^2\delta(E - E_\psi)$ *behaves like* $\rho(E) \propto E^{2\Delta-1}$.

Because Euclidean correlators have power-law divergences as times become coincident, similar statements hold for transitions between non-vacuum states $\langle\psi|\mathcal{O}(0)|\psi'\rangle$. When $\tau_1 > \tau_2$, power-law growth in energy is not strong enough to overcome exponential damping from the evolution operator $e^{-(\tau_1-\tau_2)H}$, and thus the correlator (2) is finite.[a]

By contrast, suppose that the $\mathcal{O}_i$ were not time-ordered, for example $\tau_2 > \tau_1$. Then $e^{-(\tau_1-\tau_2)H}$ would be a wildly unbounded operator, causing high-energy states to be exponentially enhanced. This exponential enhancement easily overwhelms the transition amplitudes $\langle\psi_i|\mathcal{O}_i|\psi_{i+1}\rangle$, making the correlator infinite.

For these reasons, we usually only consider time-ordered Euclidean correlators:

$$\begin{aligned}\langle\mathcal{O}_{1E}(\tau_1)\cdots\mathcal{O}_{nE}(\tau_n)\rangle &= \langle\Omega|T_E\{\mathcal{O}_{1E}(\tau_1)\cdots\mathcal{O}_{nE}(\tau_n)\}|\Omega\rangle \\ &= \langle\Omega|\mathcal{O}_{1E}(\tau_1)\cdots\mathcal{O}_{nE}(\tau_n)|\Omega\rangle\theta(\tau_1 > \cdots > \tau_n) \\ &\quad + \text{permutations}.\end{aligned} \tag{5}$$

These are also the objects naturally computed by the Euclidean path integral.

2.2. *Analytic continuation and holomorphicity*

Let us consider analytically-continuing a Euclidean correlator to complex times. The θ-functions in (10) cannot be continued in a natural way. Instead, we should think of the Euclidean correlator as a collection of different functions — one for each ordering of the τ_i.[b] We can separately continue each of these functions. For example, consider

$$F(\tau_1,\ldots,\tau_n) = \langle\Omega|\mathcal{O}_{1E}(\tau_1)\cdots\mathcal{O}_{nE}(\tau_n)|\Omega\rangle. \tag{6}$$

[a] A class of operators that have an exponentially large amplitude to create high energy states are nonlocal operators (like line or surface defects) that are extended in Euclidean time. Correlators of such extended operators are only guaranteed to converge if they are sufficiently far separated in Euclidean time that they don't overlap.

[b] The functions for different orderings may or may not be related by symmetries.

Claim 2. *$F(\tau_1,\dots,\tau_n)$ can be analytically continued to a holomorphic function of its arguments in the region* $\mathrm{Re}\,\tau_1 > \cdots > \mathrm{Re}\,\tau_n$.

Let us write $\tau_i = \epsilon_i + it_i$, and consider changing the t_i away from zero. The time-evolution operators become

$$e^{-(\epsilon_1-\epsilon_2)H} \to e^{-(\epsilon_1-\epsilon_2)H-i(t_1-t_2)H}. \tag{7}$$

As long as we stay in the region $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n$, high-energy states remain exponentially-damped. The imaginary parts t_i just insert phases into already nicely-convergent sums. Thus, the correlator (6) is holomorphic in this region.

2.3. *Lorentzian correlators*

The Lorentzian time-evolution operator is e^{-itH}. We define Lorentzian Heisenberg picture operators by[c]

$$\mathcal{O}_L(t) = e^{itH}\mathcal{O}(0)e^{-itH} = \mathcal{O}_E(\tau = it). \tag{8}$$

A correlator of Lorentzian operators is given by

$$\begin{aligned}&\langle\Omega|\mathcal{O}_{1L}(t_1)\cdots\mathcal{O}_{nL}(t_n)|\Omega\rangle\\ &= \langle\Omega|\mathcal{O}_1(0)e^{-i(t_1-t_2)H}\mathcal{O}_2(0)\cdots\mathcal{O}_{n-1}(0)e^{-i(t_{n-1}-t_n)}\mathcal{O}_n(0)|\Omega\rangle.\end{aligned} \tag{9}$$

Unlike in the Euclidean case, it is reasonable to consider non-time-ordered Lorentzian correlators. Indeed, e^{-itH} is oscillatory for both signs of t — neither sign is nicer than the other. For example, it makes sense to consider a commutator $[\mathcal{O}_{1L}(t_1), \mathcal{O}_{2L}(t_2)]$. By contrast, a commutator of operators at different Euclidean times does not make sense because it involves at least one unbounded operator $e^{\tau H}$.

In quantum field theory, a Lorentzian correlator with fixed ordering like (9) is called a Wightman function. We adopt this terminology in what follows. By contrast, a time-ordered Lorentzian correlator is a sum of Wightman functions times θ-functions enforcing different orderings

$$\begin{aligned}&\langle\Omega|T_L\{\mathcal{O}_{1L}(t_1)\cdots\mathcal{O}_{nL}(t_n)\}|\Omega\rangle\\ &= \langle\Omega|\mathcal{O}_{1L}(t_1)\cdots\mathcal{O}_{nL}(t_n)|\Omega\rangle\theta(t_1 > \cdots > t_n) + \text{permutations}.\end{aligned} \tag{10}$$

[c]The choice of $-i$ vs. i in the time-evolution operator e^{-itH} is a convention that Schrödinger fixed at the beginning of the 20th century. All minus signs and factors of i follow from this convention. In particular, the relationship $t = i\tau$ between Lorentzian and Euclidean time is fixed by this convention, together with the statement that H is unbounded above.

2.3.1. *Tempered distributions*

The fact that e^{-itH} is oscillatory has an important consequence: Wightman functions are not actually functions. To make the sum over high-energy states converge, we must *smear* the times t_{iL} against smooth "test" functions

$$\int dt_1 \cdots dt_n f_1(t_1) \cdots f_n(t_n) \langle \Omega | \mathcal{O}_{1L}(t_1) \cdots \mathcal{O}_{nL}(t_n) | \Omega \rangle. \tag{11}$$

Provided the functions $f_i(t_i)$ are sufficiently smooth, the rapidly oscillating phases from high-energy states will average out to small values, causing the smeared correlator to converge.

The technical statement is that Wightman functions are *tempered distributions*. Let us define this term. Firstly, given a space of functions $\mathcal{F}$, a distribution T on $\mathcal{F}$ is a continuous linear function

$$T : \mathcal{F} \to \mathbb{C}. \tag{12}$$

"Continuous" means that if $f_1, f_2, \cdots \in \mathcal{F}$ is a convergent sequence (under some appropriate norm), then the sequence $T(f_1), T(f_2), \cdots \in \mathbb{C}$ is convergent as well. We often write formally

$$T(f) = \int dt f(t) T(t) \in \mathbb{C}, \tag{13}$$

though the value $T(t)$ might not make sense. An example is the Dirac δ-distribution $T(t) = \delta(t)$, which is only defined by its integral against a test function

$$\int dt f(t) \delta(t) = f(0). \tag{14}$$

The space of test functions relevant for QFT is the Schwartz space $\mathcal{S}$ of rapidly decreasing functions. Functions $f \in \mathcal{S}$ have the property that[d]

$$t^m \left(\frac{\partial}{\partial t} \right)^n f(t) \tag{15}$$

is bounded as a function of t for any $m, n \in \mathbb{Z}_{\geq 0}$. Distributions on $\mathcal{S}$ are called *tempered distributions*. Our claim is that Wightman functions must be integrated against Schwartz functions to obtain finite numbers.[e] Henceforth, we use the term "test function" synonymously with Schwartz function.

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[d] **Say something about the norm on $\mathcal{S}$.**

[e] The more technically correct term is "Wightman distributions".

2.3.2. *Boundary values of holomorphic functions*

To understand why Wightman functions are tempered distributions, we must complete the dictionary between Euclidean and Lorentzian correlators. The key observation is that the Wightman function (9) is a boundary value of the holomorphic function (6). Specifically, let us set

$$\tau_i = \epsilon_i + it_i, \quad \text{where} \quad \epsilon_i, t_i \in \mathbb{R}, \tag{16}$$

where $\epsilon_1 > \cdots > \epsilon_n$.

When $t_i = 0$, we have a Euclidean correlator. The function F is holomorphic in this region, so we can safely continue t_i from 0 to any desired value. Afterwards, we can try to take the limit $\epsilon_i \to 0$, maintaining the ordering $\epsilon_1 > \cdots > \epsilon_n$. Formally, this produces the desired Wightman function

$$\lim_{\epsilon_i \to 0} \langle \Omega | \mathcal{O}_{1E}(\epsilon_1 + it_1) \cdots \mathcal{O}_{nE}(\epsilon_n + it_n) | \Omega \rangle = \langle \Omega | \mathcal{O}_{1L}(t_1) \cdots \mathcal{O}_{nL}(t_n) | \Omega \rangle. \tag{17}$$

However, the limit is not always well-defined because it requires approaching the boundary of the region of holomorphicity. We claim that if we smear against test functions $f_i(t_i)$, and subsequently take $\epsilon_i \to 0$, then the limit becomes well-defined. This is how the boundary value of F is defined as a tempered distribution.

For example, consider a correlator $F(\tau)$ that is holomorphic in a single variable $\tau = \epsilon + it$ in the region $\epsilon > 0$. Assume that $F(\tau)$ has at most a power-law divergence as we approach the boundary of the regime of holomorphicity[f]

$$|F(\epsilon + it)| \leq \epsilon^{-k} P(t), \tag{18}$$

where $P(t)$ is polynomially bounded for large t. Consider the integral of F against a test function $f(t)$ at finite nonzero ϵ,

$$a(\epsilon) = \int_{-\infty}^{\infty} dt F(\epsilon + it) f(t). \tag{19}$$

We would like to argue that $\lim_{\epsilon \to 0} a(\epsilon)$ is finite, and furthermore it does

[f]A power law divergence ϵ^{-k} is typical for CFT correlators, where k can be related to operator dimensions.

not depend discontinuously on f. Note that

$$a^{(n)}(\epsilon) = \left(\frac{\partial}{\partial\epsilon}\right)^n a(\epsilon) = \int_{-\infty}^{\infty} dt \left(-i\frac{\partial}{\partial t}\right)^n F(\epsilon+it)f(t)$$
$$= \int_{-\infty}^{\infty} dt F(\epsilon+it)\left(i\frac{\partial}{\partial t}\right)^n f(t), \qquad (20)$$

so that (18) and $f \in \mathcal{S}$ imply

$$\left|a^{(n)}(\epsilon)\right| \leq \frac{C_n}{\epsilon^k}, \qquad (21)$$

where the constants C_n depend continuously on f. We can recover $a(\epsilon)$ by integrating its derivatives. However, each time we integrate, we lower the strength of divergence in ϵ

$$\left|a^{(n-1)}(\epsilon)\right| = \left|a^{(n-1)}(\epsilon_0) + \int_{\epsilon_0}^{\epsilon} a^{(n)}(\epsilon)d\epsilon\right| \leq \frac{C_n}{k-1}\frac{1}{\epsilon^{k-1}} + \text{const.} \qquad (22)$$

Starting with (21) for $n = k+1$ and integrating $k+1$ times, we find that $\lim_{\epsilon\to 0} a(\epsilon)$ is finite.

2.4. *Example: two-point function in 1d*

As an example, consider a two-point function in a 1d CFT,

$$\langle\Omega|\mathcal{O}_E(\tau)\mathcal{O}_E(0)|\Omega\rangle = \frac{1}{\tau^{2\Delta}}. \qquad (23)$$

The corresponding Wightman distribution is[g]

$$\langle\Omega|\mathcal{O}_L(t)\mathcal{O}_L(0)|\Omega\rangle = \lim_{\epsilon\to 0}\frac{1}{(\epsilon+it)^{2\Delta}}. \qquad (24)$$

When integrating against a test function $f(t)$, the correct definition of the distribution on the right-hand side is that one should integrate $\int dt f(t)$ with ϵ finite and nonzero, and subsequently take $\epsilon \to 0$.

Even when Δ is enormous, (24) is a tempered distribution. This might seem surprising because the limit $\epsilon \to 0$ naively leads to an enormous singularity at $t = 0$. However, this singularity becomes innocuous when integrated against a test function. As a trivial example, if the test function is holomorphic, then we can deform the integration contour away from the singularity at $t = i\epsilon$, obtaining a manifestly finite result.

