Generalised Symmetries and Line Operators RG flows

DESY String theory Journal Club

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Abstract

In this Journal Club we continue our presentation of recent advances in the topic of Generalised Symmetries. Today we focus on one-form symmetries in four dimensional gauge theories and the corresponding selection rules on RG flows of Line Observables thereof.

Contents

1 Introduction

1.1 A new Renaissance of symmetries

Symmetries are arguably the most important guiding principle in physics. They are ubiquitous in all branches of theoretical physics and provide organising principles and selection rules to characterise observables. Even more importantly, symmetries constraint in a non-trivial way the dynamics of quantum systems, allowing us to push the understanding of Field Theories beyond the perturbative regime to probe strongly coupled systems. As a rule of thumb, given a physical system the more symmetries we can identify the more information regarding the dynamics we can infer. This paradigm, already fully employed in quantum mechanics, has found its paramount application in Quantum Field Theories where symmetries represent the main ingredient to building any physical theory. Yet, for many years, the interest of the high energy community has been mostly relegated to the particular class of continuous symmetries. Only in the '90, the works of Dijkgraaf and Witten on discrete symmetries laid the foundations for a new wave of active research in Condensed Matter as a tool to study topological phases of matter and critical phenomena. Yet, a new *renaissance* of symmetries in High Energy Physics resulted from the work of Gaiotto-Kapustin-Seiberg-Willet [13] of 2014 that absorbed and extended the progress of the previous two decades. The generalised notion of *higher-form symmetries* proposed in the paper provides a formalism to describe symmetries under which operators supported over higher-dimensional manifolds are charged. Non-local operators are central in modern theory: the Gauge Theories are not specified by the local physics, but rather depend crucially on the global properties. This Generalised symmetries provide us with new tools to probe these non-local aspects that would not be accessible through perturbation theory that is blind to them. Furthermore thier anomalies, being rigid under the Renormalization Group, allow us to flow between the different regimes of Quantum Field Theories. Further generalisations have been investigated in the last few years and are now a very active research area. So far we learnt that Higher-form symmetries can combine and mix into *Higher-Group structures* and, even more drastically, people are currently studying *non-invertible symmetries* that do not have any underlying group structure. Indeed, the picture that is currently arising is that the most general symmetric structure of a Quantum field theory is a intricate sum of all those possibility.

1.2 Today's talk

As one could expect, the subject is rich, vast and rapidly expanding. The goal of this short note is surely not the one of trying to cover the entire subject. Rather, our main interest is focusing on the case of 1-form symmetries to elucidate how this new formalism is **not** just a rewriting of previous knowledge in a new, more general and systematic language, but it is actually a new tool that allow us to access new data that was previously inaccessible, hoping that, by the end of this note, you will be able to have in your hands a concrete example thereof. To do that, we have decided to discuss the very exciting programm of trying to understand the fate of Line Operators under the RG flows, i.e. studying the behaviour of the theory in the long-distance regime.

These notes are organised as follows: hoping that it might serve as a motivation, we will start with presenting the physical problem we are interested in studying using higher-form symmetries. Then we will try to build up the necessary dictionary to navigate through this generalised symmetry revolution. Successively, we will focus our attention on the particular case of 1-form symmetries, discussing the presence thereof in the concrete case of unitary gauge theories possibly coupled with matter fields. Then we will conclude by illustrating how we can apply this new tools to study Line Operators RG flows following the recent results obtained by Komargodski and friends.

2 Motivation

Consider the physical case of a material in d+1 dimension, for instance, you can think of a metal or of a graphene sheet. It is very well known that the behaviour of such a material in its (quantum) critical point (second order phase transition) at zero temperature can be described as a CFT. In the last decades a lot has been achieved in employing the conformal symmetries to constraint correlation function of local operators, up to the point that there are cases in which we can completely solve the interacting theory of local excitation only by exploiting the symmetry of the theory. Yet, as remarked, local operators are not the end of the story. Indeed, already from our very physical condensed matter example, we can understand the experimental interest in studying the response of system once it is coupled with extended line operators. Indeed, imagine that at any point of the lattice, we insert a point-like *impurity*. This will change dramatically the theory modifying its Hilbert Space. In space-time this is nothing but the insertion of a (time-like) 1 dimensional defect into the theory. Now, imagine that the bulk theory is already a CFT, it is of crucial interest to understand what is the fate of this impurity in the IR physics, or more physically, what is the effect on the local physics at very large distance from the impurity. Such a point-like impurity is, in general, a non-conformal line inserted in a conformal bulk and preserves only the subgroup of $SO(d+1,1)$:

$$\mathbb{R} \times SO(d-1) \tag{2.1}$$

at long distance, this line will flow to a critical point of some sort where it becomes a Conformal line preserving:

$$SL(2,\mathbb{R}) \times SO(d-1) \tag{2.2}$$

that together with the bulk will be now described by a DCFT. Again, a lot of effort has been dedicated to study DCFT and here in DESY we have many people that made huge advances in the interesting program of developing tools to extract new CFT data from this setup. Today we are not interested in making prediction on the physics of these critical points, rather we are interested in showing how this flow can be constrained by using 1-form symmetries. Indeed, albeit we know that there is a critical point of some sort, the qualitative essence of this critical point might be dramatically different. Indeed in the IR, the impurity might be completely screened, and hence the non-conformal line flows to the trivial line (unit operator), or it might flow to some non-trivial conformal defect. As we will see, 1-form symmetries provides us with selection rules telling us precisely when the impurity **does not** flow to a trivial line in the IR. This is a very physical and interesting question with a plethora of applications both in condensed matter (qunatum computing error-correction, spin-impurities in graphene etc) and in high energy, where the fate of line operators is striclty connected with confinement and other interesting strongly coupled phenomena. Not only this is insteresting, but there have been some **experimental results** on graphene and other materials that we can use to directly test this predictions.

3 Higher-form Symmetries

Symmetries are crucial in quantum field theory as they constrain the spectrum of states and operators. While gauge symmetries are redundancies in the description of the theory, global symmetries are intrinsic properties of QFTs. This implies that, even though the same field theory may have equivalent presentations in terms of different gauge theories, the global symmetries must be unambiguously the same, providing a probe for dualities. Global symmetries are a powerful tool applicable in all the regimes of a field theory and even in the case of theories where a Lagrangian description is not available at all. In this chapter we study the generalisation of to higher-form global symmetry proposed in [13]. The charged operators are no longer point-like operators, rather than are extended objects supported on higher-dimensional manifolds.

3.1 Standard Symmetries

We are all familiar with the concept of standard global symmetries in Lagrangian QFT as groups of functional transformations acting on the fields of the theory that leave the action invariant and under which local (point-like) operators "*particles*" are charged. Yet, the problem with this formulation is that it makes hard to detach the intrinsic nature of the global symmetry from the specific lagrangian description of the theory. Hence, it is useful to formalise the notion of global symmetry from a more modern perspective that, not only is particularly suitable for the generalisation we are aiming to achieve, but also allow us to study generalised symmetries of non-Lagrangian theories. Specifically, an ordinary global symmetry can be described abstractly as a set of topological operators $U_g(\mathcal{M}^{(d-1)})$ associated with elements g of the symmetry group G supported on codimension 1 manifolds and obeying its multiplication law:

$$U_g(\mathcal{M}^{(d-1)}) \circ U_{g'}(\mathcal{M}^{(d-1)}) = U_{g\circ g'}(\mathcal{M}^{(d-1)}) \tag{3.1}$$

This law, as always in the following, should be intended as an operator identity valid inside correlation functions:

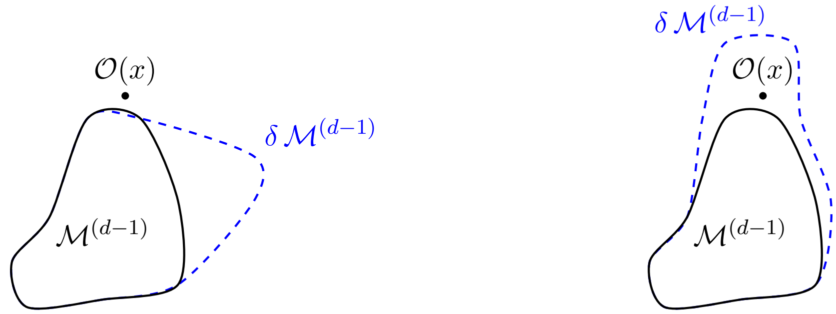
$$\langle U_g(\mathcal{M}^{(d-1)}) \circ U_{g'}(\mathcal{M}^{(d-1)})\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle = \langle U_{g\circ g'}(\mathcal{M}^{(d-1)})\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle \tag{3.2}$$

The topological nature of the operator means that any correlation function involving $U_g(\mathcal{M}^{(d-1)})$ is independent of deformations of $\mathcal{M}^{(d-1)} \to \mathcal{M}^{(d-1)} + \delta\,\mathcal{M}$ as long as we do not cross any other operator of the theory as in Fig 1. This topological property implies the (3.1) can be extended to consider composition of topological operators supported on different manifolds, as long as we can deform one into the other as in Fig 2.

These topological operators arise very naturally whenever we have a continuous symmetry group G. In this case, we have a Noether (vector/one-form) conserved current j, for each generator of the group and we can integrate its Hodge dual $d-1$ closed form $*j$ on any $\mathcal{M}^{(d-1)}$ manifold giving a *charge*:

$$\mathcal{Q}(\mathcal{M}^{(d-1)}) = \oint_{\mathcal{M}^{(d-1)}} *j \tag{3.3}$$

When $\mathcal{M}^{(d-1)}$ is the entire space this is the usual conserved charge at a fixed time $\mathcal{Q}(t)$, but in general we can take $\mathcal{M}^{(d-1)}$ to be any (compact or non-compact) manifold. We can construct



(a) Allowed (b) Not allowed

Figure 1: Deformations of topological operators

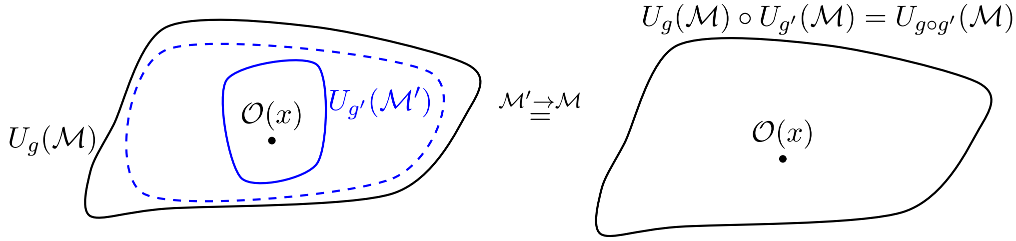


Figure 2: Composition of two topological operators supported on $\mathcal{M}$ and $\mathcal{M}'$

surface operators labelled by group elements by exponentiation of the charges:

$$U_g(\mathcal{M}^{(d-1)}) \equiv \exp\Big(i\, g\; \mathcal{Q}(\mathcal{M}^{(d-1)})\Big), \qquad g \in G \tag{3.4}$$

To prove that indeed the operators as constructed are topological, we use that j conserved implies $*j$ closed $\mathrm{d} * j = 0$. Then, we consider a small deformation $\mathcal{M}^{(d-1)} \to \widetilde{\mathcal{M}}^{(d-1)}$ and, letting $\mathcal{N}^{(d)}$ be a d-dimensional manifold with boundaries such that $\partial\mathcal{N}^{(d)} = \mathcal{M}^{(d-1)} \cup \widetilde{\mathcal{M}}^{(d-1)}$, it follows from Stoke's theorem that:

$$\mathcal{Q}(\mathcal{M}^{(d-1)}) - \mathcal{Q}(\widetilde{\mathcal{M}}^{(d-1)}) = \oint_{\mathcal{M}^{(d-1)}} *j - \oint_{\widetilde{\mathcal{M}}^{(d-1)}} *j = \oint_{\mathcal{N}^{(d)}} \mathrm{d}(*j) = 0 \tag{3.5}$$

Hence, the description of global symmetries in term of topological operators is equivalent to the standard treatment based on Noether theorem. To see how this topological operators implement the symmetry action on charged operators, we pursue our continuous symmetry case before generalising it. We recall the Ward identity (T_a generator associated with j_a):

$$\mathrm{d}\,\langle *j_a(x)\,\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle = \sum_{i=1}^{n} \delta(x-x_i)\langle\Phi_1(x_1)\cdots T_a\Phi_i(x_i)\cdots\Phi_n(x_n) \tag{3.6}$$

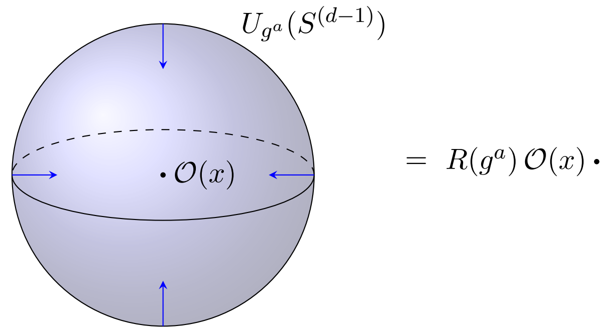


Figure 3: Linking of $U_g(S^{(d-1)})$ and $\mathcal{O}(x)$

implies, when a charged operator is inserted in the position x_i, the operatorial identity[1]:

$$\mathrm{d} * j_a = \delta^{(d)}(x - x_i) T_a \qquad \Longrightarrow \qquad \mathrm{d} * j_a\, \mathcal{O}(x) = -i\delta^{(d)}(x - x_i) T_a\, \mathcal{O}(x) \tag{3.7}$$

Hence, considering a topological operator U_{g^a} (the notation g^a explicitly refer to the fact I can have more than one generator in my symmetry group) supported on (WLOG) a sphere $S^{(d-1)}$ centred in the operator $\mathcal{O}(x)$ living in the a representation R of the group, as in Fig 3. We can shrink the sphere down to x using the topological property, but due to Eq (3.7), we have (inside any correlation function):

$$U_{g^a}(S^{d-1})\, \mathcal{O}(x) = \exp\left(i g^a \int_{S^{d-1}} *j \right) \mathcal{O}(x) = \exp(i g^a T_a)\, \mathcal{O}(x) = R(g)\, \mathcal{O}(x) \tag{3.8}$$

where $R(g)$ is the group element in the representation R. In other words, unlinking the topological operator and a charged operator accounts for the action of a symmetry group element on the charged operator itself. In the basic example of a $U(1)$ symmetry $R(g)$ this is just a phase factor is just $R(g) = e^{igq}$ with q the abelian charge of $\mathcal{O}(x)$ under the symmetry. Another key identity can be obtained from the Ward identity. Namely, we can integrate over the Ward identity (3.6), over a "pillbox" bounded by two distinct times t_+ and t_-, extending to spatial infinity in all the other directions and containing only the operator Φ_1. Upon converting the volume integral into a surface integral $\Sigma_+ \cup (-\Sigma_-)$ (the minus sign is due to the surface orientation) we obtain:

$$\begin{aligned} \langle \oint_{*\Sigma_+ \cup - \Sigma_-} (*j \Phi_1(x_1) \Phi_2(x_2) \cdots \Phi_n(x_n)\rangle &= \langle Q_a(t_+) \Phi_1(x_1) \cdots \rangle - \langle Q_a(t_-) \Phi_1(x_1) \cdots \rangle \\ &= \langle T_a \Phi_1(x_1) \cdots \rangle \end{aligned} \tag{3.9}$$

that, after taking the limit $t_+ \to t_-$ and exponentiating, implies at equal time:

$$U_g(\Sigma)\, \mathcal{O}(x) = R(g)\, \mathcal{O}(x) U_g(\Sigma) \tag{3.10}$$

[1] We note that this does not contradict the that $*j$, for instance consider the form $\mathrm{d}\theta = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$, that is closed but still singular in the origin.