[g]Note that we have a non-integer power of a complex number in the denominator. However, its sign is fixed unambiguously by saying that we should analytically continue in t from $t = 0$ (where the two-point function is real and positive).

Even if the test function is not holomorphic, we can understand why (24) is a tempered distribution by taking its Fourier transform. The Fourier transform

$$(\mathscr{F}f)(\omega) = \int_{-\infty}^{\infty} dt\, e^{it\omega} f(t) \tag{25}$$

famously takes Schwartz functions to Schwartz functions

$$\mathscr{F} : \mathcal{S} \to \mathcal{S}. \tag{26}$$

It can be defined on tempered distributions by exploiting this property

$$(\mathscr{F}T)(f) = T(\mathscr{F}f), \tag{27}$$

which generalizes the usual statement for functions

$$\int_{-\infty}^{\infty} dt f(t)(\mathscr{F}g)(t) = \int_{-\infty}^{\infty} d\omega (\mathscr{F}f)(\omega) g(\omega). \tag{28}$$

We have

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\, e^{it\omega}\frac{1}{(\epsilon+it)^{2\Delta}} &= \theta(\omega) i \left(e^{-2i\pi\Delta} - e^{2i\pi\Delta}\right)\int_0^{\infty} d\tau e^{-\omega\tau}\frac{1}{\tau^{2\Delta}} \\
&= 2\theta(\omega)\sin(2\pi\Delta)\Gamma(1-2\Delta)\omega^{2\Delta-1} \\
&= \frac{2\pi}{\Gamma(2\Delta)}\omega^{2\Delta-1}\theta(\omega).
\end{aligned} \tag{29}$$

In the first line, we used that when $\omega < 0$, the integrand is holomorphic in the lower half-plane, and the contour can be deformed there to give zero. This leads to the theta function $\theta(\omega)$.[h] When $\omega > 0$, we can wrap the contour around the singularity in the vertical direction, writing $t = i\tau$, giving the above result.

Recall that the Fourier transform of a Schwartz function is itself a Schwartz function. The Fourier-space two-point function (29) is obviously integrable and polynomially bounded for all $\Delta > 0$. Thus, it is finite when integrated against a Schwartz function $\widetilde{f}(\omega)$ in frequency space.[i]

When $\Delta \leq 0$, the Wightman two-point function is still a tempered distribution, but this fact is not captured by the expression (29). Instead, we can look at the position-space correlator, which for $\Delta < 1$ can clearly be integrated against any Schwartz function $f(t)$ in position space.

[h] The factor $\theta(\omega)$ reflects the fact that the spectrum of the theory is bounded from below. Indeed, note that

$$\langle\Omega|\mathcal{O}(t) = \langle\Omega|\mathcal{O}(0)e^{-iHt} \tag{30}$$

is a sum of exponentials e^{-iEt} with positive E.

[i] The condition $\Delta \geq 0$ is exactly the unitarity bound in 1d. As we explain shortly, this is not a coincidence.

2.4.1. *Comments about smearing and Fourier transforms*

In Euclidean signature, smearing correlation functions sometimes seems like a bad idea. Because Euclidean correlation functions are time-ordered, smearing generally involves linear combinations of operators with different orderings. Furthermore, in a CFT, Euclidean smearing can move operators outside the regime of convergence of some OPE. This mixing of different OPE regions and operator orderings can make it difficult to understand the structure of the correlator. Finally, Euclidean singularities are often badly non-integrable as a function of the times τ_i. Thus, they can only be smeared against test functions if some prescription is given for dealing with singularities.[j] All of these difficulties arise if one attempts Fourier analysis for CFT correlators in Euclidean signature.[k]

By contrast, the situation for Wightman functions in Lorentzian signature is totally different. Wightman functions can be smeared without changing operator orderings. As we will see later, this also means that smearing does not move one outside the regime of OPE convergence. Furthermore, the singularities of Wightman functions are totally innocuous when integrated against test functions, as we saw in the above example. In particular, Fourier analysis works great for Wightman functions, and is an especially good idea when it plays well with symmetries.

From the point of view of smearing and Fourier analysis, time-ordered Lorentzian correlators are more similar to Euclidean correlators. In fact, they are related to Euclidean correlators by the usual Wick rotation $\tau_i \to e^{\frac{i\pi}{2}} t_i$. From the Lorentzian point of view, the θ-functions in (10) destroy the nice properties of coincident-point singularities in Wightman functions. To smear time-ordered Lorentzian correlators, one again must give a prescription for dealing with coincident-point singularities. However, near lightcone singularities, time-ordered correlators behave like tempered distributions.

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[j] These problems are sometimes unavoidable. For example, in conformal perturbation theory one smears Euclidean operators against classical background functions. In this case, a regulator must be introduced to deal with singularities.

[k] A case where Euclidean Fourier analysis is not so problematic is for de Sitter correlators, which are invariant under the Euclidean conformal group, but have different types of singularities from those of reflection-positive Euclidean CFTs.

2.5. *Generalizing to QFT*

Let us finally introduce spatial directions and study QFT. We work in "mostly plus" signature

$$ds^2 = -(dx^0)^2 + d\mathbf{x}^2 = -(dx^0)^2 + \sum_{i=1}^{d-1}(dx^i)^2. \tag{31}$$

Heisenberg picture operators are defined by

$$\mathcal{O}(x) = e^{-iP\cdot x}\mathcal{O}(0)e^{iP\cdot x}, \tag{32}$$

where P^μ are the energy-momentum generators. The signs above are fixed by requiring that the t-dependence of the right-hand operator is e^{-itH}, where $H = P^0$ and $t = x^0$. Henceforth, operators $\mathcal{O}(x)$ without "E" or "L" subscripts are Lorentzian operators.

Given a Wightman function

$$\langle\Omega|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|\Omega\rangle, \tag{33}$$

we can give the positions x_k real and imaginary parts

$$x_k = y_k + i\zeta_k, \quad y_k, \zeta_k \in \mathbb{R}^{d-1,1}. \tag{34}$$

The operators in the Wightman correlator are then separated by

$$e^{iP\cdot x_{12}} = e^{iP\cdot y_{12} - P\cdot\zeta_{12}}, \tag{35}$$

and similarly for x_{23}, x_{34}, etc..

The generalization of the statement that H is bounded from below is that the spectrum of P is contained inside the future null cone. Consequently, the the real part of the exponential (35) is negative provided that ζ_{12} is past-directed, which we write as $\zeta_{12} < 0$ or $\zeta_1 < \zeta_2$. When this condition holds, high energy states are exponentially damped. Thus, positivity of energy implies that the Wightman function is holomorphic in the region

$$\zeta_1 < \zeta_2 < \cdots < \zeta_n. \tag{36}$$

Wightman distributions in real space are defined as boundary values of holomorphic functions in the region (36). We can compute them with the following recipe:

- Start with x_i real and mutually spacelike. For example, place them at times $t_i = 0$.
- Give the x_i small imaginary parts $\zeta_i = \operatorname{Im} x_i$ satisfying (36). For example, if all points start at Lorentzian time $t_i = 0$, we can assign them times $x_i^0 = -i\epsilon_i$ with $\epsilon_1 > \cdots > \epsilon_n$.

- Continue the real parts $y_i = \mathrm{Re}\, x_i$ to the desired values.
- Take the imaginary parts to ζ_i zero, treating the result a distribution.

The Osterwalder-Schrader reconstruction theorem states that correlators in a reflection positive Euclidean QFT can be analytically continued to give Wightman functions that are tempered distributions on Minkowski space $\mathbb{R}^{d-1,1}$.

2.6. *Example: two-point function in d dimensions*

Consider a two-point function of a scalar $\mathcal{O}$ in a d-dimensional CFT. The Euclidean correlator is

$$\langle \mathcal{O}_E(\tau_1, \mathbf{x}_1) \mathcal{O}_E(\tau_2, \mathbf{x}_2) \rangle = \frac{1}{(\tau_{21}^2 + \mathbf{x}_{21}^2)^\Delta}, \tag{37}$$

where Δ is the dimension of $\mathcal{O}$. Let us compute the Wightman function

$$\langle \Omega | \mathcal{O}(t_2, \mathbf{x}_2) \mathcal{O}(t_1, \mathbf{x}_1) | \Omega \rangle. \tag{38}$$

We start with the operators $\mathcal{O}_E$ at Euclidean times $\tau_2 = \epsilon_2 > \tau_1 = \epsilon_1$ and then continue $\epsilon_i \to \epsilon_i + it_i$, staying in the region $\epsilon_2 > \epsilon_1$,

$$\langle \Omega | \mathcal{O}_E(\epsilon_2 + it_2, \mathbf{x}_2) \mathcal{O}_E(\epsilon_1 + it_1, \mathbf{x}_1) | \Omega \rangle = \frac{1}{(-t_{21}^2 + \mathbf{x}_{21}^2 + 2i\epsilon_{21} t_{21} + \epsilon_{21}^2)^\Delta}. \tag{39}$$

Finally, we take $\epsilon_{21} = \epsilon \to 0$,

$$\langle \Omega | \mathcal{O}(t_2, \mathbf{x}_2) \mathcal{O}(t_1, \mathbf{x}_1) | \Omega \rangle = \lim_{\epsilon \to 0} \frac{1}{(x_{21}^2 + i\epsilon t_{21})^\Delta}, \tag{40}$$

where we have used the Minkowski norm $x^2 = -t^2 + \mathbf{x}^2$.

The denominator involves a fractional power of a complex number. However, its phase is fixed by our prescription for analytic continuation.

Exercise 2.2. *Show that the phases in different causal configurations are*

$$\langle \Omega | \mathcal{O}(t_2, \mathbf{x}_2) \mathcal{O}(t_1, \mathbf{x}_1) | \Omega \rangle = \frac{1}{|x_{21}^2|^\Delta} \times \begin{cases} e^{-i\pi\Delta} & x_2 > x_1 \\ 1 & x_1 \approx x_2 \\ e^{i\pi\Delta} & x_1 > x_2 \end{cases}. \tag{41}$$

Here, we write $x_i > x_j$ to denote that x_i is in the future of x_j, and $x_i \approx x_j$ to denote that x_i is spacelike from x_j.

To compute the other ordering $\langle\Omega|\mathcal{O}(t_1,\mathbf{x}_1)\mathcal{O}(t_2,\mathbf{x}_2)|\Omega\rangle$, we choose $\epsilon_{12}=\epsilon>0$, which leads to

$$\langle\Omega|\mathcal{O}(t_1,\mathbf{x}_1)\mathcal{O}(t_2,\mathbf{x}_2)|\Omega\rangle = \lim_{\epsilon\to 0}\frac{1}{(x_{12}^2+i\epsilon t_{12})^{\Delta}}, \tag{42}$$

which differs from (40) only in its $i\epsilon$ prescription. In particular, the expectation value of the commutator $\langle\Omega|[\mathcal{O}(t_1,\mathbf{x}_1),\mathcal{O}(t_2,\mathbf{x}_2)]|\Omega\rangle$ vanishes at spacelike separation, as it should by microcausality. Henceforth, when writing Wightman distributions, we often leave $\lim_{\epsilon\to 0}$ implicit.

3. Lorentzian proof of the unitarity bounds

In Slava's lectures, you saw a derivation of the unitarity bounds for primary operators in a CFT

$$\Delta \geq \begin{cases} \frac{d-2}{2} & J=0, \\ J+d-2 & J>0, \end{cases}$$

where Δ and J are dimension and spin of $\mathcal{O}$, respectively. The derivation involved studying norms of descendant states in radial quantization $P_{\mu_1}\cdots P_{\mu_n}|\mathcal{O}\rangle$ and demanding that they are positive. The bound for $J=0$ comes from level $n=2$, while the bound for $J>0$ comes from level $n=1$.

One catch is that positivity of a finite number of levels only gives *necessary* conditions for unitarity. To argue that these conditions are also *sufficient*, one must determine that higher levels $n\geq 0$ don't give new constraints. By contrast, there is a beautiful Lorentzian derivation of the unitarity bounds due to Mack that immediately gives both necessary and sufficient conditions.