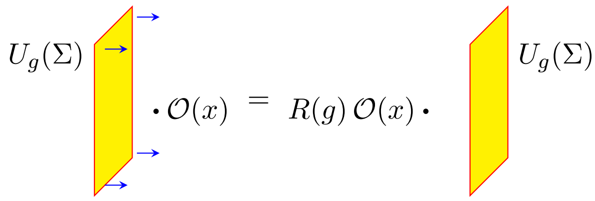


Figure 4: Operator crossing of a non-compact topological operator supported on a domain wall Σ

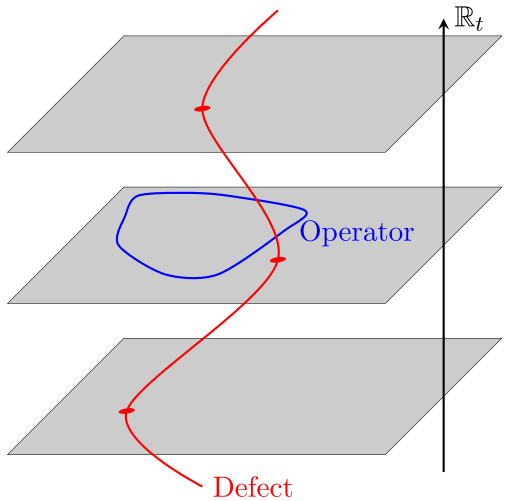


Figure 5: Defects vs Operators

This equation can be interpreted in a very natural way: the crossing of a topological operator supported on a *domain wall* Σ (we took the "pillbox" limit) implements the action of the symmetry as in Fig 4. In (3.8), nothing stopped us to inserting the topological operator in spacetime since we had not chose any "time" direction, while in (3.10) we insert topological operator inside a spatial slide. There is a subtle distinction between the two pictures in Lorentzian signature where have the natural foliation $\mathcal{M}^{(d)} = \mathbb{R}_t \times \mathcal{M}^{(d-1)}$. For each time t, there is an Hilbert Space $\mathscr{H}(\mathcal{M}^{(d-1)})$ supported on $\mathcal{M}^{(d-1)}$; an insertion of U_g corresponds to preparing a state in this Hilbert space and U_g is therefore a genuine operator of the theory. Instead an insertion of U_g such that its support extends in the time direction modify the Hilbert spaces associated with different slices (turning a QFT in a Defect QFT). In this case U_g is not strictly speaking an operator of the theory but a *topological defect*.

For this reason, we limit ourselves only to insert symmetry **operators**. Even though, this is not relevant for this section where charged operators are local operators, we stress here that instead charged object can be either operators or defects and we will refer to them interchangeably.

We showed that declaring a set of topological (equivalent to current conservation) operators together with a group composition and "crossing" (Ward identity) laws completely describes

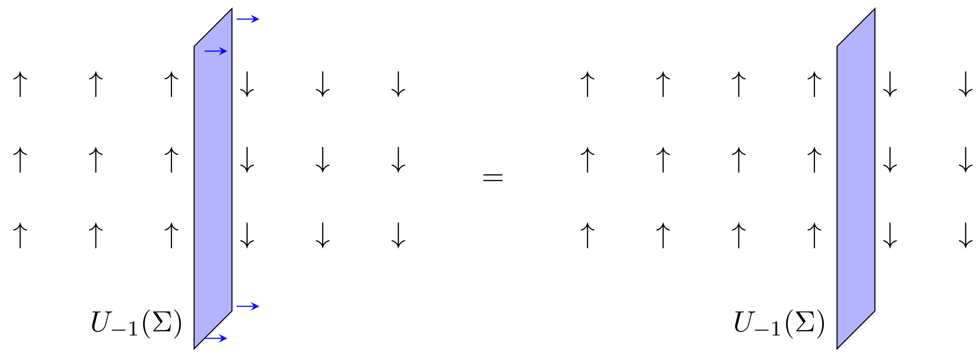


Figure 6: Ising model's $\mathbb{Z}_2$ symmetry implemented by a topological surface operator inserted in spatial slice with "time" flowing perpendicular to it and the 2d lattice extending parallel to it.

a continuous global symmetry. The crucial point is that now this formulation is completely free from assuming the existence of a conserved current and even of a Lagrangian at all! Symmetries are intrinsic and completely independent of any Lagrangian description of a theory and topological operators allow us to describe them accordingly. At this point, nothing stops us to include **discrete global symmetries** by simply assuming G to be a discrete group. The fact continuous and discrete symmetries can be described in an identical fashion may seem surprising. After all, we are taught that for discrete symmetries we do not have neither a Noether theorem nor a Ward identity and therefore we are tempted to believe we possess less tools to analyse them. Yet, since these two results follow from the existence of a Lagrangian and manipulation thereof, we would lack of them even in the case of continuous symmetries of non Lagrangian theories, suggesting that this apparent difference between discrete and continuous symmetry is only a result of the existence of a Lagrangian description rather than of the nature of the symmetries themselves. Furthermore, in the case of discrete symmetries, topological operators as generators of discrete symmetries are even more natural! For instance, consider the Ising model [6]. There is a $\mathbb{Z}_2$ global "spin-flipping" symmetry. I can detect and implement a $\mathbb{Z}_2$ transfomation as in Fig 6 by pulling the spins over a domain wall Σ. Yet, this time, $U_g(\Sigma)$ is not the integral of any local quantity. This idea of domain walls as symmetry generators can be readily generalised to any discrete symmetry, allowing us to treat them in the same fashion of continuous symmetry.

In this entire section we have considered charged operators to be local "point-like" operators, e.g. $\mathcal{O}(x)$ or a spins $\uparrow$ / $\downarrow$ in a lattice point. We are now ready for the key generalisation.

3.2 Higher Form Symmetries

Point-like objects are not the only charged objects in Gauge Theories: for instance in the previous chapter we have analysed line operators associated to 1-dimensional manifolds and we discussed that they are characterized by a lattice of *charges*. Hence, we aim to extend the formalism of the previous section to extend the notion of "symmetry" to include charged observables that are supported on are higher dimensional manifolds. We define a **Higher q-form symmetry** as characterized by a set of topological operators supported on codimen-

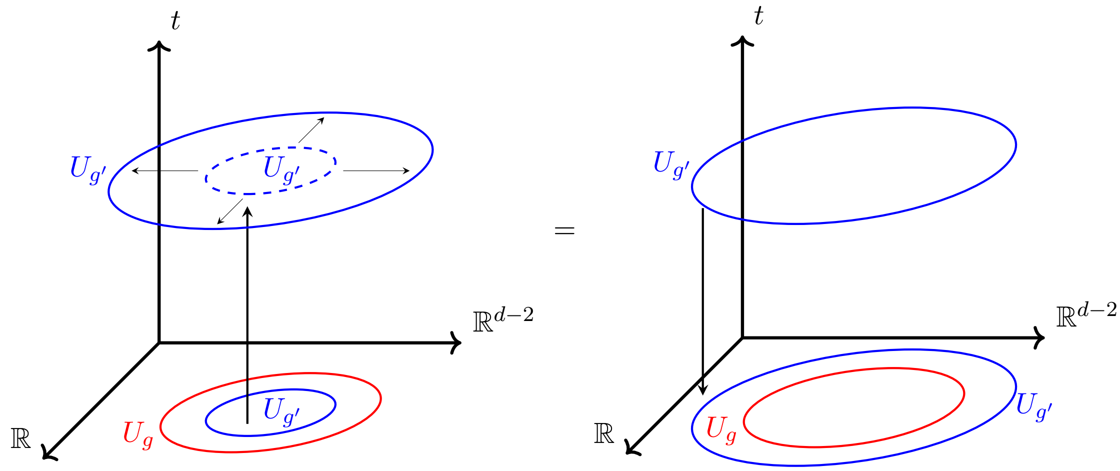


Figure 7: Example for $q = 1$: operators supported on $\mathcal{M}^{(d-2)} = S^{d-2}$.

sion $q + 1$ manifolds without boundaries: $U_g(\mathcal{M}^{(d-q-1)})$, satisfying the group multiplication property:

$$U_g(\mathcal{M}^{(d-q-1)}) \circ U_{g'}(\mathcal{M}^{(d-q-1)}) = U_{g\circ g'}(\mathcal{M}^{(d-q-1)}) \tag{3.11}$$

Obviously for $q = 0$ we recover the standard symmetries. Albeit, (3.11) looks very similar to its standard-symmetries counterpart (3.1), the former implies a very important consequence, namely: Higher-Form symmetries can only be abelian. This follows from the simple observation that in a d-dimensional space, there is no way of defining an ordering for manifolds of dimension $\delta < d - 1$. To see that, consider the insertion of two defects $U_g(\mathcal{M})$ and $U_{g'}(\mathcal{M})$ on a spatial slice at $t = t^*$ implementing their composition $U_g \circ U_{g'} = U_{g\circ g'}$; then deform $U_{g'}$, such that $\mathcal{M} \subset \mathcal{M}'$. Now, perform a slight shift such that $\mathcal{M}'$ lies in the $t = t^* + \delta t$ spatial slice. Since $\underbrace{d-1}_{\text{space}} - \underbrace{\delta}_{\text{support of } \mathcal{M}'} > 0$, there is at least one further dimension that we can use to deform $\mathcal{M}'$ such that at $\delta t = 0$, $\mathcal{M}' \subset \mathcal{M}$ (see Fig7). Thus, for any element of the group and for any manifold (as long as deformations are allowed):

$$U_{g\circ g'}(\mathcal{M}^{(d-q-1)}) = U_{g'\circ g}(\mathcal{M}^{(d-q-1)}) \tag{3.12}$$

i.e. the higher-form symmetry groups must be abelian. Actually, there are some flaws in this argument: namely there could be topologies of the base space $\mathcal{M}^{(d)}$ such that we might have obstructions to freely perform the procedure above on the topological operators. In that case, the higher form symmetries may fail to be abelian (*non-commuting fluxes*[13]). This is indeed the case for theories defined on spaces having non-trivial torsion cycles, e.g. $\mathcal{M}/\mathbb{Z}_k$ and some examples are provided in [13]. Another thing that may happen is that some of the operators inserted of the theory, viewed as states of the Hilbert Space at fixed time cannot be simultaneously diagonalised[2]. In this case the operators are intrinsically non-commuting. We will see an example of this in subsection 3.2.4 for topological operators on $\mathrm{AdS}_5 \times S^5$.

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[2]Thanks to Lakshya Bhardwaj for the clarification on this point

The reason for which they are called "q-form symmetries" is that, in the continuous case, the symmetry parameter is a q-form $\Lambda^{(q)}$, to be confronted with the $0-$form (function) of standard symmetries. This means that the fields charged under the symmetry are q-forms as well transforming as $A^{(q)} \to A^{(q)} + \Lambda^{(q)}$ for an infinitesimal global transformation. A charged operators of charge ρ can be therefore constructed by taking the flux of these q-form on q-dimensional manifolds (without boundaries):

$$\mathcal{V}_\rho(\mathcal{C}^{(q)}) = \exp\left(i\rho \oint_{\mathcal{C}^{(q)}} A^{(q)}\right), \qquad \mathcal{V}(\mathcal{C}^{(q)}) \to \exp\left(i\rho \oint_{\mathcal{C}^{(q)}} \Lambda^{(q)}\right) \mathcal{V}(\mathcal{C}^{(q)}) \tag{3.13}$$

Hence, we can derive an analogous of the (classical) Noether theorem. Let $\mathcal{L}$ be a Lagrangian of containing some higher dimensional operators $\mathcal{V}_I(C_I^{(q)}), \quad I = 1, \cdots N$, $\mathcal{L} = \mathcal{L}[\mathcal{V}_I(C_I^{(q)})]$, then $\forall \delta_\Lambda(A)$:

$$\underbrace{\delta_\Lambda \mathcal{L}}_{d} = \underbrace{\frac{\delta \mathcal{L}}{\delta\, \mathrm{d}A}}_{d-q-1} \wedge \underbrace{\delta_\Lambda(\mathrm{d}A)}_{q+1} + \underbrace{\frac{\delta \mathcal{L}}{\delta A}}_{d-q} \wedge \underbrace{\delta_\Lambda(A)}_{q} \qquad \text{(Integrating } \delta_\Lambda S \text{ by part)} \tag{3.14a}$$

$$\stackrel{!}{=} \mathrm{d}\left(\underbrace{\delta_\Lambda(A)}_{q} \wedge \underbrace{\frac{\delta \mathcal{L}}{\delta\, \mathrm{d}A}}_{d-q-1}\right) - \delta_\Lambda(A) \underbrace{\cancel{\left(\mathrm{d}\frac{\delta \mathcal{L}}{\delta\, \mathrm{d}A} - \frac{\delta \mathcal{L}}{\delta A}\right)}}_{=0 \text{(on shell)}} = \underbrace{\mathrm{d}(\star(f^{(q+1)}) \wedge \delta_\Lambda(A))}_{d} \tag{3.14b}$$

where the last bit is a convenient way of expressing a (possible) total derivative term. Thus, on shell we have a closed $(d-q-1)$-form dual to a conserved $(q+1)$-form:

$$\mathrm{d}\left(\frac{\delta \mathcal{L}}{\delta\, \mathrm{d}A} - *f^{(q+1)}\right) := \mathrm{d} * j^{(q+1)} = 0, \qquad j^{(q+1)} = *\frac{\delta \mathcal{L}}{\delta\, \mathrm{d}A} - f^{(q+1)} \tag{3.14c}$$