Let us give the argument for a primary Hermitian scalar $\mathcal{O}$. The conformal multiplet of $\mathcal{O}$ is densely spanned by states of the form

$$|\Psi_f\rangle \equiv \int d^dx f(x)\mathcal{O}(x)|\Omega\rangle, \tag{43}$$

where the integral runs over points $x\in\mathbb{R}^{d-1,1}$ in Minkowski space, and $f\in\mathcal{S}(\mathbb{R}^{d-1,1})$ are test functions. Unitarity is the statement that the norms of all such states are positive,

$$\langle\Psi_f|\Psi_f\rangle = \int d^dx_1 d^dx_2 f^*(x_2)f(x_1)K(x_2-x_1)\geq 0, \tag{44}$$

where

$$K(x_2-x_1)\equiv\langle\Omega|\mathcal{O}(x_2)\mathcal{O}(x_1)|\Omega\rangle = \frac{1}{(x_{12}^2+i\epsilon t_{12})^{\Delta}}. \tag{45}$$

This is equivalent to the statement that $K(x_2 - x_1)$ is a positive-definite integral kernel.

Note that $K(x_2 - x_1)$ is translationally invariant, and is thus diagonalized (as a kernel) in momentum space

$$\int d^d x_1 d^d x_2 f^*(x_2) f(x_1) K(x_2 - x_1) = \int \frac{d^d p}{(2\pi)^d} |\widetilde{f}(p)|^2 \widetilde{K}(p), \tag{46}$$

where

$$\widetilde{K}(p) = \int d^d x \langle \Omega | \mathcal{O}(x) \mathcal{O}(0) | \Omega \rangle e^{-ip\cdot x} = \int d^d x \frac{1}{(x^2 + i\epsilon t)^\Delta} e^{-ip\cdot x}. \tag{47}$$

Thus, unitarity is equivalent to positivity of $\widetilde{K}(p)$ as a tempered distribution on momentum space.

Let us calculate $\widetilde{K}(p)$. By positivity of energy, it is proportional to $\theta(p)$, a θ-function imposing that p lies inside the forward lightcone. Using Lorentz symmetry, we can fix $p = (p^0, 0, \ldots, 0)$. We then have

$$\begin{aligned} \lim_{\epsilon\to 0} \int d^d x \frac{1}{(-t^2 + \mathbf{x}^2 + i\epsilon t)^\Delta} e^{ip^0 t} &= \operatorname{vol} S^{d-2} \int \frac{r^{d-2} dr dt}{(-t^2 + r^2 + i\epsilon t)^\Delta} e^{ip^0 t} \\ &= \frac{2^{d-2\Delta} 2\pi^{\frac{d+2}{2}}}{\Gamma(\Delta - \frac{d-2}{2})\Gamma(\Delta)} (p^0)^{2\Delta - d} \theta(p^0), \end{aligned} \tag{48}$$

where we did the r-integral first, and used (29) to do the integral over t. Restoring Lorentz covariance gives

$$\widetilde{K}(p) = \frac{2^{d-2\Delta} 2\pi^{\frac{d+2}{2}}}{\Gamma(\Delta - \frac{d-2}{2})\Gamma(\Delta)} (-p^2)^{\Delta - \frac{d}{2}} \theta(p). \tag{49}$$

Exercise 3.1. *Verify that $\widetilde{K}(p)$ vanishes if p is spacelike or past-directed. You may find it helpful to use Lorentz invariance to fix p in each case.*

We see that $\widetilde{K}(p)$ is positive if $\Delta \geq \frac{d-2}{2}$, which is the correct unitarity bound for scalar operators. When $\Delta = \frac{d-2}{2}$, the Γ function in the denominator contributes a zero. However, this combines with a lightcone singularity $(-p^2)^{-1}$ to give a delta-function localized on the light cone $\delta(p^2)$ (as appropriate for a free theory). This distribution is still positive, so that $\Delta = \frac{d-2}{2}$ is an allowed value in a unitary theory. As Δ varies below the unitarity bound, there are regions where the product of Γ functions becomes positive again. However, in this region, the expression (49) has negative powers of p^2 which lead to singularities on the lightcone. One can still make sense of this as a distribution, but it is not guaranteed to be positive.

3.1. *Generalization to spinning operators*

We can perform a similar exercise for a spinning two-point function. We must impose that

$$\widetilde{K}^{\mu_1\cdots\mu_J}{}_{\nu_1\cdots\nu_J}(p) = \int d^dx e^{-ip\cdot x}\langle\Omega|\mathcal{O}^{\mu_1\cdots\mu_J}(x)\mathcal{O}_{\nu_1\cdots\nu_J}(0)|\Omega\rangle \tag{50}$$

is a positive-definite kernel. It is natural to decompose $\widetilde{K}$ into projectors onto irreducible representations of the $\mathrm{SO}(d-1)$ that preserves p. For example, in the spin-1 case, we can write

$$\begin{aligned}
\widetilde{K}^{\mu}{}_{\nu}(p) &= (-p^2)^{\Delta-\frac{d}{2}}\theta(p)\sum_{s=0,1}\mathcal{A}_s(\Delta, J=1)\Pi_s(p)^{\mu}{}_{\nu},\\
\Pi_1(p)^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} - \frac{p^{\mu}p_{\nu}}{p^2},\\
\Pi_0(p)^{\mu}{}_{\nu} &= \frac{p^{\mu}p_{\nu}}{p^2}.
\end{aligned} \tag{51}$$

In general, $(-1)^{J+s}\Pi_s(p)$ is a positive-definite bilinear form.

Exercise 3.2. *Show that*

$$(-1)^{1+0}\mathcal{A}_0(\Delta,1) = \frac{2^{d-2\Delta}2\pi^{\frac{d+2}{2}}(\Delta-d+1)}{\Gamma(\Delta-\frac{d-2}{2})\Gamma(\Delta+1)}, \tag{52}$$

$$(-1)^{1+1}\mathcal{A}_1(\Delta,1) = \frac{2^{d-2\Delta}2\pi^{\frac{d+2}{2}}(\Delta-1)}{\Gamma(\Delta-\frac{d-2}{2})\Gamma(\Delta+1)}. \tag{53}$$

When $\Delta > d-1$ (which is the unitarity bound for $J=1$), both factors are positive. For $\Delta = d-1$ we find

$$\mathcal{A}_0(\Delta,1) = 0, \qquad \mathcal{A}_1(\Delta,1) = \frac{2^{3-d}\pi^{\frac{d+2}{2}}(d-2)}{\Gamma(\frac{d}{2})\Gamma(d)}. \tag{54}$$

This is consistent with the fact that spin-1 operators with $\Delta = d-1$ are conserved currents, i.e. they transform in a short multiplet. The condition $\mathcal{A}_0 = 0$ simply says that the scalar $s=0$ component vanishes,

$$p_\mu\mathcal{O}^\mu(p) = 0. \tag{55}$$

In position space this is the conservation equation

$$\partial_\mu\mathcal{O}^\mu(x) = 0. \tag{56}$$

This pattern persists for higher-spin operators: at the unitarity bound $\Delta = J+d-2$ only the $s=J$ component of the operator survives, i.e. only $\mathcal{A}_J(\Delta,J)$ is non-zero.

4. The embedding formalism and the Lorentzian cylinder

To continue Euclidean correlators to Lorentzian signature, we choose a Euclidean time direction τ and analytically continue $\tau = \epsilon + it$, staying inside the region of holomorphicity. The choice of τ is equivalent to a choice of Hamiltonian. In CFT, we have the freedom to choose other generators of the conformal algebra as Hamiltonians. A particularly nice choice is the Luscher-Mack Hamiltonian $\frac{1}{2}(P^0 - K^0)$, which Slava described in his lectures.[1] This is equal to Q_ξ, where ξ is the conformal Killing vector

$$\xi = \frac{1}{2}(p^0 - k^0) = \frac{1}{2}(1 + x^2)\partial^0 - x^0(x \cdot \partial). \tag{57}$$

To understand what happens when we continue with the Luscher-Mack Hamiltonian, it is helpful to use the embedding space formalism.

4.1. *Euclidean embedding space*

The idea of the embedding formalism is to embed spacetime in a space where the conformal group acts linearly. The Euclidean conformal group is $\mathrm{SO}(d+1, 1)$, and it acts linearly on the Euclidean embedding space $\mathbb{R}^{d+1,1}$. On $\mathbb{R}^{d+1,1}$, we choose coordinates

$$X = (X^+, X^-, X^\mu) \tag{58}$$

and metric

$$X^2 = -X^+X^- + \sum_{\mu=0}^{d-1}(X^\mu)^2. \tag{59}$$

The conformal compactification of $\mathbb{R}^d$, namely S^d, is equivalent to the projective null cone

$$S^d \cong \frac{\{X \in \mathbb{R}^{d+1,1} \text{ such that } X^2 = 0\}}{X \sim \lambda X \text{ where } \lambda \in \mathbb{R}}. \tag{60}$$

The denominator means that rescaling of X is a gauge redundancy. We can identify a copy of $\mathbb{R}^d$ by making the gauge choice $X^+ = 1$, called the Poincare section. A null vector on the Poincare section has the form

$$X(x) = (1, x^2, x), \qquad x \in \mathbb{R}^d. \tag{61}$$

[1] Our convention for the sign of K is from [], which is different from the one in the Luscher-Mack paper.

The induced metric on the Poincare section is

$$dX(x) \cdot dX(x) = \sum_{\mu=0}^{d-1} (dx^\mu)^2 = ds^2_{\mathbb{R}^d}, \tag{62}$$

which is the usual Euclidean metric on $\mathbb{R}^d$. The Poincare section misses one point on the projective null cone, which up to rescaling is $X(\infty) = (0, 1, 0)$.

The action of the conformal group on $\mathbb{R}^d$ arises as follows. We start with $X(x)$ and act with an element of the Euclidean conformal group $g \in \mathrm{SO}(d+1, 1)$ as a $(d+2) \times (d+2)$ matrix. This can take us off the Poincare section, so we rescale by $1/(gX(x))^+$ to get back to the Poincare section

$$X(x) \to gX(x) \to gX(x)/(gX(x))^+ = X(x'). \tag{63}$$

Overall this gives a nonlinear map $x \mapsto x'$ that implements g.

Different gauge fixings of the redundancy $X \sim \lambda X$ correspond to different conformal classes of metric on $\mathbb{R}^d$, which we call "conformal frames." Consider the gauge $X^+ = \Omega(x)$. The induced metric is

$$d(\Omega(x)X(x)) \cdot d(\Omega(x)X(x)) = \Omega(x)^2 ds^2_{\mathbb{R}^d}, \tag{64}$$

where we used that $X \cdot X = 0$ and $X \cdot dX = 0$. Correlation functions in different conformal frames are related by

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{ds^2} = \prod_i \Omega(x_i)^{\Delta_i} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\Omega^2 ds^2}. \tag{65}$$

4.2. *Continuation to Lorentzian signature*

Henceforth, we use a subscript "E" to denote Euclidean coordinates, for example $x_E \in \mathbb{R}^d$ and $X_E \in \mathbb{R}^{d+1,1}$. Consider analytically continuing $x_E = (x_E^0, \mathbf{x})$ to Lorentzian signature

$$x_E^0 \to i x_L^0, \tag{66}$$

Let us define a Lorentzian embedding space vector $X_L \in \mathbb{R}^{d,2}$ by

$$\begin{aligned} X_E^0 &= i X_L^0, \\ X_E^j &= X_L^j \quad (j = +, -, 1, \ldots, d-1). \end{aligned} \tag{67}$$

The metric on the Lorentzian embedding space $\mathbb{R}^{d,2}$ is

$$X_L^2 = -X_L^+ X_L^- - (X_L^0)^2 + \sum_{i=1}^{d-1} (X_L^i)^2. \tag{68}$$

After the continuation (66), X_L lies on a Lorentzian version of the Poincare section

$$X_L = (1, x_L^2, x_L) \in \mathbb{R}^{d,2}, \tag{69}$$

where $x_L = (x_L^0, \mathbf{x}) \in \mathbb{R}^{d-1,1}$.

We would like to show that by analytically continuing in a different time coordinate, we can reach a larger region than just Minkowski space. Let us start with a different Euclidean conformal frame, which we parametrize as

$$\begin{aligned} X_E(\eta, \vec{n}) = (X_E^+, X_E^-, X_E^0, X_E^i) &= \left(\cosh\eta + n^d, \cosh\eta - n^d, \sinh\eta, n^i\right), \\ \eta &\in \mathbb{R}, \\ \vec{n} &= (n^1, \ldots, n^d) \in S^{d-1}. \end{aligned} \tag{70}$$

Exercise 4.1. *Show that the induced metric in the frame (70) is*

$$dX_E(\eta, \vec{n}) \cdot dX_E(\eta, \vec{n}) = d\eta^2 + d\Omega_{d-1}^2, \tag{71}$$

i.e. it is the metric on a Euclidean cylinder $\mathbb{R} \times S^{d-1}$. Find the relationship between $\eta, \vec{n}$ and x_E, and verify that the cylinder metric is a Weyl rescaling of the flat metric. Show that $\eta = \pm\infty$ corresponds to $x_E = (\pm 1, 0, \ldots, 0)$.

In our new frame, η becomes radial time for radial quantization between the points $x_E = (\pm 1, 0, \ldots, 0)$.

The purpose of the conformal frame (70) is that it turns the conformal Killing vector $\xi = \frac{1}{2}(p^0 - k^0)$ into a regular Killing vector, i.e. the isometry ∂_η. This makes it much easier to understand the consequences of analytic continuation. The Hamiltonian corresponding to ∂_η is precisely the Luscher-Mack Hamiltonian Q_ξ. We also see that Q_ξ has the same spectrum as the dilatation operator. In particular, it is positive and bounded from below, so our analytic continuation machinery applies.

Let us understand where we end up when we continue $\eta \to i\sigma$. Making the replacement $\eta = i\sigma$ in (70) and using (67), we find

$$X_L(\sigma, \vec{n}) = (X_L^+, X_L^-, X_L^0, X_L^i) = \left(\cos\sigma + n^d, \cos\sigma - n^d, \sin\sigma, n^i\right). \tag{72}$$

This parametrizes $S^1 \times S^{d-1}$ inside $\mathbb{R}^{d,2}$, with induced metric

$$ds^2 = -d\sigma^2 + d\Omega_{d-1}^2. \tag{73}$$

$S^1 \times S^{d-1}$ is another gauge slice of the projective null cone in $\mathbb{R}^{d,2}$.

However, $S^1 \times S^{d-1}$ is not a good spacetime for a QFT because it contains closed timelike curves. Relatedly, the spectrum of the Luscher-Mack Hamiltonian is the set of dimensions Δ of the CFT. Time evolution inserts phases $e^{-i\sigma\Delta}$ into correlation functions, which are not necessarily periodic under $\sigma \to \sigma + 2\pi$. Thus, Wightman functions obtained by continuing $\eta \to i\sigma$ do not live on $S^1 \times S^{d-1}$, but instead on its universal cover $\widetilde{\mathcal{M}}_d = \mathbb{R} \times S^{d-1}$, called the Lorentzian cylinder. We have arrived at

Theorem 1 (Luscher and Mack). *Correlation functions in a unitary (reflection positive) CFT can be analytically continued to Wightman distributions on the Lorentzian cylinder* $\widetilde{\mathcal{M}}_d$.

We can find the relationship between the Minkowski coordinate x_L and $(\sigma, \vec{n})$ by rescaling $X_L(\sigma, \vec{n})$ to the Poincare section

$$(1, x_L^2, x_L^0, x_L^i) = \left(1, \frac{\cos\sigma - n^d}{\cos\sigma + n^d}, \frac{\sin\sigma}{\cos\sigma + n^d}, \frac{n^i}{\cos\sigma + n^d}\right). \tag{74}$$

Exercise 4.2. *Show that that* $x_L \in \mathbb{R}^{d-1,1}$ *covers the region* $\cos\sigma + n^d > 0$ *and* $\pi < \sigma < \pi$. *These are the points spacelike separated from* $\sigma = 0, \vec{n} = (0, \ldots, 0, -1)$, *which is spatial infinity in* $\mathbb{R}^{d-1,1}$.

We refer to this region as the (first) Minkowski patch $\mathcal{M}_d \subset \widetilde{\mathcal{M}}_d$.

The full Lorentzian cylinder $\widetilde{\mathcal{M}}_d$ is tiled by an infinite number of Minkowski patches (figure 1).[m] Luscher and Mack tell us that there is an entire world to explore beyond Minkowski space! Using (65), the dictionary between Wightman functions on $\mathcal{M}_d$ and $\widetilde{\mathcal{M}}_d$ (in their natural metrics) is

$$\begin{aligned} &\langle\Omega|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|\Omega\rangle_{\mathcal{M}_d} \\ &= \prod_{i=1}^{n}(\cos\sigma_i + n_i^d)^{\Delta_i}\langle\Omega|\mathcal{O}_1(\sigma_1, \vec{n}_1)\cdots\mathcal{O}_n(\sigma_n, \vec{n}_n)|\Omega\rangle_{\widetilde{\mathcal{M}}_d}. \end{aligned} \tag{75}$$

The Lorentzian conformal group is $\widetilde{\mathrm{SO}}(d,2)$: the universal cover of $\mathrm{SO}(d,2)$. Taking the universal cover is necessary because the transformation $\sigma \to \sigma + 2\pi$ is the identity element in $\mathrm{SO}(d,2)$. It is a 2π rotation in the $\mathrm{SO}(2)$ part of the maximal compact subgroup $\mathrm{SO}(d) \times \mathrm{SO}(2) \subset \mathrm{SO}(d,2)$. This transformation becomes nontrivial in $\widetilde{\mathrm{SO}}(d,2)$.

To every point $p \in \widetilde{\mathcal{M}}_d$, there is an associated point $\mathcal{T}p$ obtained by shooting light rays in all future directions from p and finding the point

[m] The embedding space vector X_L does not completely specify a point on $\widetilde{\mathcal{M}}_d$ because it only tells us the patch mod 2.

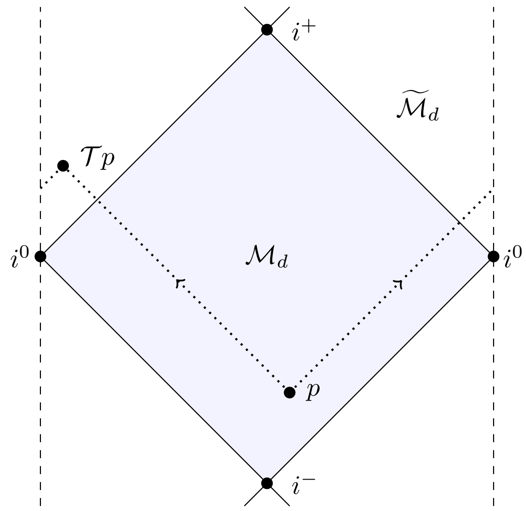


Fig. 1.: Minkowski patch $\mathcal{M}_d$ (blue, shaded) inside the Lorentzian cylinder $\widetilde{\mathcal{M}}_d$ in the case of 2 dimensions. Spacelike infinity of $\mathcal{M}_d$ is marked by i^0 and future/past infinity are marked by $i^{\pm}$. The dashed lines should be identified. The point $\mathcal{T}p$ is obtained from p by shooting light-rays in all possible future directions (dotted lines) and finding the first point where they converge.

where they converge in the next patch. In embedding coordinates, $\mathcal{T}$ takes $X_L \to -X_L$. For example, $\mathcal{T}$ takes spatial infinity i^0 with embedding coordinate $X_{i^0} = (0, 1, 0)$ to future infinity i^+ with embedding coordinate $X_{i^+} = (0, -1, 0)$. We sometimes write $p^+ \equiv \mathcal{T}p$ and $p^- \equiv \mathcal{T}^{-1}p$.

5. Null-integrated operators

Let us now describe some observables that are difficult to make sense of in Euclidean signature, but have many interesting physical applications. Consider the integral of a local operator along a null line

$$\int_{-\infty}^{\infty} dv\, \mathcal{O}_{v\cdots v}(u = 0, v, \vec{y}), \tag{76}$$

where we use lightcone coordinates

$$\begin{gathered} ds^2 = -du\, dv + d\vec{y}^2, \\ v = x^0 + x^1, \quad u = x^0 - x^1, \quad \vec{y} \in \mathbb{R}^{d-2}. \end{gathered} \tag{77}$$

Null-integrated operators have applications in collider physics, information theory, and holography, as well as consequences for the conformal bootstrap. When $\mathcal{O}_{\mu\nu} = T_{\mu\nu}$, (76) becomes the average null energy operator, which we discuss in section 6.

5.1. *Index-free notation*

Firstly, we should understand how (76) transforms under conformal transformations. Let us introduce some technology that makes it easy to understand the transformation properties of (76).

To describe operators with spin, it is helpful to use index-free notation. Given a traceless symmetric tensor $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$, we can contract its indices with a future-pointing null "polarization" vector z_μ to form

$$\mathcal{O}(x,z) \equiv \mathcal{O}^{\mu_1\cdots\mu_J}(x) z_{\mu_1}\cdots z_{\mu_J}. \tag{78}$$

When $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$ is an integer-spin local operator, $\mathcal{O}(x,z)$ is a homogeneous polynomial of degree J.

When J is an integer, we can easily go back from $\mathcal{O}(x,z)$ to the underlying tensor $\mathcal{O}^{\mu_1\cdots\mu_J}(x)$. We simply strip off the factors of z. When we do so, we have ambiguities proportional to $\eta_{\mu\nu}$ because z is null. However, these ambiguities are fixed by demanding that $\mathcal{O}$ is traceless.

Exercise 5.1. *Suppose that $f^{\mu\nu}$ is traceless and symmetric, and satisfies*

$$f^{\mu\nu} z_\mu z_\nu = (v\cdot z)^2 \tag{79}$$

for all null vectors z. Show that

$$f^{\mu\nu} = v^\mu v^\nu - \frac{1}{d} v^2 \eta^{\mu\nu}. \tag{80}$$

Note that $\mathcal{O}(x,z)$ is by construction a polynomial in z satisfying the homogeneity condition

$$\mathcal{O}(x,\lambda z) = \lambda^J \mathcal{O}(x,z). \tag{81}$$

Furthermore, if $\mathcal{O}$ transforms like a primary operator, then $\mathcal{O}(x,z)$ has a particular transformation law under conformal transformations:

$$\begin{aligned} \Longrightarrow \qquad & U_g \mathcal{O}(x,z) U_g^{-1} = \Omega(x')^\Delta \mathcal{O}(x', R(x')z), \\ \text{where} \qquad & \frac{\partial x'}{\partial x} = \Omega(x') R^\mu{}_\nu(x'), \qquad g \in \mathrm{SO}(d,2). \end{aligned} \tag{82}$$

With index-free notation, it is easy to describe a primary with non-integer spin. We simply drop the requirement that $\mathbb{O}$ be a polynomial in z, and require only that it be a homogeneous function on the null cone

$$\mathbb{O}(x, \lambda z) = \lambda^J \mathbb{O}(x, z), \qquad \lambda \in \mathbb{R}_+, \tag{83}$$

where $J \in \mathbb{C}$ is not necessarily an integer. It is not yet obvious why we should consider such an object, but that will hopefully become clear shortly.

5.2. *Lifting to the embedding space*

In the embedding formalism, the operator $\mathcal{O}(x, z)$ gets lifted to a homogeneous function $\mathcal{O}(X, Z)$ of coordinates $X, Z \in \mathbb{R}^{d,2}$, subject to the relations $X^2 = X \cdot Z = Z^2 = 0$. It is defined by

$$\mathcal{O}(X, Z) = (X^+)^{-\Delta} \mathcal{O}\left(x^\mu = \frac{X^\mu}{X^+}, z = Z^\mu - \frac{Z^+}{X^+} X^\mu\right), \tag{84}$$

$$\mathcal{O}(x, z) = \mathcal{O}\left(X = (1, x^2, x), Z = (0, 2x \cdot z, z)\right). \tag{85}$$

where Δ is the dimension of $\mathcal{O}$. Note that $\mathcal{O}(X, Z)$ satisfies

$$\mathcal{O}(X, Z) = \mathcal{O}(X, Z + \beta X) \qquad \text{gauge invariance}, \tag{86}$$

$$\mathcal{O}(\lambda X, \alpha Z) = \lambda^{-\Delta} \alpha^J \mathcal{O}(X, Z) \qquad \text{homogeneity}. \tag{87}$$

A vector $Z \in \mathbb{R}^{d,2}$ has $d + 2$ degrees of freedom. The transverseness condition $X \cdot Z = 0$ and the gauge redundancy $Z \sim Z + \beta X$ reduce the number of degrees of freedom by 2, which is the correct number to describe the polarization vector $z \in \mathbb{R}^{d-1,1}$. Finally, the condition $Z^2 = 0$ is the embedding-space version of the condition $z^2 = 0$.