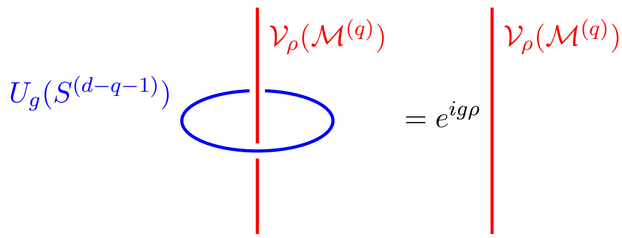
In the quantum theory, assuming for now the invariance of the integration measures (i.e. there are no anomalies), we get the corresponding Ward identity[3], $(\delta_\Lambda A = \Lambda)$:

$$\begin{aligned}
0 &\stackrel{\forall \Lambda^{(q)}}{=} \delta_\Lambda \langle \mathcal{V}_{\rho_1}(C_1^{(q)}) \cdots \mathcal{V}_{\rho_N}(C_N^{(q)}) \rangle \\
&= \int \mathcal{D}_A \exp\left(iS[A] - i\int_{\mathcal{M}^{(d)}} \Lambda^{(q)} \wedge \mathrm{d} * j^{(q+1)}\right) \exp\left(i\rho_1 \oint_{\mathcal{C}_1^{(q)}} \Lambda^{(q)}\right) \mathcal{V}_1 \cdots \exp\left(i\rho_N \oint_{\mathcal{C}_N^{(q)}} \Lambda^{(q)}\right) \mathcal{V}_N \\
&= \int \mathcal{D}_A \exp(iS[A]) \sum_{I=1}^{N} \left[1 + i\left(\rho_I \oint_{\mathcal{C}_I^{(q)}} \Lambda^{(q)} - \int_{\mathcal{M}^{(d)}} \Lambda^{(q)} \wedge \mathrm{d} * j^{(q+1)}\right) + \mathcal{O}(\Lambda \wedge \Lambda)\right] \mathcal{V}_{\rho_I}
\end{aligned}$$

from which we deduce:

$$\mathrm{d} * \langle j^{(q+1)} \mathcal{V}_I \rangle = \sum_{I=1}^{N} \rho_I\, \delta^{(d-q)}(\mathcal{C}_I^{(q)})\, \langle \mathcal{V}_I \rangle \tag{3.14d}$$

[3] To the best of our knowledge there is no explicit derivation of the Ward identity for Higher Form Symmetries in the literature. A similar derivation has been carried in the sole case of 1-form symmetries in the thesis [30], where the derivation is original[29].



(3.17)

Figure 8: Linking of a q-form symmetry operator with an extended dim q charged operator

where $\delta(\mathcal{C}_I^{(d-1q)})$ is a q-form generalisation of the Dirac delta function:

$$\int_{\mathcal{M}^{(d)}} \omega^{(q)} \wedge \delta^{(d-q)}(\mathcal{C}^{(q)}) = \oint_{\mathcal{C}^{(q)}} \omega^{(q)}, \qquad \forall \omega^{(q)} \tag{3.14e}$$

Given the conserved currents, we can construct the topological operators by integrating the closed $(d-q-1)$-form $*j$ over $(d-q-1)$ manifolds without boundaries:

$$U_g(\mathcal{M}^{(d-q-1)}) = \exp\left(ig \oint_{\mathcal{M}^{(d-q-1)}} *j \right) \tag{3.15}$$

As in the standard symmetry case, they are indeed topological as $*j$ is closed. Furthermore, from (3.14d) we deduce the topological operator Ward identity as in Fig 8:

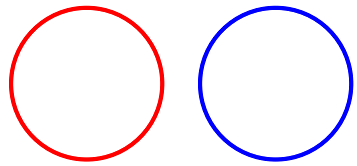
$$U_g(S^{(d-q-1)})\mathcal{V}_\rho(\mathcal{C}^{(q)}) = \exp(ig\rho)\mathcal{V}_\rho(\mathcal{C}^{(q)}) = R_{\mathcal{V}}(g)\mathcal{V}_\rho(\mathcal{C}^{(q)}) \tag{3.16}$$

The fact $R_{\mathcal{V}}(g)$ is simply a phase is consistent with the higher-form symmetries being always abelian. The specific form of the phase in the equation above does not depend on the form of the charged operators. In fact, by using the group multiplication of the topological operators $U_g = U_h \circ U_k$ and considering their composition $\phi(g,\rho) = \phi(h,\rho)\phi(k,\rho)$ we deduce that $\phi(g,\rho)$ has to be the irreps ρ of dimension 1 (abelian) of g: $\phi(g,\rho) = e^{ig\rho}$. This also implies that the set of allowed charges C ρ must be elements of the Pontryagin dual group of the q-form symmetry $C \cong \widehat{G}$ as they provide a map $C \times G \to \mathrm{U}(1)$ that is injective upon restriction of C to the independent charges. As before, we can also deduce the analogue of the equal time commutator:

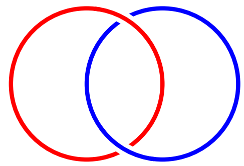
$$U_g(\mathcal{M}^{(d-q-1)})\mathcal{V}(\mathcal{C}^{(q)}) = R_{\mathcal{V}}(g)^{(\mathcal{C}^{(q)},\mathcal{M}^{(d-q-1)})}\mathcal{V}(\mathcal{C}^{(q)})U_g(\mathcal{M}^{(d-q-1)}) \tag{3.18}$$

Respect to the standard symmetry (3.10), we introduced a factor due to the intersection number between the two manifolds. In the standard symmetry case this term was not needed as the linking of a dimension 0 can happen only in the trivial way, i.e. a crossing at the point. Rather, for higher dimensional objects, this is not true in general. As a matter of fact, from our discussion it is not clear if the two manifolds can be actually linked at all!

For instance two knots (strings) in $d = 4$ are always unlinked, meaning that we can always unknot them, while in $d = 3$ they can be linked non trivial ways (Fig 9).Luckily, in our case this question have a really neat answer due to the Alexander Duality[17] $H^p(M) \cong H_{d-p-1}(\mathbb{R}^d/M)$



(a) Unknotted $L = 0$



(b) Linked $L = 1$

Figure 9: Knots in $d = 3$

that allows us to generalise the concept of linking number from knots in $d = 3$ to p and q dimensional (orientable) manifolds embedded in an $\mathbb{R}^d$ manifold satisfying the relation: $d = p + q + 1$. But this is exactly our case as: $\underbrace{(q)}_{\mathcal{C}^{(q)}} + \underbrace{(d-q-1)}_{\mathcal{M}^{(d-q-1)}} + 1 = d$. Let us sketch how you can define it for two (compact, connected, oriented and without boundary) manifolds $\mathcal{M}^{(p)}$ and $\mathcal{N}^{(q)}$[39, 17] included in $\mathbb{R}^d$. The inclusion map: $i : \mathcal{N} \to \mathbb{R}^d / \mathcal{M}$ induces on the homology:

$$i_* : H_q(\mathcal{N}) \to H_q(\mathbb{R}^d / \mathcal{M}) \cong H^p(M) \cong \mathbb{Z} \tag{3.19}$$

where we used firstly Alexander duality and then $H^p(\mathcal{M}) \equiv \mathbb{Z}$ when $\mathcal{M}$ is orientable. Also $H_q(\mathcal{N}) \equiv \mathbb{Z}$ and therefore this i^* is characterized by a single integer ℓ that corresponds exactly to the linking number $\ell = (\mathcal{M}, \mathcal{N})$ between the two manifolds. To see that one can check that it reduces in $d = 3$ to the usual definition and to the linking number of spheres S^p and S^q that can be computed by other means([32] and [39] for a thorough discussion). As a result, not only (3.18) is well defined but also the linking number is **topological** as defined by maps on the (co)homology, so that we do not have to worry about deforming the topological operators. As before, once the Ward Identity and the equal time commutator are formulated in terms of topological operators, nothing depends on the conserved current and we can extend them to the case of discrete symmetry group which are by far the most common cases of Higher Form Symmetries. In particular, the linking property hold in the form of (3.18). In Tab 1 a dictionary standard $\longleftrightarrow$ higher form symmetries.

3.2.1 1-Form Symmetries of U(1) gauge theory

One of the most enlightening examples of theories possessing Higher Form symmetries is given by the $d = 4$ Maxwell Free U(1) gauge theory:

$$\mathcal{L} = -\frac{1}{4} F^{(2)} \wedge \star F^{(2)} \qquad F^{(2)} = \mathrm{d}A^{(1)} \tag{3.20}$$

	Standard symmetries	Generalised Symmetries
Parameter	$\alpha(x)$	$U_g(\mathcal{M}^{(d-q-1)})$
Charged operator	$\mathcal{O}(x)$	$A^{(q)}$ supported on $\mathcal{C}^{(q)}$
Transformations	$\mathcal{O}(x) \to \mathcal{O}(x) + \alpha(x)$	Crossing of U_g and $\mathcal{C}$
Noether theorem	$\partial_\mu j^\mu = 0$	$U_g(\mathcal{M}^{(d-q-1)})$ topological
Ward identity	$\mathrm{d} * j_a(x)\, \mathcal{O}(y) = \delta(x-y) T_a\, \mathcal{O}(y)$	$U_g \mathcal{V}_\rho = \exp(ig\rho)\mathcal{V}_\rho$
Commutator	$[Q, \mathcal{O}(x)] = -iT_a\, \mathcal{O}(x)$	$U_g \mathcal{V} = R_{\mathcal{V}}(g)^\ell \mathcal{V} U_g$

Table 1: Dictionary standard-higher form symmetries

. With $A^{(1)}$ being an U(1) gauge field. The equations of motion and the Bianchi identity imply that there are 2 U(1)-valued independent and closed 2-form currents:

$$\text{E.o.m}: \qquad \mathrm{d} * F^{(2)} = 0 \qquad \Longrightarrow \qquad j_e^{(2)} = F^{(2)} \tag{3.21}$$

$$\text{Bianchi}: \qquad \mathrm{d}F^{(2)} = 0 \qquad \Longrightarrow \qquad j_m^{(2)} = *F^{(2)} \tag{3.22}$$

Therefore theory has the one form symmetry $\Gamma^{(1)} = \mathrm{U}(1)_e^{(1)} \times \mathrm{U}(1)_m^{(1)}$ generated by the topological (Gukov-Witten [16]) operators supported on $d-1-1=2$ dimensional spheres[4]:

$$U_g^{(e)}(S^{(2)}) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} *j_e^{(2)}\right) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} *F^{(2)}\right) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} E_\perp\right) \tag{3.23}$$

$$U_g^{(m)}(S^{(2)}) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} *j_m^{(2)}\right) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} F^{(2)}\right) = \exp\left(\frac{ig}{2\pi}\oint_{S^{(2)}} B_\perp\right) \tag{3.24}$$

The charged defects under these symmetries are line ($q=1$) operators. But we know what those operators are! They are exactly the Wilson W and 't Hooft T lines (and mixings thereof) we studied in the previous chapter. For a Dyonic defect $L_{n_e,n_m}(\mathcal{L}) = W^{n_e} T^{n_m}$ the linking rule explicitly gives:

$$U_g^{(e)}(S^{(2)}) L_{n_e,n_m}(\mathcal{L}) = e^{\frac{ign_e}{2\pi}} L_{n_e,n_m}(\mathcal{L})\,, \qquad U_g^{(m)}(S^{(2)}) L_{n_e,n_m}(\mathcal{L}) = e^{\frac{ign_m}{2\pi}} L_{n_e,n_m}(\mathcal{L}) \tag{3.25}$$

As $U_g^{(e)/(m)}$ measure the electric/magnetic flux through the sphere $S^{(2)}$ that is proportional to the electric/magnetic charge of the ∞-mass particle generating the worldline $\mathcal{L}$. It is very interesting to look at this from a Lagrangian point of view. Let us focus on the electric 1-form symmetry. One can be tempted to say that the action of this symmetry on the field is exactly given by a shift $A^{(1)} \to A^{(1)} \to \Lambda^{(1)}$, with $\Lambda^{(1)}$ being a flat connection (closed 1-form) as this is indeed a symmetry of the action. Yet, this would be premature as the theory is a gauge theory and such a shift could be reabsorbed by a gauge transformation $A^{(1)} \to A^{(1)} + \mathrm{d}\alpha^0$. Hence, to actually check the action of the symmetry on the fields, we firstly need to fix a gauge, say $A_0 = 0$ (following[6]). Now, consider the simple case of a Gukov-Witten operator defined on a constant x_3 plane Σ. Then, then only relevant component in the flux of $*F$ on this surface is $(*F) \subset \partial_0 A_3\, \mathrm{d}x_1 \wedge \mathrm{d}x_2$, which is nothing but the the conjugate momentum to A_3. Therefore the operator $\exp(i\alpha\partial_0 A_3)$ implements the translation by $\alpha^{(0)}$ of the $\mathrm{d}x_3$ component of the

[4]In computing the pullbacks we recall that the operators are inserted at a fixed time.