The advantage of lifting to the embedding space is that conformal transformations act linearly on the coordinates. Specifically, by combining (82) and (84), one can show

$$U_g \mathcal{O}(X, Z) U_g^{-1} = \mathcal{O}(gX, gZ) \qquad g \in \mathrm{SO}(d, 2). \tag{88}$$

5.3. *The light transform*

Null integrated operators like (76) can be understood in terms of a conformally-invariant integral transform called the “light-transform.” In embedding space language, the light-transform is defined by

$$\mathbf{L}[\mathcal{O}](X, Z) \equiv \int_{-\infty}^{\infty} d\alpha\, \mathcal{O}(Z - \alpha X, -X). \tag{89}$$

It is invariant under $\widetilde{\mathrm{SO}}(d,2)$ because (89) only depends on the embedding-space vectors X, Z, and furthermore its arguments $X' = Z - \alpha X, Z' = -X$ satisfy the conditions $X'^2 = X' \cdot Z' = Z'^2 = 0$. It respects the gauge-redundancy (86) because a shift $Z \to Z + \beta X$ can be compensated by shifting $\alpha \to \alpha + \beta$ in the integral. The initial point of the integration contour in (89) is X, since $Z - (-\infty)X$ is projectively equivalent to X. The final point is $\mathcal{T}X = -X$.

Furthermore, if $\mathcal{O}(X,Z)$ has homogeneity (87), then the light-transform has homogeneity

$$\mathbf{L}[\mathcal{O}](\lambda X, \alpha Z) = \lambda^{-(1-J)}\alpha^{1-\Delta}\mathbf{L}[\mathcal{O}](X,Z). \tag{90}$$

Thus, $\mathbf{L}[\mathcal{O}]$ transforms like a primary at X with dimension $1-J$ and spin $1-\Delta$:

$$\mathbf{L} : (\Delta, J) \to (1-J, 1-\Delta). \tag{91}$$

In order for $\mathbf{L}$ to be conformally-invariant, it must preserve the Casimirs of the conformal group

$$\begin{aligned} C_2(\Delta,J) &= \Delta(\Delta-d) + J(J+d-2) \\ C_4(\Delta,J) &= (\Delta-1)(d-\Delta-1)J(2-d-J). \end{aligned} \tag{92}$$

Indeed, one can check that it does. Transformations that preserve the Casimirs are called *affine Weyl reflections*. The light-transform is part of a dihedral group D_8 worth of Lorentzian integral transforms that preserve the conformal Casimirs.[n] Note that the light-transform naturally gives rise to operators with non-integer spin.

In Minkowski coordinates, $\mathbf{L}$ becomes

$$\begin{aligned} \mathbf{L}[\mathcal{O}](x,z) &= \int_{-\infty}^{\infty} d\alpha\, \mathcal{O}(Z-\alpha X, -X)\Big|_{\substack{X=(1,x^2,x)\\ Z=(0,2x\cdot z,z)}} \\ &= \int_{-\infty}^{\infty} d\alpha\, (-\alpha)^{-\Delta-J}\mathcal{O}\left(X - \frac{Z}{\alpha}, Z\right)\Big|_{\substack{X=(1,x^2,x)\\ Z=(0,2x\cdot z,z)}} \\ &= \int_{-\infty}^{\infty} d\alpha\, (-\alpha)^{-\Delta-J}\mathcal{O}\left(x - \frac{z}{\alpha}, z\right). \end{aligned} \tag{93}$$

The integration contour in (93) starts at x when $\alpha = -\infty$ and reaches the boundary of Minkowski space when $\alpha = 0$. The correct prescription there is to continue the contour into the next Minkowski patch to the point $\mathcal{T}x \in \widetilde{\mathcal{M}}_d$. The expression (93) makes it clear that $\mathbf{L}[\mathcal{O}]$ converges whenever

[n] Only one of them — the "shadow transform" — also makes sense in Euclidean space.

$\Delta + J > 1$, as long as there are no other operators at x or $\mathcal{T}x$. Note that $\mathbf{L}[\mathcal{O}](x, z)$ is not a polynomial in z and thus cannot be written in terms of an underlying tensor with $1 - \Delta$ indices.

Let us finally make contact with the null-integrated operator (76) we described before. We can choose

$$\begin{aligned} X_0 &= -(0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \vec{0}), \\ Z_0 &= (1, \vec{y}^2, 0, 0, \vec{y}), \end{aligned} \tag{94}$$

where $\vec{0}, \vec{y} \in \mathbb{R}^{d-2}$. Note that these satisfy the conditions $X_0^2 = X_0 \cdot Z_0 = Z_0^2 = 0$. The light-transform becomes

$$\mathbf{L}[\mathcal{O}](X_0, Z_0) = \int_{-\infty}^{\infty} d\alpha \, \mathcal{O}_{v\cdots v}(u = 0, v = \alpha, \vec{y}), \tag{95}$$

Specifically, we learn that the null integral of the stress tensor $\mathbf{L}[T](X_0, Z_0)$ transforms like a primary with dimension -1 and spin $1 - d$, located at the point X_0 at past null infinity.

5.4. *Annihilating the vacuum*

Lemma 1. *For any local operator $\mathcal{O}$ satisfying $\Delta + J > 1$, the light-transform $\mathbf{L}[\mathcal{O}]$ annihilates the vacuum $|\Omega\rangle$.*

Consider a Wightman function $\langle\Omega|V_1 \cdots V_n \mathcal{O}(x, z)|\Omega\rangle$. It is convenient to pick coordinates (94), so that the light transform becomes

$$\langle\Omega|V_1 \cdots V_n \mathbf{L}[\mathcal{O}]|\Omega\rangle = \int_{-\infty}^{\infty} dv \, \langle\Omega|V_1(x_1) \cdots V_n(x_n)\mathcal{O}(u = 0, v, \vec{y})|\Omega\rangle. \tag{96}$$

Recall that positivity of energy guarantees that the correlator is holomorphic in the region

$$\operatorname{Im} x_1 < \operatorname{Im} x_2 < \cdots < \operatorname{Im} x_n < (0, \operatorname{Im} v, \vec{0}) \tag{97}$$

In order to define the Wightman correlator in the first place, the $\operatorname{Im} x_k$ must be chosen to satisfy the above holmorphicity conditions when v is real. Thus, if we move v into the upper half plane, the correlator will remain analytic. One can additionally argue that the correlator dies like $v^{-\Delta-J}$ at large v. Thus, if $\Delta + J > 1$, we can deform the contour into the upper half-plane to get zero. If all Wightman functions (96) vanish, the state $\mathbf{L}[\mathcal{O}]|\Omega\rangle$ vanishes.

Conformal invariance implies that the light-transform of a Wightman three-point function just exchanges the quantum numbers according to (91)

$$\langle 0|\phi_1 \mathbf{L}[\mathcal{O}_{\Delta,J}]\phi_2|0\rangle \propto \langle 0|\phi_1 \mathcal{O}_{1-J,1-\Delta}\phi_2|0\rangle \tag{98}$$

Here, we use the notation that a three-point function in the fictitious state $|0\rangle$ represents the unique conformally invariant structure for the given representations.

Exercise 5.2. *Compute the coefficient in (98).*

6. The average null energy condition

The average null energy condition (ANEC) says that the average null energy operator $\mathbf{L}[T]$ is positive-definite. There are two recent proofs of the ANEC — one using information theory, and another using causality. We will sketch a version of the causality proof in section 7.6 if there's time.

6.1. *Conformal collider physics*

One justification for positivity of the average null energy operator in CFT comes from a thought experiment due to Hofman and Maldacena. Imagine a "conformal collider" event in the middle of Minkowski space. This could be the insertion of some operator that creates a nontrivial state. To measure the event, we place a calorimeter (a stress tensor insertion) at a faraway position $\vec{x} = r\vec{n}$ where $\vec{n} \in S^{d-2}$ and integrate the energy deposited in the calorimeter over time. We expect the energy measured in the calorimeter to be positive.

More precisely, the calorimeter measures the radial flux of energy along the $\vec{n}$ direction,

$$\mathcal{E}(n) = \lim_{r\to\infty} r^{d-2} \int_0^\infty dt\, n^i T^0{}_i(t, r\vec{n}). \tag{99}$$

The factor r^{d-2} is necessary to obtain a finite result as $r \to \infty$. We claim that the expectation value

$$\langle \Psi|\mathcal{E}(n)|\Psi\rangle \tag{100}$$

is positive for any state $|\Psi\rangle$.

Let us see why $\mathcal{E}(n)$ is related to $\mathbf{L}[T]$.

Exercise 6.1. *Let $z^\mu = (1, -\vec{n})$ and $\overline{z}^\mu = (1, \vec{n})$ be null vectors. Show that*

$$n^i T^0{}_i = \frac{1}{4}(z^\mu z^\nu - \overline{z}^\mu \overline{z}^\nu)T_{\mu\nu}. \tag{101}$$

The $\overline{z}^\mu$ term will drop out when we take $r \to \infty$, so let us focus on the contribution proportional to z. In embedding-space language, we have

$$\mathcal{E}(n) = \frac{1}{4} \lim_{r\to\infty} r^{d-2} \int_0^\infty dt\, T(X, Z) + \overline{z} \text{ term}, \tag{102}$$

where

$$\begin{aligned} X &= (1, x^2, x) = (1, r^2 - t^2, t, r\vec{n}), \\ Z &= (0, 2x \cdot z, z) = (0, -2(r+t), 1, -\vec{n}). \end{aligned} \tag{103}$$

If we take the limit $r \to \infty$ at fixed t, we end up at the point at spatial infinity. However, this clearly doesn't capture the whole contribution of the integral. We should change variables to retarded time $\alpha = t - r$. At large r, we have

$$\begin{aligned} X &= (1, r^2 - (r+\alpha)^2, r+\alpha, r\vec{n}) \sim r(0, -2\alpha, 1, \vec{n}) = r(Z_\infty - 2\alpha X_\infty) \\ Z &= (0, -2(2r+\alpha), 1, -\vec{n}) \sim -4r(0, 1, 0, 0) = -4rX_\infty, \end{aligned} \tag{104}$$

where

$$\begin{aligned} X_\infty &= (0, 1, 0), \\ Z_\infty &= (0, 0, z). \end{aligned} \tag{105}$$

Here, X_∞ is the point at spatial infinity.

Exercise 6.2. *Show that* $\overline{Z} = (0, 2x \cdot \overline{z}, \overline{z})$ *is lower-order in* r *than* Z*, and thus does not contribute in the limit* $r \to \infty$.

Using the homogeneity of $T(X, Z)$ in X and Z (87), we find

$$\begin{aligned} \mathcal{E}(n) &= \frac{1}{4} \lim_{r\to\infty} r^{d-2} \int_{-r}^\infty d\alpha\, r^{-d} (4r)^2 T(Z_\infty - 2\alpha X_\infty, -X_\infty) \\ &= 2 \int_{-\infty}^\infty d\alpha\, T(Z_\infty - \alpha X_\infty, -X_\infty) \\ &= 2\mathbf{L}[T](X_\infty, Z_\infty) \\ &\equiv 2\mathbf{L}[T](\infty, z) \end{aligned} \tag{106}$$

where in the second line we redefined $2\alpha \to \alpha$.

6.2. *Hofman-Maldacena bounds*

Thus, $2\mathbf{L}[T](\infty, z)$ represents the energy flux from an event, measured at null infinity at the point $\vec{n}$ on the celestial sphere. We see that it transforms like a primary operator at spatial infinity ∞ with dimension $1-2=-1$ and spin $1-d$. This is useful for constraining its matrix elements.

In particular, consider a three-point function

$$\langle\Omega|\mathcal{O}_i(x_2)\mathbf{L}[T](\infty,z)\mathcal{O}_j(x_1)|\Omega\rangle. \tag{107}$$

Note that a primary operator at infinity is killed by translation generators

$$[P^\mu,\mathbf{L}[T](\infty,z)]=0. \tag{108}$$

Thus, the above three-point function is translationally invariant

$$\langle\Omega|\mathcal{O}_i(x_2)\mathbf{L}[T](\infty,z)\mathcal{O}_j(x_1)|\Omega\rangle = K_{ij}(x_2-x_1). \tag{109}$$

To study positivity of $\mathbf{L}[T]$, we can use the same methods as in the Lorentzian proof of the unitarity bounds in section 3. Positive definiteness of the ANEC operator is equivalent to positive-definiteness of the translationally-invariant integral kernel $K_{ij}(x_2-x_1)$. The kernel can be partially diagonalized by going to Fourier space

$$\widetilde{K}_{ij}(p)=\int d^dx K_{ij}(x)e^{-ip\cdot x}\equiv\langle\mathcal{O}_i(p)|\mathbf{L}[T](\infty,z)|\mathcal{O}_j(p)\rangle. \tag{110}$$

Thus, positive-definiteness of $\mathbf{L}[T]$ is equivalent to positive-definiteness of $\widetilde{K}_{ij}(p)$ as a matrix-valued tempered distribution.