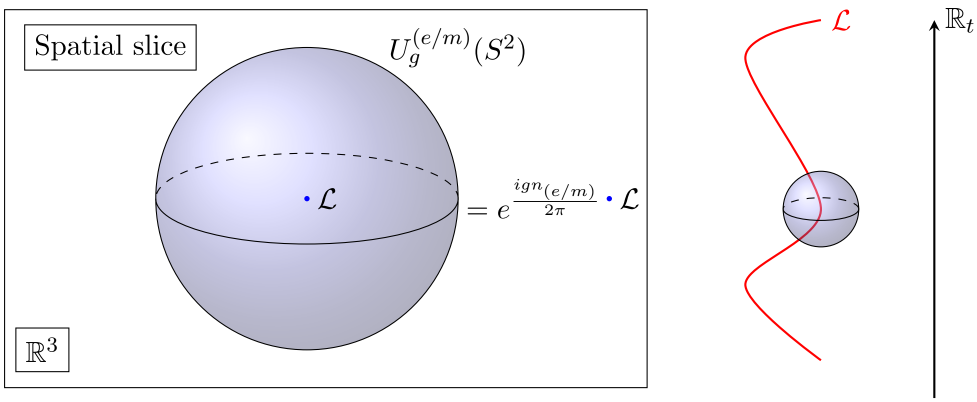


Figure 10: Gukov-Witten operators in a 4d Maxwell theory are inserted in the 3 dimensional spatial slide where a line defect corresponds to a point.

connection (the *delta*'s follow from the definition of Σ):

$$U_\alpha(\Sigma)\, A^{(1)}\, U_\alpha^{-1}(\Sigma) = A^{(1)} + \alpha\, \delta(t)\delta(x_3)\, \mathrm{d}x_3 \tag{3.26}$$

Thus, the 1-form symmetry indeed acts by shifting the fields by a flat connection (a 1-form symmetry parameter) that cannot be reabsorbed into a Gauge transformation (we have already fixed the gauge). Deriving explicitly the action of a magnetic 1-form symmetry is not possible in this Lagrangian description as the electric and magnetic connections cannot made explicit simultaneously. Still, one can appeal directly to the electromagnetic duality, or explicitly choose a purely magnetic base, where connections are *magnetic* connections and 't Hooft lines are simply generated by their holonomy. In this basis, the situation is completely analogous: the magnetic connection is shifted by a flat connection by the action of $F^{(2)}$.

3.2.2 Adding Matter

So far charged operators were supported on manifolds without boundaries. Let us generalise this supposing that there are **endable** charged operators $\mathcal{V}(\mathcal{C}^{(q)})$. We note straight away that this is not always the case[18]: for instance in a pure YM theory Wilson lines supported on open curves are not operators of the theory not being gauge-invariant. Assuming that there are such endable operators in the spectrum, we can consider the linking between one of this operators $\mathcal{V}(\mathcal{C}^{(q)})$ and a topological operator implementing the q-form symmetry $U_g(\mathcal{M}^{(d-q-1)})$. Given that $\mathcal{C}^{(q)}$ has boundaries, we can unlink it from $\mathcal{M}^{(d-q-1)}$ via a deformation of the latter (Fig 11). But, since the U_g is topological:

$$U_g(\mathcal{M}^{(d-q-1)})\, \mathcal{V}(\mathcal{C}^{(q)}) = R_{\mathcal{V}}(g)\mathcal{V}(\mathcal{C}^{(q)}) = U_g(\widetilde{\mathcal{M}}^{(d-q-1)})\, \mathcal{V}(\mathcal{C}^{(q)}) = \mathcal{V}(\mathcal{C}^{(q)}) \tag{3.27}$$

this results in a contradiction unless $R_{\mathcal{V}}(g) = \mathrm{Id}$, . Hence, in presence of endable operators, higher-form symmetries are broken down to their subgroup where all endable operators links trivially with the topological operators. As the title of the section suggest, this has non-trivial consequences on non-pure gauge theories with matter in a given representation of the gauge group. In fact, in presence of matter fields, line operators ending on matter field in their same

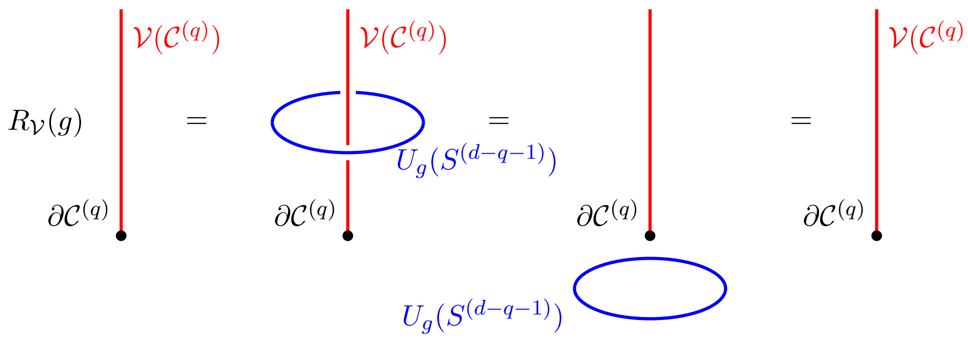


Figure 11: Unlinking of endable operator

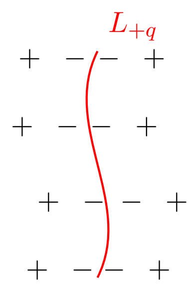


Figure 12: Screened line

representation become genuine gauge invariant operators and we are in the situation we have just discussed: Higher-form Symmetries are broken down to the subgroup that act trivially on all the matter fields. Furthermore, this argument, constrains the kind of fusions we may have among line defects in a theory with a 1-form symmetry: namely we can only have network of line operators, such that the charge at every vertex is "conserved", meaning that independently of where I contract the topological operator before and after the intersection vertex, the phase must be the same. Interacting **U(1)** gauge theory To see the physical interpretation of this, we go back to the U(1) theory.

But this time we take the interacting theory with matter fields in the representation with abelian charge q. The U(1) is none but quantum electrodynamics where we know that the insertion of a Line with charge $+q$ induces a polarization of the vacuum pairs that screens the line, see Fig 12. As a result, the line acquires a position-dependent effective charge that is eventually zero at very long distances. Hence, we expect the Gukov-Witten operators to become non topological (the charge depends on the position) and therefore a breaking of the symmetry. Furthermore, we also expect lines having charge integer multiple of q to become uncharged under the broken symmetry as they are completely screened by the charged matter. To see that explicitly, consider a small deformation $\mathcal{M} \to \widetilde{\mathcal{M}}$ with $\partial\mathcal{N} = \mathcal{M} \cup \widetilde{\mathcal{M}}(\mathrm{d} * F^{(2)} = *J_{\text{matter}})$:

$$\exp\left(ig \oint_{\mathcal{M}^{(2)}} *F^{(2)}\right) \exp\left(-ig \oint_{\widetilde{\mathcal{M}}^{(2)}} *F^{(2)}\right) = \exp\left(ig \int_{\mathcal{N}} \star J_{\text{matter}}\right) = \exp(igq) \tag{3.28}$$

Thus, the operators are topological if and only if: $g = \frac{2\pi k}{q}, \quad k \in \mathbb{Z}$, i.e. we have the electric

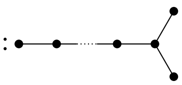
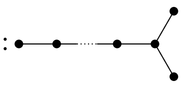
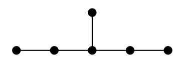
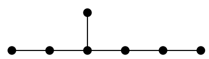
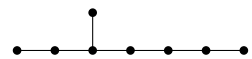
Algebra		Center	Algebra		Center
$\mathfrak{su}(N)$	A_{N-1}:	$\mathbb{Z}_N$	$\mathfrak{sp}(2N)$	C_{2N}:	$\mathbb{Z}_2$
$\mathfrak{so}(4N)$	D_{2N}: 	$\mathbb{Z}_4$	$\mathfrak{so}(4N+2)$	D_{2N+1}: 	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{so}(2N+1)$	B_N :	$\mathbb{Z}_2$	$\mathfrak{e}_6$		$\mathbb{Z}_3$
$\mathfrak{e}_7$		$\mathbb{Z}_2$	$\mathfrak{e}_8$		$\{e\}$
$\mathfrak{g}_2$		$\{e\}$	$\mathfrak{f}_4$		$\{e\}$

Table 2: Center of the simply connected group relative to the 9 Lie algebra series.

one form symmetry explicit breaking: $\mathrm{U}(1) \longrightarrow \mathbb{Z}_q$. Indeed, lines with charge q, and any local fusion thereof, are uncharged under this symmetry. We may also considered more involved examples: for instance take a theory with one form symmetry $\mathrm{U}^{(1)} \times \mathrm{U}^{(1)}$, adding matter with opposite charge in the two groups results in the broken *diagonal* $\mathrm{U}^{(1)}(1)$ 1-form symmetry group.

3.2.3 1-Form Symmetries of non-Abelian gauge theories

We have now all the tools to discuss 1-form symmetries of non-Abelian gauge theories. We avoid, for now, the non simply connected groups (e.g. $\mathrm{SU}(N)/\mathbb{Z}_N$, $\mathrm{SO}(N)$ ...), so take only simply connected groups G (e.g. $\mathrm{Spin}(N), \mathrm{SU}(N)$). We have already studied the spectrum of the line operators of the theory finding that the electric lines are in bijection with the center of the gauge group $Z(G)$. This suggests that the theory has a $1-$form symmetry valued in the Pontryagin dual of the center of the group $\widehat{Z}(G)^{(1)}$. To show that, we can interpret it as the screening effect due to adding matter to the $\mathrm{U}(1)_e^{(1)}$ symmetry that shift the non-abelian connection by a flat abelian (higher form symmetries must be abelian) connection. In fact, pure non-Abelian Gauge theories have matter fields (*gluons*) valued in the Adjoint representation of the group. According to what discussed in section 3.2.2, they screen the symmetry to the subgroup that acts trivially on the Adjoint. The Adjoint rep $Ad_g(h) = g \circ h \circ g^{-1}$ is only defined up to elements of the center $\mathcal{Z}(G)$ (that commutes with all elements). Then, the subgroup of U(1) that acts trivially on Adjoint is exactly the Pontryagin dual of the center $\mathrm{Hom}(\mathcal{Z}(G), \mathrm{U}(1)) := \widehat{\mathcal{Z}}(G)$. We conclude that pure non-abelian gauge theories have an electric 1-form symmetry $\widehat{\mathcal{Z}}(G)^{(1)}$. Again, the physical interpretation is straightforward: the electric charge of a Wilson line in the representation R can only be measured up to gluon screening. In Table 2 a list of the centers and therefore of the of the one form symmetry of the simply connected groups relative to the 9 Lie algebra series. For instance, in our favourite example of $\mathrm{SU}(N)$ YM, there is a $\mathbb{Z}_N$ symmetry under which the Wilson lines are charged. The charge of a Wilson line under is the charge under the center that is the the N-ality of the representation (see Tab 13): the number of boxes $\mod N$ of the Young tableaux of the irrep (the center acts on each box as $e^{2\pi i q/N}, q = 0, \cdots, N-1$).

The story is analogous e.g. for $\mathrm{Spin}(N)$ groups (simply connected with $\mathfrak{so}(N)$ algebra) and all the others. We have already discussed the fate of 1-form symmetries once matter is added to the theory: adding matter fields with N-ality $k_1, \cdots, k_n$ explicitly breaks the $\mathbb{Z}_N^{(1)} \to \mathbb{Z}_K^{(1)}$ with $K = \gcd(k_1, \cdots, k_n)$. In the next chapter we will explain how to obtain all the other

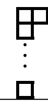
Rep	Tableaux	N-ality
$\underline{\mathbf{N}}$		1
$\underline{\mathbf{S^2}}$, $\underline{\mathbf{\Lambda^2}}$	 , 	2
$\underline{\mathbf{Adj}}$		N

Figure 13: N-ality of some SU(N) irreps

non-simply connected theories via gauging of a subgroup of the 1-form symmetry. For now, let us just do the example of a PSU(N) YM theory that will be useful later in this chapter. The spectrum of the 't Hooft lines is $\mathbb{Z}_N$, while the electric lines are only the trivial ones. Hence, we expect (and it will be the case) this theory to be the first example of a theory with a magnetic $\mathbb{Z}_N$ 1-form symmetry and no electric 1-form symmetry.

3.2.4 Holography and 1-form symmetries

According to the very well known AdS/CFT conjecture (that complete description is beyond our scopes), Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ with N unit of five-form $F_5 = \mathrm{d}C_4$ flux on S^5 is dual to $\mathcal{N} = 4$ SYM with gauge **algebra** $\mathfrak{su}(N)$. On the gauge side, we know that theory posses a 1-form symmetry, which group depends on the center of the specific global version of $\mathfrak{su}(N)$(all the matter fields are in the adjoint in $\mathcal{N} = 4$). In this section we want to discuss how how this symmetry is matched through holography[1]. The key is the topological Chern-Simon term in type IIB[2](C_2 is the RR 2-form and B_2 is the NS-NS B-field):

$$S_{IIB} \supset \frac{1}{4\pi} \int_{\mathrm{AdS}_5 \times S^5} F_5 \wedge B_2 \wedge \mathrm{d}C_2 \tag{3.29}$$

Given that we are interested in the 4 dimensional dual theory, we take the dimensional reduction on S^5 of (3.29). By expanding the fields around the background and integrating out the S^5 modes, among other terms, we obtain the topological term containing the fluctuations of the B and C fields[37]:

$$\frac{N}{4\pi} \int_{\mathrm{AdS}_5} b_2 \wedge \mathrm{d}c_2 \tag{3.30}$$

This implies that b_2 and c_2 are flat (EoM) and that their components, once quantized, are canonically conjugated variables $[(b_2)_{\mu\nu}(x), (c_2)_{\rho\sigma}(y)] = -\epsilon_{\mu\nu\rho\sigma} \frac{2\pi i}{N} \delta(x-y)$ (the antisymmetric symbol comes from the $\wedge$ product). Furthermore, shifting any of the two form by a two form λ such that $N\lambda$ is pure gauge, leaves the topological action invariant. So, b_2 and c_2 are really classes in $H^2(\mathcal{M}, \mathrm{U}(1))$ subjected to the property of being defined only modulo N. By taking their flux over compact 2-manifolds, we can construct the **topological** operators $U_b(\mathcal{M}^{(2)})$ and $U_c(\mathcal{M}^{(2)})$ which are not mutually local as a result of the two fluxes being canonically conjugated. This is one of the *non commuting fluxes* situations considered in Appendix F of [13]). Having constructed the topological surface operators, we now proceed to identify the objects dual to line operators of the $4d$ theory living on $\partial\mathrm{AdS}_5$. To do that, we just need to consider manifolds with boundaries $\partial\mathcal{M}^{(2)}_\gamma = \gamma \in \partial\mathrm{AdS}_5$. Ba doing so, $U_b(\mathcal{M}^{(2)}_\gamma)$ corresponds exactly with an operator supported on the worldsheet of a fundamental $F1$-brane (electric B

field). But this is exactly the object conjectured to be dual to the Wilson Loop $W[\gamma]$ in the $4d$ theory[11, 31][5]. Analogously, $U_c(\mathcal{M}^{(2)}_\gamma)$ is dual to the a 't Hooft lines γ on ∂AdS_5 as it comes from the worldsheet of a *magnetic* D1-brane. This shed a new light on the non-mutual locality condition, corresponding in the bounded manifold case with the non-mutual locality of the full set of (Wilson + 't Hooft lines) before imposing the constraints of Chapter 1. Hence, in order to have a consistent theory, we must impose boundary conditions (Dirichlet or Neumann) on the operators b_2 and c_2 such that the resulting spectrum of lines in $4d$ is coherent with one of the global version of $\mathfrak{su}(N)$. For instance, choosing Dirichlet on ∂AdS_5 for b_2 and Neumann for c_2 gives the SU(N) theory with topological operators U_c and *electric lines* U_b with linking implemented by the non-mutual locality of U_c and U_b, while the opposite gives the PSU(N) theory with topological operators U_b and *magnetic lines* U_c. To prove that the one generated by the topological operators is indeed the symmetry dual to the 1-form center symmetry of $\mathfrak{su}(N)$, we still need to show that the topological operators define $\mathbb{Z}_N$ gauge fields as the other global version will just restrict this condition. Indeed, a stack of N F1-brane with one extremum on ∂AdS_5 can terminate on an D5-brane wrapped on S^5 (the so-called *Baryon vertex*) that is integrated out in the effective theory (for details cfr. [2, 38]); an analogous holds for the magnetic counterpart. Hence, N bounded $U_{b/c}$ are screened by the D5 branes and therefore the topological symmetry generated by the U's corresponds to a $\mathbb{Z}_N$ 1-form symmetry. Of course upon *mixed* choices of boundary conditions we get the global SU(N)/$\mathbb{Z}_K$ theory.