The indices i,j run over all conformal multiplets of the theory, and all spin states within each conformal multiplet. Thus, it is difficult to study sufficient conditions for positivity, as this involves an infinite-dimensional matrix. However, we can derive necessary conditions by studying simple submatrices.

An important example is when $\mathcal{O}_i,\mathcal{O}_j$ are themselves stress tensors. We must demand positivity of the matrix

$$\widetilde{K}_{\mu\nu,\rho\sigma}(p)=\langle T_{\mu\nu}(p)|\mathbf{L}[T](\infty,z)|T_{\rho\sigma}(p)\rangle \tag{111}$$

Note that $\widetilde{K}$ has support inside the future lightcone of p, by positivity of energy. We can find a Lorentz transformation that sets $p=(p^0,0,\ldots,0)$. By homogeneity in z, we can then rescale z to have the form $(1,-\vec{n})$. A primary operator with dimension Δ, placed at infinity has mass dimension $-\Delta$.[o] In particular $\mathbf{L}[T](\infty,z)$ has mass dimension 1, which is appropriate because it measures energy. Thus, $\widetilde{K}$ has mass dimension $2d-d+1=d+1$. This allows us to set $p^0=1$ and restore the correct powers of p^0 later. Finally, since $p^\mu T_{\mu\nu}(p)=0$, $\widetilde{K}$ is only nonzero when it has purely spatial indices.

[o] **Explain why.**

Making these specializations $p = (1, 0, \ldots, 0)$, $z = (1, -\vec{n})$ and restricting to spatial indices, the possible tensor structures that can appear are[p]

$$\begin{aligned}\widetilde{K}_{ij,kl} \propto \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{t_2}{4}\left(\delta_{ik}P_{jl} + \delta_{il}P_{jk} + \delta_{jl}P_{ik} + \delta_{jk}P_{il}\right) \\ + t_4 P_{ijkl} - \text{traces in } ij \text{ and } kl, \end{aligned} \tag{112}$$

where

$$\begin{aligned} P_{ij} &= n_i n_j - \frac{\delta_{ij}}{d-1}, \\ P_{ijkl} &= n_i n_j n_k n_l - \frac{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{(d+1)(d-1)}. \end{aligned} \tag{113}$$

The presence of three different tensor structures reflects the fact that there are three conformally-invariant, conserved structures for three-point functions of stress tensors $\langle T_{\mu\nu} T_{\rho\sigma} T_{\alpha\beta}\rangle$. The coefficient of one of the structures is fixed by Ward identities. The remaining unfixed OPE coefficients can be taken to be t_2 and t_4. The precise relationship between t_2, t_4 and the coefficients of your favorite position-space structures must be worked out by explicitly taking the light transform and then the Fourier transform.

Finally, we demand that $\widetilde{K}_{ij,kl}$ is a positive-definite bilinear form, and this gives bounds on t_2 and t_4. These are the famous Hofman-Maldacena bounds. In 4 dimensions, they are

$$\begin{aligned} 1 - \frac{t_2}{3} - \frac{2t_4}{15} &\geq 0 \\ 2\left(1 - \frac{t_2}{3} - \frac{2t_4}{15}\right) + t_2 &\geq 0 \\ \frac{3}{2}\left(1 - \frac{t_2}{3} - \frac{2t_4}{15}\right) + t_2 + t_4 &\geq 0. \end{aligned} \tag{114}$$

The coefficients t_2 and t_4 are related to the anomaly coefficients a and c. In particular, (114) imply

$$\frac{31}{18} \geq \frac{a}{c} \geq \frac{1}{3}. \tag{115}$$

By studying other conformal multiplets $\mathcal{O}_i, \mathcal{O}_j$, one can derive generalizations of the Hofman-Maldacena bounds. One finds more constraints on OPE coefficients, and sometimes also improvements on unitarity bounds for operator dimensions. The recent proofs of the ANEC have spurred new ongoing work in this area.

[p]The funny δ_{ij} terms in P_{ij} and P_{ijkl} are to give them nice tracelessness properties. You can think of the three allowed structures as essentially $\delta_{ik}\delta_{kl}$, $\delta_{ik}n_j n_l$, $n_i n_j n_k n_l$, and their symmetrizations.

7. Analyticity in spin

7.1. *The Regge limit*

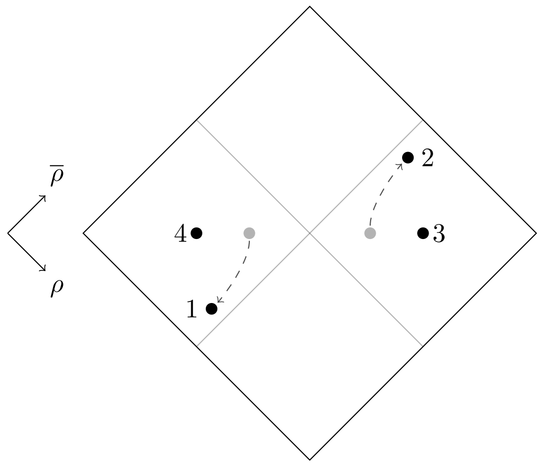


Fig. 2.: The Regge limit in the configuration (118). We boost points 1 and 2 while keeping points 3 and 4 fixed. This configuration is related by a boost to the one in figure 3.

The Regge limit of a four-point function is obtained by boosting two operators $\mathcal{O}_1, \mathcal{O}_2$ by a large amount relative to the other operators $\mathcal{O}_3, \mathcal{O}_4$ (figures 2 and 3). This gives a CFT version of a high-energy scattering process: operators $\mathcal{O}_1, \mathcal{O}_3$ create excitations that propagate from past null infinity, interact at high energies near the origin, and are eventually measured by $\mathcal{O}_2, \mathcal{O}_4$ at future null infinity. An important and nontrivial fact is that CFT four-point functions are bounded in the Regge limit.

For simplicity, let us specialize to identical scalars $\phi(x_i)$. The four-point function has a conformal block expansion

$$\begin{aligned}\langle\Omega|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle &= \frac{g(z,\overline{z})}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\\ g(z,\overline{z}) &= \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 G_{\Delta,J}(z,\overline{z}),\end{aligned}\tag{116}$$

where $G_{\Delta,J}$ are conformal blocks and $f_{\phi\phi\mathcal{O}}$ are OPE coefficients of operators appearing in the $\phi\times\phi$ OPE. Using conformal transformations, the four operators can be placed on a plane with lightcone coordinates $u = t-x, v =$

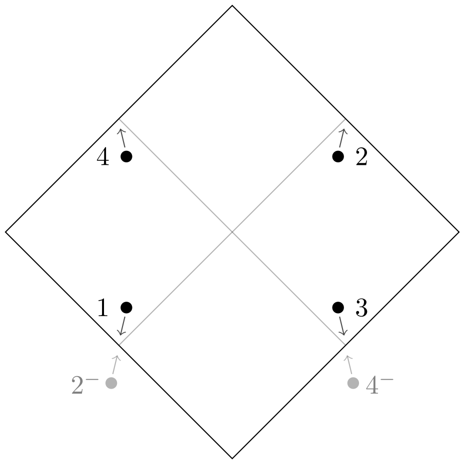


Fig. 3.: Another description of the Regge limit is $x_1 \to x_2^-$ and $x_3 \to x_4^-$. The points x_2^-, x_4^- are shown in gray. The cross-ratios $z, \bar{z}$ associated with the points $1, 2, 3, 4$ are the same as those associated with $1, 2^-, 3, 4^-$.

$t + x$. The cross-ratios are

$$z = \frac{u_{12}u_{34}}{u_{13}u_{24}}, \qquad \bar{z} = \frac{v_{12}v_{34}}{v_{13}v_{24}}. \tag{117}$$

Let us understand what happens to the cross ratios as we continue to the Regge limit.

We begin with the configuration

$$x_4 = (1, -1), \quad x_3 = (-1, 1), \quad x_2 = (-r, r), \quad x_1 = (r, -r) \tag{118}$$

in lightcone coordinates (u, v), with $0 < r < 1$, so that all four points are spacelike and the cross ratios satisy $0 < z, \bar{z} < 1$. We would then like to move x_1 into the past of x_4 and x_2 into the future of x_3 (figure 2). In doing so, we encounter branch cut singularities when x_2 becomes lightlike from x_3 and when x_1 becomes lightlike from x_4. The way we move around the branch cuts depends on the operator ordering.

Consider first the ordering

$$\langle \Omega | \phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3) | \Omega \rangle. \tag{119}$$

This is equal to the Lorentzian time-ordered correlator in the kinematics we're considering. To achieve this ordering, we should take

$$\begin{aligned} t_{23} &\to t_{23} - i\epsilon \\ t_{41} &\to t_{41} - i\epsilon \end{aligned} \tag{120}$$

with $\epsilon > 0$. This ensures that x_2 is later in Euclidean time ($\tau = it$) than x_3 and that x_1 is earlier in Euclidean time than x_4. Note that we don't have to specify the imaginary parts of the other t_{ij}'s because all other pairs of operators are spacelike separated and thus commute.

We can fix the u coordinates of the operators during our continuation, so we only have to keep track of the v coordinates. Our $i\epsilon$ prescription becomes

$$\begin{aligned} v_{23} &\to v_{23} - i\epsilon \\ v_{41} &\to v_{41} - i\epsilon. \end{aligned} \tag{121}$$

Exercise 7.1. *Show that after continuing x_2 to the future of x_3 and x_1 to the past of x_4, the different Wightman functions are given by*

$$\begin{aligned} x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle\Omega|\phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3)|\Omega\rangle &= g^{\circlearrowleft}(z,\overline{z}) \\ x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle\Omega|\phi(x_4)\phi(x_1)\phi(x_3)\phi(x_2)|\Omega\rangle &= g(z,\overline{z}) \\ x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle\Omega|\phi(x_1)\phi(x_4)\phi(x_2)\phi(x_3)|\Omega\rangle &= g(z,\overline{z}) \\ x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle\Omega|\phi(x_1)\phi(x_4)\phi(x_3)\phi(x_2)|\Omega\rangle &= g^{\circlearrowright}(z,\overline{z}). \end{aligned} \tag{122}$$

where $\circlearrowleft$ or $\circlearrowright$ means we continue $\overline{z}$ around 1 in the indicated direction and then back to the real axis.[q] *Argue that this is the complete set of cases we need to consider, since spacelike-separated operators commute.*

In the Regge limit, the cross ratios $z, \overline{z}$ approach zero. A nice way to see this is to note that cross-ratios are the same after acting with $\mathcal{T}$ on any of the points. By acting with $\mathcal{T}^{-1}$ on 2 and 4, we can transform the Regge limit into something that looks kinematically like an OPE limit (figure 3), though the physics is different if the cross-ratios undergo monodromy. More explicitly, we can apply a boost

$$(-u_1, -v_1) = (u_2, v_2) = (-e^{-\eta} r, e^{\eta} r). \tag{123}$$

Exercise 7.2. *Show that z and $\overline{z}$ both behave as $e^{-\eta}$ as $\eta \to \infty$.*

[q]The fact that the middle two orderings don't involve any monodromy can be understood by an old result of Mack. In both cases, we can rearrange the operators to get either $\phi(x_1)\phi(x_2)|\Omega\rangle$ or $\langle\Omega|\phi(x_1)\phi(x_2)$. Mack showed that the OPE converges whenever both operators act on the vacuum. Thus the 1×2 OPE is valid in both of these cases, which means that $\overline{z}$ never leaves the regime of convergence (i.e. it never crosses the cut) in these cases.

Thus, we see that the Wightman functions $\langle 4132\rangle$ and $\langle 1423\rangle$ are bounded in the Regge limit — for them we can simply use the leading term in the conformal block expansion $g(z,\overline{z}) \sim 1 +$ higher powers of $z, \overline{z}$.