3.3 Spontaneous symmetry breaking

Ordinary global symmetry may be spontaneously broken (SB); this is captured by a local operator (*Landau order parameter*) gaining a non-zero vacuum expectation value $\langle\mathcal{O}(x)\rangle$. Whenever the spontaneously broken symmetry is continuous then in the IR we have one Nambu-Goldstone boson for each broken generator. We also recall the Mermin-Wagner theorem stating that continuous symmetries cannot be broken in $d \leq 2$. Instead whenever a discrete symmetry is SB, the low energy description of a theory has degenerate vacua (*superselection sectors*). The situation is completely analogous for Higher Form Symmetries[28, 13, 20] that can be Spontaneously Broken as well. Hence, following the Landau paradigm, Higher Form symmetries furnish an important tool to characterize low-energy phases of gauge theories. Here, the order parameter is the large volume behaviour of the charged operators. The set-up is very similar to the rectangular Wilson Loop we considered in Chapter 1 Fig **??**. Namely, we take the expectation value of charged defect supported on a *large* compact[6] dimension-q surface $\mathcal{V}(\mathcal{C}^{(q)})$. We might have two distinct situations. The VeV can exhibit:

- an **area law**, i.e. it scales with the volume of the $q+1$-dimensional region $\mathcal{A}^{(q+1)}$ enclosed by the compact surface $\mathcal{C}^{(q)}$. Then, $\langle\mathcal{V}(\mathcal{C})\rangle \sim e^{-T\,\text{Area}} \to 0$ upon sending its size to infinity and the symmetry is *unbroken*. The defects have a non-zero tension T in the large surface limit (before taking the infinite limit). A line operator exhibiting an area law is the benchmark for confining vacua as we saw in Chapter 1, so that in this case we say the the theory is in a **confinement** phase.

[5] To be more precise, in the non-effective theory, a string in $\text{AdS}_5 \times S^5$ defining a contour on ∂AdS_5 is dual to the supersymmetric Wilson Loop that we can think of the result of dimensional reduction from $\mathcal{N}=1$ susy in $10d$.

[6] Endable operators cannot be charged under an higher symmetry.

- or a **perimeter law** (or even milder), i.e. it scales with the q-dimensional surface area of the defect itself: $\langle \mathcal{V}(\mathcal{C})\rangle \sim e^{-\text{Perimeter}}$. This time, the *perimeter* divergence can be reabsorbed by the introduction of a local counter term on the defect proportional to the surface element [6], so that $\langle \mathcal{V}(\mathcal{C})\rangle \neq 0$ and the symmetry is spontaneously broken. Here, the defects are tensionless and fluctuate at all scales and we say that the phase the theory is in a **deconfined** phase.

More generally, we can have a SSB $G^{(q)} \to H^{(q)} \subset G^{(q)}$ when all **but** the defects charged under the subgroup $H^{(q)}$ exhibit a perimeter law, while the others scale with their area.

3.3.1 SSB of discrete 1-form symmetry

We promised that Symmetries highly constrain the dynamics of gauge theories. Indeed, 1−Form Symmetries breaking are very interesting and rich to study as these new order parameters allow us to define and probe *confinement* in gauge theories. Vacua that preserve the 1-form symmetry must be confining vacua, while vacua where the symmetry is spontaneously broken are deconfining. So, 1-form symmetries provide a very powerful diagnostic of confinement. Take as an example a SU(N) gauge theory. Here, we have a $\mathbb{Z}_N$ one form symmetry and Wilson lines W_e^n with electric charge $n_e \in \mathbb{Z}_N$. Each of these operator can exhibit a perimeter or area scaling law, specifically the **minimum** L such that $\langle W^L\rangle \neq 0$ define a confinement index that regulates the SSB $\mathbb{Z}_N \to \mathbb{Z}_L$. More interestingly, 1-form symmetries allow us to go beyond the realm of pure SU(N) theories. Consider the case of PSU(N), where there are no purely electric dyonic lines, but rather only $(n_e, n_m) = (p, 1)^{n_m}$ for $n_m \in \mathbb{Z}_N$ and $p = 0, \cdots, N-1$ the discrete theta angle parameter. It is very well known that magnetic monopoles condensate in the vacuum of the theory[1]. To identify what is the broken subgroup, we note that for each p the line with $n_m = K = \frac{N}{\gcd(p,N)}$ has a trivial electric component (is a power of W^N) but a non trivial magnetic one, so it must behave as a monopole and therefore exhibit a non-zero VeV. Hence the magnetic symmetry is spontaneously broken to $\mathbb{Z}_K$ and the theory realises a phase with L degenerate vacua.[7]

3.3.2 Higher Nambu-Goldstone Theorem

When the spontaneously broken symmetry is continuous, we have an generalisation of the Nambu-Goldstone Theorem, i.e. the SSB af a q-form symmetry results in a gapless q-form (q-spin) Goldstone mode. This massless excitations arises as the transverse diffeomorphisms of the tensionless p-defects populating the deconfined phase of the theory (VeV = 0 in the vacuum). Let us sketch the proof (adapted with some modifications from [28]). To see that the spectrum has a massless excitation we can consider the quantity:

$$\mathcal{K} = \langle 0| \left[\mathcal{Q}(\mathcal{S}^{(d-q-1)}), \mathcal{V}(\mathcal{C}(q))\right] |0\rangle \neq 0 \tag{3.31}$$

that is not zero upon choosing that $\mathcal{S}^{(d-q-1)}$ links $\mathcal{C}(q)$, where $|0\rangle$ is the vacuum state of the spatial Hilbert Space. This quantity can be readily proven to be independent of the time

[7] There could be also a phase in which are dyons HW^k that condensate (opposite to magnetic monopoles) and analogously we easy can compute the unbroken symmetry to be $Z_{N/\gcd(p-k,N)}$[13].

insertion and the charge can be rewritten as an integral over the entire space slice as:

$$\mathcal{Q} = \oint_{\mathcal{M}^{(d-1)}} *j^q \wedge \delta^{(q-1)}(\mathcal{S}^{(d-q-1)}) \tag{3.32}$$

that is now simply the pullback of a zero form. Hence, we can proceed by inserting a complete set of states and, after integrating over the spatial slice we get:

$$\mathcal{K} = \sum_n (2\pi)^{d-1}\delta(p)\left(\langle 0|*j^{(q)} \wedge \delta^{(q-1)}|n\rangle \langle n|\mathcal{V}|0\rangle e^{-iE_n t} - \langle 0|\mathcal{V}|n\rangle \langle n|*j^{(q)} \wedge \delta^{(q-1)}|0\rangle e^{iE_n t}\right)$$

that, being non vanishing, is time independent if and only if the theory contains an excitation with $E = 0$ at zero momentum energy[8]. The Goldstone bosons are the p-form fields that shift linearly under the symmetry. Furthermore since higher form symmetries are always abelian, in the IR theory they can only be coupled through an effective abelian gauge theory kinetic terms:

$$S \subset \int_{\mathcal{M}^{(d)}} F^{(q+1)} \wedge *F^{(q+1)} \tag{3.33}$$

as a direct higher-form generalisation of the standard abelian Maxwell kinetic term.

Continuous 1-Form symmetries breaking In this case particular case, the expectation value of Wilson Loops is the order parameter for the 1-Form symmetry breaking. When the symmetry is spontaneously broken the spin-1 Goldstone boson can only couple through a standard Maxwell Kinetic Term in the IR theory. This means that the low energy phase is populated by free propagating spin-1 bosons: photons! This means that we can interpret a free Mawell theory in the IR as the low energy realisation of a spontaneously broken continuous 1-form symmetry in the UK with the photon being the associated Goldstone boson[28, 13].

3.3.3 Higher Coleman-Mermin-Wagner

Quite interestingly, there is generalization of the Coleman-Mermin-Wagner theorem to higher-form symmetries[13]. Namely: continuous $q-$form symmetries in d dimension (at finite temperature) are never broken when $d-2 \leq q$. Of course for $q = 0$ we recover the standard result. When we consider discrete symmetries, then the critical dimension is shifted by 1: $d - 1 \leq q$ (For instance in the $2d$ Ising model we have a SSB of the $\mathbb{Z}_2^{(0)}$ spin-flipping global symmetry indeed). We will not discuss the proof of the theorem that only consists in showing that for $d - 2 \leq q$ the order parameter expectation value would have a (non renormalizable) logarithmic IR divergence (due to the traverse modes $\equiv$ Goldstone boson) unless unbroken (cfr. [28]). An as application we can deduce that in $d = 3$ there is no confined/deconfined phase transition.

[8]Here we have assumed that there are no other massless excitations

4 RG flow of Line Operators

5 Appendix A: Line Operators

5.1 Spectrum of lines

Local correlation functions are not sensible to the global structure of the gauge group and the local *point-like* operators observables are completely blind to any topological feature. Yet, they are not the only operators of gauge theories. There are also extended operators supported on p-dimensional manifolds, which are inherently not local. In this section we discuss **line operators**: operators localised on 1-dimensional manifolds that preserve all the space-time symmetries preserved by the line[21]. They are of two distinct classes: Wilson Lines and 't Hooft Lines, but can mix into more generic Dyonic Lines. Being labelled by representations of the group, each Wilson Line operator can be associated by an highest weight vectors of the algebra that are in bijective correspondence with elements of the Weyl chamber: $\Lambda_w/\mathcal{W}$[1, 3][9]. Still, this does not imply that all the representation of the algebra are realised as Wilson Line.

Wilson operators are not the sole line operators. It is natural to consider their electromagnetic duals: the **'t Hooft Lines**. These *magnetic* operators are defined to be curves with a Dirac monopole singularity of the ambient Gauge Field along them. Namely, this means that a 't Hooft line insertion imposes to perform the path integral over configurations subjected to the Dirac quantization conditions of the Dirac monopole singularity along the line:

$$e^{im\cdot H} = 1 \quad H \in Z(\tilde{G}) \quad \Rightarrow \quad m \in \Lambda_{cw}/\mathcal{W} \tag{5.1}$$

Where the $2\pi \cdot 1$ of the abelian Dirac quantization, is "replaced" in the non-abelian case by an element of the Center $Z(\tilde{G})$ of the simply connected group $\tilde{G}$ uniquely determined [14] by the exponentiation of the algebra $\mathfrak{g}$ of G. In the dual electromagnetic basis, where the connection is a magnetic connection, the 't Hooft lines are nothing but the Holonomy of the magnetic connection. Yet, a basis in which they are both "diagonalized" does not exists and at least one of them have to carry the topological definition. As stated in (5.1), 't Hooft Lines are in correspondence with elements of the Weyl chamber of the **Langland dual** algebra ${}^\vee\mathfrak{g}$: $\Lambda_{cw}/\mathcal{W}$, with $\Lambda_{cw} \subset \mathfrak{t}$ being the co-weights lattice. [10][26, 5]. This is another manifestation of the electromagnetic duality: under Langlands duality $g \leftrightarrow {}^\vee\mathfrak{g}$, the lattices $\Lambda_w \leftrightarrow \Lambda_w$, and therefore Wilson and 't Hooft lines, are exchanged as well. [11] Analogously, 't Hooft Lines may be interpreted as worldline of magnetic monopoles and are magnetic probes. We can consider mixed Wilson-'t Hooft operators: the **Dyonic Lines** by simply taking the local fusion thereof. Their representations are labelled are labelled by [21]:[12]

$$(\lambda_e, \lambda_m) \in \mathcal{L} = (\Lambda_w \times \Lambda_{cw})/\mathcal{W} \tag{5.2}$$

As we mentioned earlier, not all these couples are in one-to-one correspondence with the realised of Dyonic Lines and indeed they are subjected to various constraints:

[9] Λ_w is the weight lattice contained in the dual $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R}/\mathbb{Z})$ of the cartan subalgebra $\mathfrak{t}$.

[10] We recall that the Langland dual algebra is the algebra generates by the co-roots ${}^\vee\mathfrak{g} = \frac{\alpha}{(\alpha,\alpha)}$

[11] In the case of U(1) it is nothing but $(F, \star F) \leftrightarrow (\star F, -F)$ that exchanges electric monopole of charge e with magnetic monopoles of charge $\frac{1}{e} = {}^\vee e$.

[12] One should note that this is more restrictive than $(\lambda_e, \lambda_m) \in (\Lambda_w/\mathcal{W}) \times (\Lambda_{cw})/\mathcal{W}$, we remand to the reference [21] for details.