Something more complicated happens to the other Wightman functions, for example $\langle\Omega|\phi(x_4)\phi(x_1)\phi(x_2)\phi(x_3)|\Omega\rangle$. Under the continuation $\circlearrowleft$, a spin-J conformal block mixes with another solution of the conformal Casimir equation,

$$G^{\circlearrowleft}_{\Delta,J}(z,\overline{z}) = (\cdots)\, G_{\Delta,J}(z,\overline{z}) + (\cdots)\, G_{1-J,1-\Delta}(z,\overline{z}), \tag{124}$$

where $(\cdots)$ are coefficients that we are not writing explicitly. Note that the quantum numbers in the other solution are the same as those appearing in the light-transform. The appearance of the light-transformed quantum numbers is bad news for the naive conformal block decomposition because

$$G_{1-J,1-\Delta}(z,\overline{z}) \sim (z\overline{z})^{\frac{1-J}{2}} \sim e^{(J-1)\eta} \qquad z,\overline{z} \ll 1, \tag{125}$$

so naively the sum over spins in (116) becomes badly divergent at large boost $\eta \to \infty$. The problem is that continuation to the Regge limit does not commute with the naive conformal block expansion.

7.2. *Rindler positivity*

Nevertheless, even the problematic correlators $g^{\circlearrowleft,\circlearrowright}(z,\overline{z})$ are bounded in the Regge limit. For a four-point function of identical scalars, one can argue this using the conformal block decomposition in a different channel. However, the general statement requires Rindler positivity.

Any Lorentz-invariant QFT has an antiunitary symmetry CRT satisfying $\mathsf{CRT}^2 = 1$ that acts on spacetime by reflecting time and one spatial direction

$$\mathsf{CRT} : (t, x^1, x^2, \ldots, x^{d-1}) \mapsto \overline{x} = (-t, -x^1, x^2, \ldots, x^{d-1}). \tag{126}$$

We define the "Rindler conjugate" of an operator $\mathcal{O}$ by

$$\overline{\mathcal{O}} = (\mathsf{CRT})\mathcal{O}(\mathsf{CRT}). \tag{127}$$

For traceless-symmetric tensors, it is given by

$$\overline{\mathcal{O}(x,z)} = \mathcal{O}^\dagger(\overline{x},\overline{z}). \tag{128}$$

Note that Rindler conjugation preserves operator ordering

$$\overline{AB} = (\mathsf{CRT})AB(\mathsf{CRT}) = (\mathsf{CRT})A(\mathsf{CRT})(\mathsf{CRT})B(\mathsf{CRT}) = \overline{A}\,\overline{B}. \tag{129}$$

The statement of Rindler positivity is that

$$\langle\Omega|\overline{A}A|\Omega\rangle \geq 0 \tag{130}$$

for all products of operators A localized in the right Rindler wedge, e.g. $A = \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$ with $x_1,\ldots,x_n$ in the right Rindler wedge. **Explain the proof?** Thus, we can define a positive-definite Hermitian inner product

$$(B,A) \equiv \langle\Omega|\overline{B}A|\Omega\rangle. \tag{131}$$

In particular, we have the Cauchy-Schwarz inequality

$$|(B,A)|^2 \leq (A,A)(B,B). \tag{132}$$

Let us return to our four-point function and denote $\phi(x_i) \equiv \phi_i$ for brevity. Consider a Rindler-symmetric configuration of the four points, such that $\phi_1 = \overline{\phi_2}$ and $\phi_3 = \overline{\phi_4}$. Note that such configurations still allow us to access arbitrary values of the cross-ratios $z, \overline{z}$. The Cauchy Schwarz inequality implies

$$\begin{aligned} |\langle\Omega|\phi_4\phi_1\phi_2\phi_3|\Omega\rangle|^2 &= |(\phi_3\phi_2, \phi_2\phi_3)|^2 \\ &\leq (\phi_2\phi_3, \phi_2\phi_3)(\phi_3\phi_2, \phi_3\phi_2) \\ &= \langle\Omega|\phi_1\phi_4\phi_2\phi_3|\Omega\rangle\langle\Omega|\phi_4\phi_1\phi_3\phi_2|\Omega\rangle. \end{aligned} \tag{133}$$

In particular, we find

$$|g^{\circlearrowleft}(z,\overline{z})|^2 \leq g(z,\overline{z})^2, \qquad 0 < z, \overline{z} < 1. \tag{134}$$

This shows that the Wightman correlator $\langle\Omega|\phi_4\phi_1\phi_2\phi_3|\Omega\rangle$ is bounded in the Regge limit. This argument is a version of the Cauchy-Schwarz argument that enters the proof of the chaos bound.

7.3. *A toy model*

How can we make sense of the fact that $g^{\circlearrowleft}(z,\overline{z})$ appears to have a divergent conformal block expansion in the Regge limit, but is actually bounded there? A nice toy model is to consider an "amplitude"

$$\mathcal{A}(w) = \sum_{J=0}^{\infty} a_J w^J. \tag{135}$$

Let us write $w = e^{\eta}$ and think of η as a boost parameter. Suppose that $\mathcal{A}(w)$ is analytic everywhere outside of a cut $w \in [1,\infty)$, and is also bounded in the Regge limit of large boost

$$\mathcal{A}(w) \leq 1, \quad w \to \infty. \tag{136}$$

This implies that there must be a delicate balance between the "partial waves" $a_J w^J$ for $J \geq 1$. If we wiggled any of the a_J a tiny amount, it would ruin the cancellation at $w \to \infty$ and violate Regge boundedness (136).

The delicate balance between a_J's can be expressed by a "Froissart-Gribov" formula. We start with an inversion formula for a_J,

$$a_J = \frac{1}{2\pi i} \oint \frac{dw}{w^{J+1}} \mathcal{A}(w). \tag{137}$$

We might call this a "Euclidean inversion formula" because it involves integrating over imaginary boosts, i.e. angles $\eta = i\theta$. Using the analyticity properties of $\mathcal{A}$, together with boundedness in the Regge limit, we can deform the contour to wrap around the cut, giving the Froissart-Gribov formula

$$\begin{aligned} a_J &= \frac{1}{2\pi i} \int_1^\infty \frac{dw}{w^{J+1}} \mathrm{Disc}[\mathcal{A}(w)] \qquad (J > 0), \\ \mathrm{Disc}[\mathcal{A}(w)] &\equiv \mathcal{A}(w + i\epsilon) - \mathcal{A}(w - i\epsilon). \end{aligned} \tag{138}$$

We can drop the arc at infinity provided $J > 0$. Thus, this formula does not apply to $J = 0$. We might call (138) a "Lorentzian inversion formula" because it involves an integral over real boosts. A remarkable feature of this formula is that it now makes sense for non-integer J. This shows that the coefficients a_J fit together into an analytic function of spin. This gives a way of understanding why there is a delicate balance between the a_J's.

Exercise 7.3. *Consider the example $\mathcal{A}(w) = \log(1 - w)$. Compute the coefficients a_J using (138) and compare them to the Taylor expansion around $w = 0$. What happens when $J = 0$?*

7.4. *The Lorentzian inversion formula*

For the four-point function to be bounded in the Regge limit, something similar must happen with its OPE coefficients as a function of spin J. Caron-Huot's Lorentzian inversion formula is the analog of (138) for CFT's. To write it, we must package the OPE coefficients into a single object. We write

$$g(z, \bar{z}) = \sum_{J=0}^{\infty} \int_{\frac{d}{2} - i\infty}^{\frac{d}{2} + i\infty} \frac{d\Delta}{2\pi i} C(\Delta, J) G_{\Delta, J}(z, \bar{z}) + \text{non-norm.}, \tag{139}$$

where $C(\Delta, J)$ has poles on the real Δ axis at the location of physical operators, with residues proportional to squared OPE coefficients

$$C(\Delta, J) \sim -\sum_k \frac{f^2_{\phi\phi\mathcal{O}_k}}{\Delta - \Delta_k}. \tag{140}$$

The terms "non-norm." are a finite sum of conformal blocks with dimensions less than $d/2$ (which includes the unit operator). When $z, \bar{z} \notin [1, \infty)$, the conformal block $G_{\Delta,J}(z, \bar{z})$ dies exponentially at large Δ in the right-half plane. In this region, we can deform the Δ contour to the right to wrap around the poles of $C(\Delta, J)$. This recovers the usual conformal block expansion. The function $C(\Delta, J)$ packages together the dynamical information in this four-point function, and we would like to compute it!

Caron-Huot's formula gives an expression for $C(\Delta, J)$ that involves an integral over Lorentzian kinematics.

$$\begin{aligned} C(\Delta, J) &= -\frac{(1 + (-1)^J)\kappa_{\Delta+J}}{8} \int_0^1 dz d\bar{z} \frac{|z - \bar{z}|^{d-2}}{(z\bar{z})^d} G_{J+d-1,\Delta-d+1}(z, \bar{z}) \\ &\qquad \times x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle \Omega | [\phi(x_4), \phi(x_1)][\phi(x_2), \phi(x_3)] | \Omega \rangle. \\ \kappa_{\Delta+J} &= \frac{\Gamma(\frac{\Delta+J}{2})^4}{2\pi^2 \Gamma(\Delta + J - 1)\Gamma(\Delta + J)} \end{aligned} \tag{141}$$

This formula holds for $\mathrm{Re}(J) \geq 1$, for similar reasons to why (138) was only valid for $J > 0$.

The quantity on the second line is a double-commutator that should be evaluated in a configuration where $4 > 1$ and $2 > 3$, with other pairs of points being spacelike separated, and such that the cross-ratios associated to the points are $z, \bar{z}$. Using (122), we have

$$\begin{aligned} x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle \Omega | [\phi(x_4), \phi(x_1)][\phi(x_2), \phi(x_3)] | \Omega \rangle &= g^{\circlearrowright}(z, \bar{z}) + g^{\circlearrowleft}(z, \bar{z}) - 2g(z, \bar{z}) \\ &\equiv -2\mathrm{dDisc}[g(z, \bar{z})] \end{aligned} \tag{142}$$

The double-discontinuity $\mathrm{dDisc}[g(z, \bar{z})]$ has some nice properties. One property is positivity, which allows one to obtain positivity conditions on OPE coefficients, and is behind the bootstrap-based proof of the ANEC (as we'll discuss below). Another is that in large-N theories, it only gets contributions from single-trace operators at leading order in $1/N^2$. This allows one to efficiently reconstruct CFT data from single-trace data. This is essentially the CFT version of unitarity methods from amplitudes, where one glues on-shell lower-point data to obtain higher-point or higher-loop data. The efficiency of the inversion formula underlies many recent advances in bootstrapping theories at large-N and/or weak coupling.

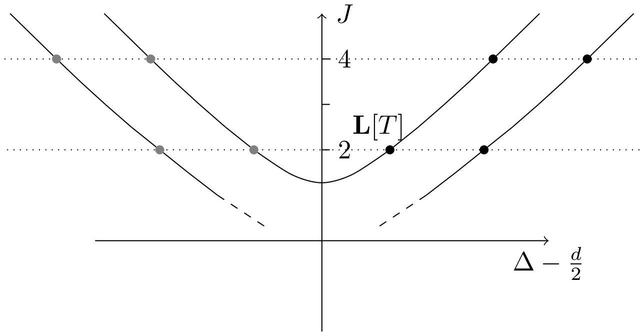


Fig. 4.: Chew-Frautschi plot of Regge trajectories of even-spin operators in a CFT. $C(\Delta, J)$ has a so-called shadow symmetry under $\Delta \to d - \Delta$ that we unfortunately do not have time to explain. Consequently, if we plot J vs $\Delta - \frac{d}{2}$, the plot is left-right symmetric.

Another remarkable feature of Caron-Huot's formula is that it makes sense for non-integer J: CFT data can be analytically continued in spin. Because of the $(-1)^J$, the analytic continuation is different for even and odd spin. Analytically-continued operator dimensions (poles of $C(\Delta, J)$) determine "Regge trajectories" — families of operators connected by a curves in Δ, J space. We can visualize the spectrum of the CFT with a Chew-Frautschi plot (figure 4).

One might ask: if we can analytically continue the CFT data, can we analytically continue the operators themselves? A bit of thought shows that there is no way to analytically continue local operators in spin. The problem is that local operators $\mathcal{O}(x, z)$ act on the vacuum to give nontrivial states $\mathcal{O}|\Omega\rangle$. (This is the state operator correspondence.) However, a non-integer spin operator must kill the vacuum. We cannot have an analytic function of J that is nonzero for integer J but zero everywhere else.