- Allowed lines form a commutative algebra, meaning that if (ε, μ) and (ε, μ) are allowed lines then also $(\varepsilon + \varepsilon', \mu + \mu')/\mathcal{W}$ is allowed. This follows from considering the local fusion of two line operators: generically, the resulting line sits in general in a reducible representation, decomposable in its irreducible content by an operator product expansion that can be shown to always contain the highest weight representation $(\varepsilon + \varepsilon', \mu + \mu')/\mathcal{W}$[22].
- Gauge fields are defined on the adjoint representation of $\tilde{G}$; hence, electric operators corresponding to them are surely allowed lines of the $\tilde{G}$ theory being their parallel transport. Since the weights of the adjoint are the roots, the weight lattice of **inequivalent** lines (obtainable up to summing others) is restricted modulo the root lattice $\Lambda_r^{\tilde{G}}$. By Langland duality, the exact analogous holds for the magnetic ones, defined only up to the co-root lattice Λ_{cr}. Therefore the Lattice of the $\tilde{G}$ is contained in:

$$\mathcal{L}^{\tilde{G}} \subset (\Lambda_w^{\tilde{G}}/\Lambda_r^{\tilde{G}} \times \Lambda_{cw}^{\tilde{G}}/\Lambda_{cr}^{\tilde{G}})/\mathcal{W}^{\tilde{G}} \cong \mathrm{Hom}(Z(\tilde{G}), \mathbb{R}/\mathbb{Z}) \times Z(\tilde{G}) \tag{5.3}$$

$$\Rightarrow \mathcal{L}^{\tilde{G}} \subset \widehat{Z}(\tilde{G}) \times Z(\tilde{G}) \qquad\qquad \widehat{Z}(\tilde{G}) = \mathrm{Hom}(Z(\tilde{G}), \mathbb{R}/\mathbb{Z}) \tag{5.4}$$

  It will be important for the following that for abelian groups $\widehat{G} \cong G$.
- As we have already stated, [14], the Lie Group G is given in terms of $\tilde{G}$ trough the quotient by an unique $\mathcal{Z} \subset Z(\tilde{G})$. Therefore, $Z(G) = Z(\tilde{G})/\mathcal{Z}$ with extension:

$$0 \to \mathcal{Z} \to Z(\tilde{G}) \to Z(G) \to 0 \tag{5.5}$$

  Purely electric lines (Wilson Lines) must correspond to representations of G, i.e. they sit in $Z(G) \subset Z(\tilde{G})$. We have also the group-theoretic relation $\Lambda_{cw}^{\tilde{G}}/\Lambda_{cr}^{\tilde{G}} = \mathcal{Z}$. From which we deduce the lattice to sit in the extension:

$$0 \to \widehat{Z}(G) \to \mathcal{L} \to \mathcal{Z} \to 0 \tag{5.6}$$

- A further constraint is given by the mutual locality imposed by requiring that the phase accumulated by a total braiding of two dyonic lines is zero. This is a generalization the non-abelian version of the Dirac quantization condition, the Dirac-Schwinger-Zwanziger condition:

$$\mathbb{Z} \ni \langle (\lambda, \mu), (\lambda', \mu') \rangle := \langle \lambda, \mu' \rangle + \langle \lambda', \mu \rangle = 0 \mod Z(\tilde{G}) \tag{5.7}$$

$$\langle \cdot, \cdot \rangle : \widehat{Z}(\tilde{G}) \times Z(\tilde{G}) \to \mathbb{Z} = \mathrm{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \tag{5.8}$$

  Where with $0 \mod Z(\tilde{G})$ means that it is zero as an element of the center, e.g. if $Z(\tilde{G}) = \mathbb{Z}_N$ then (5.7) becomes: $\langle (\lambda, \mu), (\lambda', \mu') \rangle = 0 \mod N$.

Summarising, the line operators lattice is defined by the two extensions (5.6) and (5.5) regulated by the condition(5.7). More precisely, the lattice is given the pullback:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{Z}(G) & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle Id} & & \big\uparrow & & \big\downarrow{\scriptstyle \eta} & & \\ 0 & \longrightarrow & \widehat{Z}(G) & \longrightarrow & \widehat{Z}(\tilde{G}) & \longrightarrow & \widehat{\mathcal{Z}} & \longrightarrow & 0 \end{array} \tag{5.9}$$

Note that the function $\eta : \mathcal{Z} \to \widehat{\mathcal{Z}} \cong \mathrm{Hom}(\widehat{\mathcal{Z}}, \mathrm{U}(1))$, seen as $\eta : \mathcal{Z} \times \widehat{\mathcal{Z}}$ is exactly the quadratic refinement $\sigma : \mathcal{Z} \times \widehat{\mathcal{Z}} \to U(1)$ of the Lagrangian coupling of the Brauer class, starting to establishing the correspondence with the bundle description in (**??**). With all this information, we can write down explicitly the lattice. For $G = \prod G_k = \prod \tilde{G}_k/\mathcal{Z}_k$, the lattice is specified by set of roots

$$(n_1, \cdots, n_N; m_1 \cdots, m_N) \equiv (n_k, m_k) \in (\widehat{Z}(G_1) \oplus \cdots \oplus \widehat{Z}(G_N)) \times (\mathcal{Z}_1 \oplus \cdots \oplus \mathcal{Z}_N) \quad (5.10)$$

and we can take all $\mathcal{Z}_N$ finite order groups $\mathbb{Z}_{a_k}$ as this is the case for all the Lie Algebras. We normalise the pairing so that the locality condition is:

$$\langle (n_k, m_k), (n'_k, m'_k) \rangle = \sum_k \frac{n_k m'_k - n'_k m_k}{a_k} = 0 \mod 1 \quad (5.11)$$

Then, the multiplication for magnetic charges is killed by the division by $a_k \mod 1$. Therefore, the condition that the electric line n_i in the chamber relative to $\mathcal{Z}_j$, n_i^j (defined modulo k_j) must satisfy is $\frac{n_i^j}{k_j} - \frac{n_j^i}{k_i} = 0 \mod 1$ that has solutions [1]:

$$\frac{n_i^j}{k_j} = \frac{n_j^i}{k_i} = \frac{m_{ij}}{\gcd(k_i, k_j)} \qquad \begin{cases} m_{ij} = 0, 1, \cdots, \gcd(k_i, k_j) & i \neq j \\ m_{ii} = 0, 1, \cdots, k_i - 1 & i = j \end{cases} \quad (5.12)$$

These exactly saturates the independent (non-instantonic) bundle classification, showing that it is indeeed maximal.

Overall, we found that, while the Lie Algebra determines completely the spectrum of local operators, it does not fully determine the line operators which spectrum depends on the global structure of the group. As anticipated, the global structure of the gauge group matters and determined the allowed line operators of the theory. Furthermore, the lattice and the discussion on the θ-angles are fully related, but we still need to wait until Chapter 3 to fully disclose the connection.

The presence of a S_θ term in the action produces an interesting phenomenon on dyonic lines: the **Witten effect** Lines. This accounts to:

$$\theta \to \theta + 2\pi \qquad \Longrightarrow \qquad (\varepsilon, \mu) \to (\varepsilon + \mu, \mu) \quad (5.13)$$

This resemble the Witten effect of Wilson-'t Hooft monopoles stating that any magnetic monopole with charge m, in presence of a S_θ term, gains an electric charge $q = \frac{\theta}{2\pi} m$ (more generally for a dyonic monopole $(q, m) \to (q + 2\pi m, m)$)[36, 15]. Yet, the generalization of the latter to dyonic lines is not trivial. A formal proof of (5.13) can be found in [19] where it is showed that the 't Hooft lines, under $\theta \to \theta + 2\pi$ undergo a monodromy transformation $T_R(\gamma) \to T_R(\gamma) W_R(\gamma)$ while $W_R(\gamma) \to W_R(\gamma)$, which is exactly the content (5.13).

5.1.1 Lattice of Unitary theories

In this section we want to apply our general discussion to the case of gauge theories with unitary algebras. The discussion can be repeated without any difference for any other group [1, 3]. U(1) group As a warm up, we start by considering the very simple case of an U(1)

Gauge theory. On a non-Spin manifold the maximal lattice is readily determined to be:

$$\mathcal{L} = \{\left((n + \frac{\theta}{2\pi}e)m, m\frac{2\pi}{e}\right), \quad n, m \in \mathbb{Z}\} := \{(n,m), \quad n,m \in \mathbb{Z}\} \tag{5.14}$$

Where e is the abelian coupling if the theory is Lagrangian, or a reference charge U(1) charge of the theory. We introduce a convenient notation: since the lines are generated by the two fundamental lines $(1,0) = W$ and $(0,1) = T$, we indicated the general line given by: $W^n T^m$ meaning the generic line (5.14). $\mathfrak{su}(N)$ **algebra** Consider now the less trivial example of a $\mathfrak{su}(N)$ algebra. The simply connected $\tilde{G} = \mathrm{SU}(N)$ has $Z(\mathrm{SU}(N)) = \mathbb{Z}_N$. The non-trivial subgroups of $\mathbb{Z}_N$ are $\mathcal{H} = Z_M$ with $M|N$[13], and therefore we can consider gauge theories with any gauge group $\mathrm{SU}(N)/\mathcal{H}$.

$\boldsymbol{G = \mathrm{SU}(N)}$ When the gauge group is $\mathrm{SU}(N)$, the maximal set of lines $\mathcal{L} \subset \mathbb{Z}_N \times \mathbb{Z}_N$ comprises all the purely electric lines $(n,0), \quad = 0, \cdots, N-1$ that correspond to allowed Wilson Lines. Mutual locality imposes that they are the all and sole dyons as $\langle (n,0),(0,m) = nm \mod N = 0 \quad \forall n = 1, \cdots N$ implies that the only possible m is $m = 0$, i.e. the 't Hooft lines have charges $kN, k \in \mathbb{Z}$, corresponding with the adjoint holonomy of the magnetic gauge connection in the magnetic dual picture. In this case we know that the bundles of the theory are simply the instantonic bundles.

$\boldsymbol{G = \mathrm{PSU}(N)}$ Here, $Z(G) = 1$. Then, the only purely electric line is $(0,0)$ (again only the adjoint lines). For a fixed $n = 0, \cdots, N-1$, any line $(n,1)$ is allowed by locality and therefore all the lines (nm, m), $m = 0, \cdots, N-1$ are in the spectrum.[14]. If $(n,1)$ is in the spectrum then all the other lines $(n' \neq n, 1)$ are forbidden by (5.7). Hence, there are N different theories $(SU(N)/\mathbb{Z}_N)_n$ labelled by an integer $n = 0, \cdots, N-1$ and they are all related by a shift of the *theta* angle via Witten effect:

$$(nm, m) \mapsto (nm + m \mod N, m) = ((n+1)m, m) \tag{5.15}$$

This means that under a shift of θ-angle:

$$\theta \to \theta + 2\pi : (SU(N)/\mathbb{Z}_N)_n \mapsto (SU(N)/\mathbb{Z}_N)_{(n+1) \mod N} \tag{5.16a}$$

So that a 2π shift of θ is not trivial on the bundles:

$$(SU(N)/\mathbb{Z}_N)_n^{\theta+2\pi} = (SU(N)/\mathbb{Z}_N)^{\theta}_{(n+1) \mod N} \tag{5.16b}$$

Hence, in a $SU(N)/\mathbb{Z}_N$ gauge theory the θ angle is not 2π periodic but has an enlarged $2\pi N$ periodicity:

$$(SU(N)/\mathbb{Z}_N)_n^{\theta+2\pi N} = (SU(N)/\mathbb{Z}_N)^{\theta}_n \tag{5.17}$$

This means that an $\mathrm{PSU}(N) = SU(N)/\mathbb{Z}_N$ gauge theory is made of N distinct non-instantonic bundles and indeed has a a non-trivial Brauer class $w_b \in H^2(\mathcal{M}, \mathbb{Z}_N)$. In this case, the bundles can be described by enlarging the periodicity of theta $\mathrm{PSU}(N)$. As an example consider

[13] $M|N$ means M is a divisor of N.

[14] At this point, one should be careful to treating separately the cases N generic and N prime; yet the conclusion in either cases are the same [1]

SO(3) = SU(2)/$\mathbb{Z}_2$: there are two bundles SO(3)$_-$ and SO(3)$_+$ with two different spectrum of lines. At the level of the bundle this is the consequence of the non-trivial Stieffel-Witney class $w_2 \in H^2(\mathcal{M}, \mathbb{Z}_2)$ of SO(3) bundles that is expected as SO(3) does not admit spinorial representations.

$\boldsymbol{G = \mathbf{SU}(N)/Z_K}$ Since $K|N$, call K' the integer such that $KK' = N$. Then the allowed purely electric lines are $(n_e K \mod N, 0)$ for every integer n_e. Mutual locality constraints dyons to be $n_e(K,0) + n_m(n, K') \mod N$, with $n_m \in \mathbb{Z}$ and $n = 0, \cdots, K-1$ fixed. As before, we have a family of bundles parametrized by n: $(\mathrm{SU}(N)/\mathbb{Z}_K)_n$ and related by:

$$\big(n_e(K,0) + n_m(n,K')\big) \mod N \to n_e(K,0)+n_m(n+K', K' \mod K, K') \mod N \quad (5.18a)$$

As $n + K' = n'$ is again constrained to be $n' = 0, \cdots, K-1$:

$$(SU(N)/\mathbb{Z}_K)^{\theta+2\pi}_n = (SU(N)/\mathbb{Z}_K)^{\theta}_{(n+K') \mod K} \quad (5.18b)$$

Yet, this is qualitatively different from the previous case. In fact, starting from a given n and shifting the θ angle we can only reach theories with $n' = n \mod (\gcd(K, K'))$. This means that if $\gcd(K, K') = 1$ I can simply extend the θ angle periodicity. But when N is not square-free, i.e. some of its prime factors appear more than once in its prime decomposition, then we might have $\gcd(K, K') \neq 1$ and we have distinct orbits under shifts of θ. In this case, enlarging the periodicity of θ is not enough to cover all the configurations and the theory has discrete θ-angles. As an example of this is SU(4)/$\mathbb{Z}_2$ = SO(6). We can easily verify the theory has 4 bundles, yet neither (SU(4)/$\mathbb{Z}_2$)$_0$ and (SU(4)/$\mathbb{Z}_2$)$_1$ nor (SU(4)/$\mathbb{Z}_2$)$_2$ and (SU(4)/$\mathbb{Z}_2$)$_3$ are connected through a θ periodicity, while $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$ are. Therefore the theory has 2 distinct theta angles θ_1 and θ_2 both with 4π periodicity. Again, at the bundle level this is the result of a non trivial $w_b \in H^2(\mathcal{M}, \mathbb{Z}_4)$.

6 Appendix B:Gauging and Anomalies

Exactly how standard global symmetries can, in some situations, be promoted to gauge symmetries, higher form symmetries can be gauged as well. As we will see in this chapter, not only this modifies the global structure of the gauge theory but also produce new global symmetries. Yet, it is not always possible to gauge a global symmetry: there may be obstructions to gauging, the *'t Hooft anomalies.* These anomalies are very powerful to study being protected under the renormalization group action and part of this chapter is dedicated to discussing them.

6.1 Coupling to background Gauge Fields

In this section we describe how a theory possessing an higher form global symmetry $G^{(q)}$ can be coupled to flat (closed) non-dynamical background gauge fields for the symmetry. As always, let us start by considering the simple case of a continuous higher form global symmetry. Here, the intuitive idea is generalising the standard *minimal coupling* prescription by activating

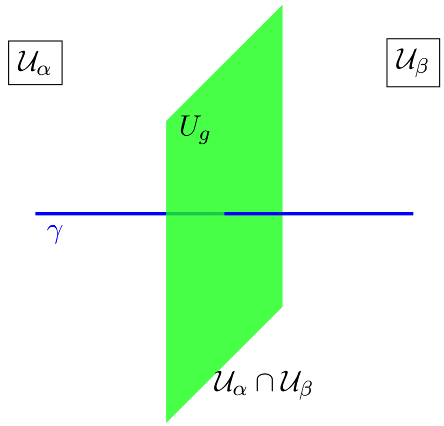


Figure 14: Symmetry defect at the junction between patches and a cycle γ linking with it.

$(q+1)$-gauge fields acting as sources for the current:

$$S \subset \int_{\mathcal{M}^{(d)}} B^{(1)} \wedge \star j^{(1)} \longrightarrow S \subset \int_{\mathcal{M}^{(d)}} B^{(q+1)} \wedge \star j^{(q+1)} \tag{6.1}$$

The correspondence between symmetry defects and background gauge fields is canonical, meaning that the two descriptions are equivalent. Namely, we can realise the correspondence between a topological defect $U_g(\mathcal{M}^{(d-q-1)})$ and a background field $B^{(q+1)}$ as the assignment of a (higher dimensional) holonomy:

$$\exp\left(i \oint_{\gamma^{(q+1)}} B^{(q+1)} \right) = g \in G \tag{6.2}$$

for each $(q+1)$-cycle of $\mathcal{M}^{(d)}$. In other words, coupling to background gauge field $B^{(q+1)}$ is equivalent to stretching a network of topological operators $U(\Sigma)$ such that the homology classes of $\Sigma \in H_{d-q-1}(\mathcal{M}^{(d)}, G)$ over which they are supported are Poincare' dual to $B^{(q+1)} \in H^{(q+1)}(\mathcal{M}^{(d)})$. Just for a sake of clarity, take the $q = 0$ case: here implementing a global symmetry accounts to stretching a network of domain walls that, in turns, induces a partition of the manifolds in a collection of d-dimensional patches $\mathcal{U}_\alpha$. Their $(d-1)$-dimensional intersections correspond exactly to the support of the symmetry operators $U_g(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. In other words, a symmetry defect network is none but the assignment of a set of transition functions $g_{\alpha\beta}$ between the patches $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$. Then we define the background gauge fields by imposing the condition (6.2) for each 1-cycle linking with the $d-1$ intersection. All of this remains true for $q > 0$. Going back to the general (continuous) discussion, by performing the *gauge transformation* $B^{(q+1)} \to B^{(q+1)} + \mathrm{d}\Lambda^{(q)}$:

$$\int_{\mathcal{M}^{(d)}} \delta_\Lambda B^{(q+1)} \wedge \star j^{(q+1)} = \int_{\mathcal{M}^{(d+1)}} \mathrm{d}\Lambda^{(q)} \wedge \star j^{(q+1)} = -\int_{\mathcal{M}^{(d+1)}} \Lambda^{(q)} \wedge \mathrm{d} \star j^{(q+1)} = 0 \tag{6.3}$$

we deduce that background gauge fields are cycle only defined up to exact forms, i.e. they are classes in the Cohomology of $\mathcal{M}^{(d)}$ with elements in G:

$$B^{(q+1)} \in H^{q+1}(\mathcal{M}^{(d)}, G^{(q)}) \tag{6.4}$$

At this point we can detach the discussion from conserved currents taking this as the definition of background gauge field for higher form symmetries. There is a very neat way of writing the linking between topological and charged operators, but one needs to be careful about the definitions. In fact, the linking between them only happens in the spatial slice at fixed time, rather than in the total space. Hence, for our purposes we must consider $[B] \in H^q(\mathcal{M}^{(d-1)}, G)$, i.e. as a q-cocycle of the spatial slice. For instance, in the case od a $(2+1)-$dimensional theory with a standard symmetry (e.g. 2d Ising model Fig[6]), the domain walls are 2-cocycle entirely contained in the spatial slice, so that the linked γ cycle over which the $B^{(1)}$ holonomy is taken, is a 1-cycle in $\mathcal{M}^{(3)}$ but only a 0-cycle (a point) in space. Analogous considerations hold in the higher-form case. At this point, we can easily write the linking making use of Poincare' duality pairing (implementing $H^k(X^d) \cong H_{d-k}(X^d)$), provided that the charged defects are associated with elements in the Homology $\beta \in H_{d-q-1}(\mathcal{M}^{(d-1)}, G)$ as:

$$U(\alpha)V(\beta) = \langle \alpha, \beta \rangle \, V(\beta), \qquad \alpha \in H^q(\mathcal{M}^{(d-1)}, G) \tag{6.5}$$

with $\langle \alpha, \beta \rangle$ being the the Poincare' duality pairing of the spatial slice $\mathcal{M}^{(d-1)}$ (already incorporating the linking number). To see that the dimension of the charged defects matches our expectations, we can use the universal coefficient theorem[17]:

$$H_{d-q-1}(\mathcal{M}^{(d-1)}, G) \cong H_q(\mathcal{M}^{(d-1)}, \widehat{G}) \tag{6.6}$$

Since the dual q-cocycle α was initially a $q+1$ cocycle of $\mathcal{M}^{(d)}$, then β is indeed a q-cycle of $\mathcal{M}^{(d)}$ with coefficients in $\widehat{G}$ according with the dimension on which the topological defects are supported.

6.2 't Hooft Anomalies

At this point background fields are strictly non-dynamical, i.e. the partition function of the theory depends explicitly on them $\mathcal{Z} = \mathcal{Z}[B^{(q)}]$. In order to promote the global symmetry to a gauge symmetry, we turn the background gauge fields into dynamical fields of the theory, i.e. we sum over them in the path integral: $\mathcal{Z} = \int \mathcal{D}_B \, \mathcal{Z}[B^{(q)}]$. Still, this is not always possible. As a matter of fact, global symmetries may be affected by 't Hooft anomalies. 't Hooft anomalies are quite tricky: they are neither anomalies of global symmetries: the symmetry is preserved at the quantum level (the Ward identity indeed holds), nor gauge anomalies which would be fatal for the theory itself as the symmetry is still a global symmetry (the backgrounds are non-dynamical) [15]. Generally, this anomalies arise as a global symmetry having a gauge redundancy on its backgrounds does not automatically imply that the partition function of the theory is invariant itself under this redundancy once we have coupled it to the background fields. Indeed, in general the theory shifts by a local functional: [27, 8, 9]

$$S[B^{(q)} + \Lambda^{(q)}] - S[B^{(q)}] = \int_{\mathcal{M}^{(d)}} \Sigma^{(d)}[B^{(q)}, \Lambda^{(q)}] \tag{6.7}$$

Whenever the *anomaly* $\Sigma[B^{(q)}, \Lambda^{(q)}]$ can be cancelled by the insertion of *local* and *gauge invariant* **counterterm** $\int_{\mathcal{M}^{(d)}} \mathcal{L}_{\text{Local}}[B, \Lambda]$ the symmetry is anomaly free. Yet, sometimes

[15] For standard continuous symmetries 't Hooft anomalies are detected through non vanishing *triangle diagrams* with global symmetry currents on all three vertices[35]

such a counterterm is not possible to write. This is the essence of a 't Hooft Anomaly: the symmetry is preserved at the quantum level but cannot be gauged as, upon gauging, a 't Hooft anomaly (6.7) would turn into a *gauge* anomaly and the theory would ceases to make sense. For this reason, we often say that 't Hooft Anomalies are **obstructions** to gauging global symmetries.

Anomaly matching The reason why we are interested in 't Hooft anomalies is that they impose important constraints on the structure of IR theories. We do not have space to discuss it here thoroughly, but the main point is that the 't Hooft anomalies satisfy **non-renormalization theorems** and are RG invariant quantities. This means that for a theory $\mathcal{T}_{\mathrm{UV}}$ with a global symmetry G_{UV}, the 't Hooft anomaly coupling κ_{UV} is rigid under the RG flow and does not depend on any coupling in the space of the parameters of the theory. This means that as we flow from the UV to the IR: $(\mathcal{T}_{\mathrm{UV}}, G_{\mathrm{UV}}) \rightsquigarrow (\mathcal{T}_{\mathrm{IR}}, G_{\mathrm{IR}})$, even though the degrees of freedom may be completely different, there must be a matching of the 't Hooft Anomaly at both scales, imposing very strong constraints on the dynamics of the IR theory. An historically relevant example is the chiral breaking anomaly matching in QCD-*like* theories giving crucial information on confinement in the IR.

$$\omega^{-1}(g_1, g_2, g_3 g_4)\, \omega^{-1}(g_1 g_2, g_3, g_4)\, \omega(g_2, g_3, g_4)\, \omega(g_1, g_2 g_3, g_4)\, \omega(g_1, g_2, g_3) = 1 \tag{6.8}$$

that is nothing by using the but the usual co-differential condition $\mathrm{d}\omega\,(g_1, g_2, g_3, g_4) = 0$. Hence, ω is a 3-cocycle in $H^3(\mathcal{B}G, \mathrm{U}(1))$. At this point we can generalise this firstly to d dimensions: $\omega \in H^{d+1}(\mathcal{B}G, \mathrm{U}(1))$ and ultimately to the higher-form case. To do that, we use again the isomorphism with the Eilemberg-MacLane homotopy classes and, with an analogous derivation, we deduce that 't Hooft anomalies of q-form symmetries $G^{(q)}$ are classified by the cohomology classes[24, 25]:

$$\omega \in H^{d+1}(\mathcal{B}^{q+1}G, \mathrm{U}(1)) \tag{6.9}$$

This has a very neat interpretation as the result extending the spacetime $\mathcal{M}^{(d)}$ to another manifold $\mathcal{N}^{(d+1)} \supset \mathcal{M}^{(d)}$ obtained by gluing to the d-simplex describing the junction, a new $(d+1)$-simplex that modify the operators' fusion on its boundary (cfr. Fig15 the $(2+1)-$dimensional case. Anomalous $(d+1)$-dimensional theory on $\mathcal{M}^{(d)}$ are classified by the means of non-anomalous $(d+1)$-dimensional theory bulk theory on $\mathcal{N}^{(d+1)}$ with $\partial\mathcal{N}^{(d+1)} = \mathcal{M}^{(d)}$. In the Lagrangian picture, we are saying that there is a gauge-invariant non anomalous and topological $\mathcal{Z}[B] = \exp\big(i \int \mathcal{J}^{(d+1)}[B]\big)$ (an *Invertible field theory*) in $d+1$ dimensions that reproduces the anomaly of the original d-dimensional theory as a boundary term (an *anomaly inflow* [10]):

$$\exp\left(i \int_{\mathcal{N}^{(d+1)}} S[B_i^{(q)} + \Lambda_i^{(q)}]\right) - \exp\left(i \int_{\mathcal{N}^{(d+1)}} S[B_i^{(q)}]\right) = \exp\left(i \int_{\mathcal{N}^{(d+1)}} \mathrm{d}\Sigma^{(d)}[B_i^{(q)}, \Lambda_i^{(q)}]\right)$$

Determining the **anomaly polynomial** $\mathcal{I}^{(d+2)}[B_i^{(q)}]$ that reproduces the anomaly of the d dimensional theory reduces to the cohomology problem:

$$\mathrm{d}\Sigma^{(d)}[B_i^{(q)}, \Lambda_i^{(q)}] = \delta_{\Lambda_i} \mathcal{J}^{(d+1)}[B_i^{(q)}], \qquad \mathcal{I}^{(d+2)}[B^{(q)}] = \mathrm{d}\mathcal{J}^{(d+1)}[B^{(q)}] \tag{6.10}$$

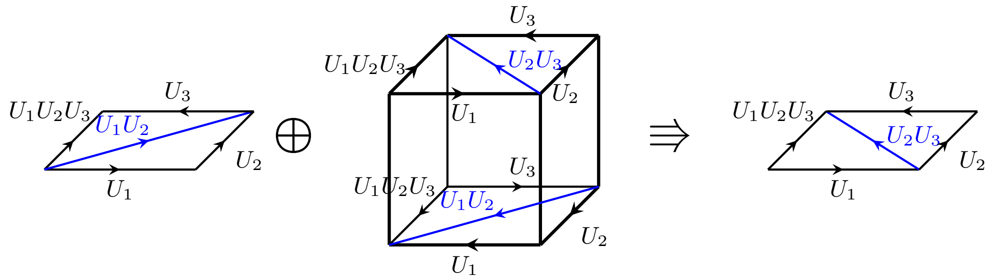


Figure 15: Attaching a $D+1$ simplex has the effect of changing the fusion algebra.