However, it turns out that we can analytically continue light-transforms of local operators $\mathbf{L}[\mathcal{O}_{\Delta,J}]$ in J, giving families of non-local "light-ray operators." The Chew-Frautchi plot is really a plot of the spectrum of light-ray operators of a theory. The integer spin points do not correspond to local operators — they correspond to their light transforms. In the very little time remaining, we will use this point of view to prove the inversion formula. Unfortunately, our discussion will require a fair amount of technology that we haven't property introduced. However, we will try to highlight the key steps of the argument.

7.5. *Light-ray operators*

To analytically continue operators in spin, we start with the "conformal partial wave"

$$P_{\Delta,J}(x,z;x_3,x_4) \propto \int_{\mathbb{R}^d} d^dx_1 d^dx_2 \langle\phi_1\phi_2\phi_3\phi_4\rangle_\Omega \langle\widetilde{\phi}_1\widetilde{\phi}_2\mathcal{O}_{\Delta,J}(x,z)\rangle. \quad (143)$$

Here, we use the shorthand $\phi_i = \phi(x_i)$. The partial wave has the property that

$$P_{\Delta,J}(x,z;x_3,x_4) = C(\Delta,J)\langle\phi_3\phi_4\mathcal{O}_{\Delta,J}(x,z)\rangle. \quad (144)$$

Here, a correlator with an "Ω" subscript $\langle\cdots\rangle_\Omega$ represents a physical four-point function. Three-point functions with no subscript represent the unique conformally invariant structure for operators in the given representations. The operators $\widetilde{\phi}$ have "shadow" dimensions $d-\Delta_\phi$, which guarantees that the integral $\int d^dx\langle\cdots\phi(x)\rangle\langle\widetilde{\phi}(x)\cdots\rangle$ is conformally-invariant.

Using (140), we can also write (144) as

$$-\operatorname*{Res}_{\Delta=\Delta_k} P_{\Delta,J}(x,z;x_3,x_4) = f_{\phi\phi\mathcal{O}_k}\langle\phi_3\phi_4\mathcal{O}_k(x,z)\rangle_\Omega, \quad (145)$$

where $\mathcal{O}_k \in \phi\times\phi$. Thus, the partial wave encodes matrix elements of local operators in the $\phi\times\phi$ OPE.

We would like to study their light-transforms, so let us Wick-rotate to Lorentzian signature and light-transform the partial wave. We simultaneously rotate all the Euclidean times

$$\tau_i \to e^{i\frac{\pi}{2}}t_i. \quad (146)$$

The Wick-rotation of a Euclidean correlator gives a time-ordered correlator. We must now light-transform the time-ordered three-point structure,

$$\mathbf{L}\langle 0|T_L\{\widetilde{\phi}_1(x_1)\mathcal{O}(x,z)\widetilde{\phi}_2(x_2)\}|0\rangle. \quad (147)$$

This is a linear combination of different Wightman three-point structures, multiplied by θ-functions. Because $\mathbf{L}[\mathcal{O}]$ kills the vacuum, the result can only be nonzero if the light-transform contour goes from the past of $\widetilde{\phi}_1$ to the future of $\widetilde{\phi}_2$ or vice versa. In these cases, the time-ordered correlator becomes equal to a Wightman correlator along the entire light-transform integration contour. Thus, we have two terms

$$\begin{aligned}\mathbf{L}\langle 0|T_L\{\widetilde{\phi}_1\mathcal{O}(x,z)\widetilde{\phi}_2\}|0\rangle &= \langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle\theta(1>x>2^-)\\ &\quad + \langle 0|\widetilde{\phi}_2\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_1|0\rangle\theta(2>x>1^-). \quad (148)\end{aligned}$$

Thus, we find

$$\mathbf{L}[P_{\Delta,J}](x,z;x_3,x_4) = \int_{1>x>2^-} d^dx_1 d^dx_2 \langle\phi_1\phi_2\phi_3\phi_4\rangle_\Omega \langle 0|\widetilde{\phi}_1 \mathbf{L}[\mathcal{O}_{\Delta,J}](x,z)\widetilde{\phi}_2|0\rangle + (1\leftrightarrow 2) \tag{149}$$

Let us define the object

$$\mathbb{O}_{\Delta,J}(x,z) \equiv \int_{1>x>2^-} d^dx_1 d^dx_2 \phi_1\phi_2 \langle 0|\widetilde{\phi}_1 \mathbf{L}[\mathcal{O}_{\Delta,J}](x,z)\widetilde{\phi}_2|0\rangle + (1\leftrightarrow 2). \tag{150}$$

By construction, when J is an even integer, we have

$$f_{12\mathcal{O}_k}\mathbf{L}[\mathcal{O}_k](x,z) = \operatorname*{Res}_{\Delta=\Delta_k} \mathbb{O}_{\Delta,J}(x,z). \tag{151}$$

However, a beautiful thing has happened: $\mathbb{O}_{\Delta,J}$ makes sense for general $J \in \mathbb{C}$. In particular, the light-transformed Wightman three-point structures are well-defined for non-integer J, and this is enough to define $\mathbb{O}_{\Delta,J}(x,z)$.

It turns out that poles in $\mathbb{O}_{\Delta,J}$ come from the region where x_1, x_2 are close to the light-ray spanned by z. Thus, when we take a residue, x_1 and x_2 get forced arbitrarily close to this region, and it is reasonable to call $\operatorname{Res}_\Delta \mathbb{O}_{\Delta,J}$ a "light-ray operator".[r]

7.6. *Proving the inversion formula*

Consider now a matrix element of $\mathbb{O}_{\Delta,J}$

$$\begin{aligned}
&\langle\Omega|\phi_4\mathbb{O}_{\Delta,J}\phi_3|\Omega\rangle \\
&= C(\Delta,J)\langle 0|\phi_4\mathbf{L}[\mathcal{O}_{\Delta,J}]\phi_3|0\rangle \\
&= \int_{2^-<x<1} d^dx_1 d^dx_2 \langle\phi_1\phi_2\phi_3\phi_4\rangle_\Omega \langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle + (1\leftrightarrow 2).
\end{aligned} \tag{152}$$

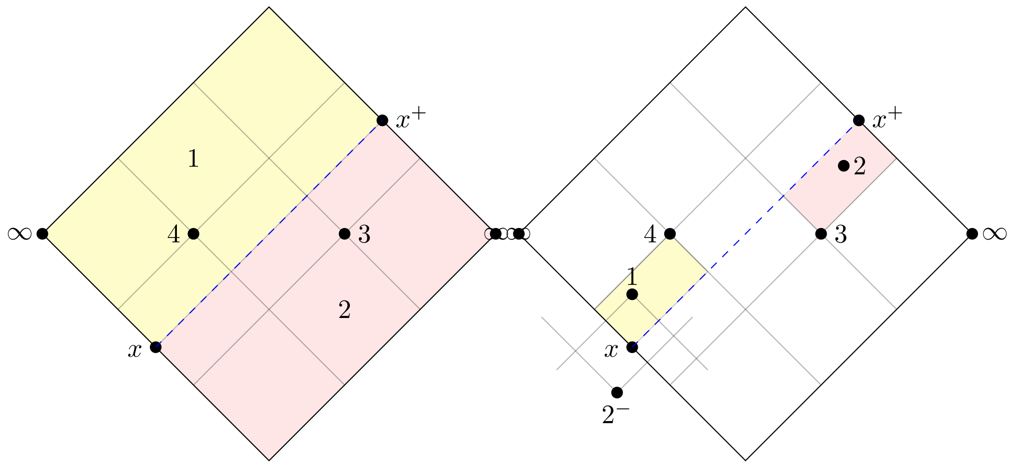
Because of the restrictions $3^- < x < 4$ and $2^- < x < 1$, the lightcone of x splits Minkowski space into two regions, with $2,3$ in the lower region and $1,4$ in the upper, see figure 5a. Thus, we can write the integrand as

$$\langle\Omega|T\{\phi_4\phi_1\}T\{\phi_2\phi_3\}|\Omega\rangle\langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle. \tag{153}$$

We can now use the reasoning in lemma 1 to obtain a double commutator. Consider a modified integrand where ϕ_1 acts on the future vacuum,

$$\langle\Omega|\phi_1\phi_4T\{\phi_2\phi_3\}|\Omega\rangle\langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle. \tag{154}$$

[r] Actually, not enough is known about the analytic structure of $\mathbb{O}_{\Delta,J}$ as a function of Δ to guarantee that it has only simple poles. In general, the discontinuity across any non-analyticity in Δ leads to a light-ray operator.



(a) After taking the light transform but before reducing to a double commutator.

(b) After reducing to a double commutator.

Fig. 5.: The configuration of points within the Poincare patch of x_∞ at various stages of the derivation. The blue dashed line shows the support of light transform of $\mathcal{O}(x,z)$. The yellow (red) shaded region shows the allowed region for 1 (2). In the right-hand figure, we indicate that x is constrained to satisfy $2^- < x < 1$. Note that after reducing to a double-commutator, the yellow and red regions are independent of x_∞ (as long as x is lightlike from x_∞).

Imagine integrating ϕ_1 over a lightlike line in the direction of z, with coordinate v_1 along the line. Because ϕ_1 acts on the future vacuum, the correlator is analytic in the lower half v_1-plane. One can argue that the integrand dies sufficiently quickly at large v_1, and thus we can deform the v_1 contour into the lower half-plane to give zero.

Consequently, the x_1 integral vanishes if we replace (153) with (154), so we can freely replace

$$T\{\phi_4\phi_1\} \;\to\; T\{\phi_4\phi_1\} - \phi_1\phi_4 \;=\; [\phi_4,\phi_1]\theta(1<4). \qquad (155)$$

By similar reasoning, we can replace

$$T\{\phi_2\phi_3\} \to [\phi_2,\phi_3]\theta(3<2). \qquad (156)$$

Overall, we find a double commutator in the integrand, together with some extra restrictions on the region of integration

$$\begin{aligned}
&\langle \phi_4 \mathbb{O}_{\Delta,J}(x,z)\phi_3\rangle_\Omega \\
&= \int_{\substack{x<1<4\\ 3<2<x^+}} d^dx_1 d^dx_2 \langle\Omega|[\phi_4,\phi_1][\phi_2,\phi_3]|\Omega\rangle\langle 0|\widetilde{\phi}_1 \mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle + (1\leftrightarrow 2).
\end{aligned} \tag{157}$$

This formula can be used to recover Hartman et. al.'s proof of the average null energy condition. The ANEC operator is $\mathbf{L}[T]$, which is obtained as a residue of the above formula at $\Delta = d, J = 2$. The key point is that the double-commutator has nice positivity properties, due to Rindler positivity. If the points are in a Rindler-symmetric configuration $\phi_4 = \overline{\phi_3}$ and $\phi_1 = \overline{\phi_2}$, then we have

$$-\langle\Omega|[\phi_4,\phi_1][\phi_2,\phi_3]|\Omega\rangle = \langle\Omega|\overline{[\phi_3,\phi_2]}[\phi_3,\phi_2]|\Omega\rangle \geq 0. \tag{158}$$

One can either argue that the three-point structure $\langle 0|\widetilde{\phi}_1\mathbf{L}[\mathcal{O}](x,z)\widetilde{\phi}_2|0\rangle$ smears the operators in a Rindler-symmetric way, or following Hartman et. al., argue that the integral is dominated by a Rindler-symmetric configuration. Expressing matrix elements of $\mathbf{L}[T]$ as the integral of something positive gives the ANEC. In this case, we obtain the ANEC in states created by ϕ, but this argument can be generalized to states created by arbitrary operators.

Formula (157) together with (152) gives an expression for $C(\Delta, J)$ as the integral of a double-commutator. The difference from Caron-Huot's formula is that the integral is over spacetime, instead of cross-ratios $z, \overline{z}$. One obtains an integral over cross-ratios by pairing both sides with a "dual" conformal three-point structure satisfying

$$\left(\langle\text{dual}\rangle, \langle 0|\phi_4\mathbf{L}[\mathcal{O}_{\Delta,J}](x,z)\phi_3|0\rangle\right)_L = 1, \tag{159}$$

where $(\cdot,\cdot)_L$ is a conformally-invariant pairing. This naturally leads to the conformal block $G_{J+d-1,\Delta-d+1}$.