The concept of anomaly polynomial is a familiar one: for instance the axial anomaly in 4 dimension $*J_A \sim c_2(F(2)) \sim F^{(2)} \wedge F^{(2)}$ can be interpreted as the addition of the Lagrangian term in 4+2 = 6 dimensions $S_A = \int \mathrm{d}B_A^{(1)} \wedge c_2(F^{(2)})$ ($B_A^{(1)}$ background gauge field of the Axial symmetry) that, on shell, gives exactly the non-conservation of the axial current. Writing the anomaly polynomials for a given theory basically accounts for writing all the possible **topological** terms ($\mathrm{d}\mathcal{I} = \mathcal{J}$) of the theory in 6 dimensions (many of them will be zero or not-independent (e.g. the gauge anomaly term must be trivial) (for a thorough review cfr. chapter 2 [7].The fact that anomalous theories are described in terms of **non-anomalous** Topological Field Theories in one dimension higher represents a touch-point with condensed matter physics and in the last decade a lot of progress has been made in the study of SPT phases whose formalism is exactly described in this terms.

6.2.1 Mixed 't Hooft anomalies

There is a very subtle type of 't Hooft anomalies. Namely, whenever we have a symmetry $G_1^{(p)} \times G_2^{(q)}$ (note that q and p can be different) we may have that separately the two symmetries are non-anomalous but, whenever I try to couple both of them to background gauge fields, the total symmetry develops a 't Hooft Anomaly. In this case we say that $G_1^{(p)}$ and $G_2^{(q)}$ have a **mixed** 't Hooft anomaly. They represent an obstruction to gauge the symmetries simultaneously, but it is not an obstruction to gauge them separately, i.e. I could gauge the former but not the latter and the gauged theory would remain perfectly meaningful. These anomalies are everywhere in QFT!

Maxwell theory For instance, in the U(1) Maxwell theory in 4 dimensions the electric and magnetic 1-form symmetries have a mixed 't Hooft Anomaly. Too see that, let us start by coupling with background fields for both the symmetries. From our general discussion we know that the backgrounds couple with the conserved currents as, i.e. we must have:

$$S \supset \int_{\mathcal{M}^{(4)}} \left(B_e^{(2)} \wedge *F^{(2)} + B_m^{(2)} \wedge F^{(2)} \right) \tag{6.11}$$

We must ensure gauge invariance $F^{(2)} \to F^{(2)} + \mathrm{d}\Lambda^{(1)}{}_e$, $B_{e/m}^{(2)} \to B_{e/m}^{(2)} + \mathrm{d}\Lambda^{(1)}{}_{e/m}$, that forces us to add to the minimal (6.11) the term $\int B_e^{(2)} \wedge *B_e^{(2)}$. Then, the total action, opportunely normalised, takes the form[34]:

$$S[A^{(1)}, B_e^{(2)}, B_e^{(2)}] = \frac{1}{2e^2} \int \left(F^{(2)} - B_e^{(2)}\right) \wedge * \left(F^{(2)} - B_e^{(2)}\right) + \frac{i}{2\pi} \int F^{(2)} \wedge B_m^{(2)} \qquad (6.12)$$

The key point is that under an electric 1-form symmetry gauge transformation, the last term in (6.12) is not invariant as $S \to S - \frac{i}{2\pi} \int \Lambda_e^{(1)} \wedge \mathrm{d}B_m^{(2)}$. To cancel this anomaly we can add the local counter-term $\int_{\mathcal{N}} B_e^{(2)} \wedge \mathrm{d}B_m^{(2)}$ defined on a $\mathcal{N}^{(5)}$ (with $\partial\mathcal{N}^{(5)} = \mathcal{M}^{(4)}$) (in accordance to our general discussion). Yet, there is **no** such local $d = 4$ counterterm: in fact, one might be tempted to integrate the $5d$ term by part and add it to the $d = 4$ theory $\sim \int_{\mathcal{M}} B_e^{(2)} \wedge B_m^{(2)}$ as this would effectively cancel the electric anomaly. But, as a result of this the theory is no longer invariant under a magnetic gauge transformation: $\delta S = \int \Lambda_m^{(1)} \wedge B_e^{(2)}$! This is the benchmark of a 't Hooft anomaly: if we had not introduced any background field for $\mathrm{U}(1)_m^{(1)}$ we would not have incurred in any anomaly (and vice-versa in the magnetic case by exchanging $e \leftrightarrow m$), but introducing background fields for both results in a mixed anomaly.

$\mathcal{N} = 1$ SYM Another interesting example is given by pure $\mathcal{N} = 1$ SYM theories. This theories have $h^\vee$ (dual coxeter number) distinct vacua; let us sketch why this is true very briefly[33]. The U(1) R-symmetry $\lambda \to e^{i\omega}\lambda$ is affected by an ABJ anomaly which computation is analogous to the axial anomaly in QED, with the only difference that, since the fermions are in the adjoint representation of the group, the trace over the *colour* dof dresses the axial anomaly $\frac{g^2}{16\pi^2} F \wedge F$ with a factor $2C_2(\mathrm{Adj}) \equiv 2h^\vee$ - equivalently one can restate the *Atiyah-Singer theorem* for Weyl fermions. This shift is reabsorbed via the transformation $\theta \to \theta + 2h^\vee\omega$ (the axial anomaly is proportional to $F \wedge F$), and therefore we have that the explicitly breaking $U(1) \to \mathbb{Z}_{2h^\vee}$ since the transformation $\lambda \to e^{2\pi i k/(2h^\vee)}\lambda$ gives a $2k\pi$ θ-angle shift that leaves the theory invariant. Furthermore, we also have that the quantity $\eta = \Lambda^{3h^\vee} \propto e^{i\tau}$ (τ complexified coupling) is invariant at all order in perturbation theory. Hence Λ transforms as $\Lambda \to \Lambda e^{2\pi i/(3h^\vee)}$ under R-symmetry (η is invariant). Hence, a mere R-charge counting implies that the VeV of the fermion bilinear must be of the form $\langle\lambda_\alpha\lambda_\alpha\rangle \propto \Lambda^3$ that implies that the theory spontaneously breaks $\mathbb{Z}_{2h^\vee} \to \mathbb{Z}_2$ acting as $\lambda \to -\lambda$. Thus, the theory has indeed $h^\vee$ degenerate vacua. In the special case of $\mathrm{SU}(N)$ $h^\vee = N$ and we also know that the 1-form symmetry is preserved (at least in the UV theory). The crucial point is that in the UV, where we have the Lagrangian description, this theory posses a mixed 't Hooft anomaly between the global zero form symmetry $\mathbb{Z}_{2N}^{(0)}$ and the one form symmetry $\mathbb{Z}_N^{(1)}$. As we stressed before, 't Hooft anomalies can also affect p and q higher symmetries with $p \neq q$ and this is an explicit example of this. To see that, we couple the theory with a background field for $\mathbb{Z}_N^{(1)}$ through the usual $F \to F + B$, as discussed in the abelian case. Hence, the action contains a term $\propto \theta \int B \wedge B$ signalling a mixed cubic 't Hooft anomaly with anomaly inflow term $\sim \int A^{(1)} \wedge B \wedge B$ [4] as $\mathbb{Z}_{2N}$ ($A^{(1)}$ its background) shifts $\theta \to \theta + 2\pi$. According to the general theory, we know that such a 't Hooft anomaly should be matched also in the IR where the R-symmetry is $\mathbb{Z}_2$, implying (see [13] for details) that the system must contain classical $\mathbb{Z}_N$ 2-form background gauge fields, i.e. an unbroken 1-from symmetry, from which we can infer that the N vacua are indeed confining, showing all the power inherited by studying higher symmetries and their anomalies!

6.3 Gauging

As anticipated, assuming that the symmetry is not anomalous, gauging an higher form symmetry $G^{(q)}$ accounts to promoting the background fields $B^{(q+1)}$ to dynamical fields of the theory, i.e. to summing over them in the path integral, schematically:

$$\begin{aligned} &\mathcal{Z}[B^{(q+1)}] = \sum_{\substack{\text{bundles} \\ B^{(q+1)}\text{fixed}}} \int \mathcal{D}_{[X^i]} \exp\Big(-S[B^{(q+1)}]\Big) \quad \xrightarrow{\text{gauging}} \\ &\Rightarrow \widetilde{\mathcal{Z}} = \int \mathcal{D}_B \sum_{\text{bundles}} \int \mathcal{D}_{[X^i]} \exp\Big(-S[B^{(q+1)}]\Big) \overset{G\,\text{discrete}}{=} \sum_{\substack{\text{bundles,} \\ B\in H^{q+1}(\mathcal{M},G)}} \int \mathcal{D}_{[X^i]} \exp(-S[B]) \end{aligned} \tag{6.13}$$

Very often the symmetry is discrete and therefore in the last line we have a sum over the value of the discrete connection. Gauging an higher form symmetry has the result of producing a new higher form symmetry in the gauged theory, analogous to the *quantum symmetry* of the discrete orbifold. As always, there are two ways of showing this. The first one is purely topological. Exactly as we have considered charged defects supported on manifolds with boundaries, we can also consider bounded topological operators $U_g(\Sigma^{(d-q-1)})$ with $\partial\Sigma^{(d-q-1)} = \gamma^{d-q-2}$. Then, the action of g is produced by the holonomy of the poincare' dual of $B^{(q+1)}$ on the spatial submanifold of $\gamma^{(d-q-2)}$ (Σ is inserted in the spatial slice so γ extends in one non space direction, where we should not take the holonomy as the Ward identity is at fixed time) each time that a charged operator links with it. Gauging the symmetry makes B dynamical and subsequently γ a charged defect: $\gamma \in H_{d-q-2}(\mathcal{M}^{(d)}, G)$ or $\equiv H_{d-q-1}(\mathcal{M}^{(d-1)}, G)$ in the spatial slice. Hence, the topological operators linking with γ in the spatial slice are valued in $H^{q-1}(\mathcal{M}^{(d-1)}, G) \cong H^{d-q-1}(\mathcal{M}^{(d-1)}, \widehat{G}) \equiv H^{d-q-1}(\mathcal{M}^{(d)}, \widehat{G})$ as γ was a cocycle extending in the time direction. Thus, gauging a q-form symmetry $G^{(q)}$ produces a new *magnetic* $(d-q-2)$-form global symmetry with values in the Pontryagin dual group $\widehat{G}^{(d-q-2)}$. In the Lagrangian description this follows from the fact that the kinetic term for the dynamical $G^{(q)}$ field must be of the *BF-like* form $\int\langle A, B\rangle \in \mathrm{U}(1)$ with $B \in H^{q+1}(\mathcal{M}^{(d)}, G)$ and $A \in H_{d-q-1}(\mathcal{M}^{(d)}, G)$; then, upon Poincare' duality and the universal coefficient theorem, we retrieve the same result, i.e. that the dynamical B field is coupled with a background field $A \in H^{d-q-1}(\mathcal{M}^{(d)}, \widehat{G})$ which is exactly a background for a $\widehat{G}^{(d-q-2)}$ symmetry [13].

6.3.1 Gauging 1-form symmetries in $\mathfrak{su}(N)$ bundles

Eq (6.13) resemble very closely the discussion of Chapter 1 on bundles in gauge theories. Of course this is not a coincidence as we have stressed many times that the global physics couples with extended defects and hence with the global symmetries thereof. There is a very neat connection of the two discussion that subsequently connect also the spectrum of the line operators with the bundles of the theories as we promised in Chapter 1. Namely, the key result is that we can construct **any** bundle with lie algebra g in $4d$, starting with the simply connected G gauge theory and gauging a subgroup of its electric 1-form symmetry. We explain this in the context of $\mathfrak{su}(N)$ bundles in 4 dimensions, but the discussion can be easily repeated for the other groups. As we have already explained, an $\mathrm{SU}(N)$ theory is divided in instantonic sectors but, being simply connected, has a fixed Brauer class $w_b \in H^2(\mathcal{M}^{(d)}, \mathbb{Z}_N) \equiv 0$. We can think of this last condition as being implemented by a Lagrange multiplier in analogy with what discussed at the end of section **??**. At the same time, in SU(N) bundle we can also introduce a background gauge field for the global $\mathbb{Z}_N^{(1)}$ symmetry: $B^{(2)}(\mathcal{M}^{(d)}, \mathbb{Z}_N)$. If we gauge the entire 1-form symmetry, the latter becomes a dynamical field that (see again section **??**) fixes $w_b \equiv B^{(2)}$ once the multiplier has been integrated out. We conclude that gauging the 1-form symmetry produces exactly the partition function of a PSU(N) bundle!

$$\mathcal{Z}^{\mathrm{SU}(N)}[B^{(2)}] \to \sum_{w_b \in H^2(\mathcal{M}, \mathbb{Z}_N)} \mathcal{Z}^{\mathrm{SU}(N)} \equiv \mathcal{Z}^{\mathrm{PSU}(N)} \tag{6.14}$$

Furthermore in 4 dimension, the gauging of a 1−form symmetry produces a new global $4-1-2=$ 1-form symmetry valued in $\mathbb{Z}_N$ that exactly matches the magnetic 1-form symmetry of PSU(N) bundles under which 't Hooft lines are charged. Alternatively, we could have started with a PSU(N) theory and by gauging the $\mathbb{Z}_N$ 1-form magnetic symmetry we would have obtained the SU(N) theory back. The discussion is very similar also for the general case: gauging any subgroup $\mathbb{Z}_K \subset \mathbb{Z}_N$ of the 1-form symmetry results in a SU(N)/$\mathbb{Z}_K$ theory.

Another, more explicit way of showing this, is making $\mathbb{Z}_K^{(1)}$ dynamical by minimally coupling the theory to a BF term for a **continuos** $\mathbb{Z}_K^{(1)}$ background to the theory, i.e. rather than describing the backgrounds with a discrete connection, we promote it to a continuos one[12]. In fact, the BF theory with continuos degrees of freedom:

$$\mathcal{L}_{BF} = \frac{iK}{2\pi} B^{(2)} \wedge \mathrm{d}\mathcal{A}^{(1)} - \frac{ipK}{4\pi} B^{(2)} \wedge B^{(2)}, \qquad \mathcal{A}^{(1)} \to \mathcal{A}^{(1)} + p\lambda^{(1)}, \quad B^{(2)} \to B^{(2)} + \mathrm{d}\lambda^{(1)}$$

and quantized integral fluxes is well-known to describe a $\mathbb{Z}_K$ gauge theory when the fields are dynamical (see [23] and Appendix C [13]) and therefore corresponds to a gauging of the background field - this is analogous to gauging a continuous symmetry: we minimally couple the theory and we introduce a kinetic term for the gauge connection. We restrict for simplicity to $K = N$ as the general case does not add particular flavour. To elucidate the emergent magnetic 1-form symmetry of the gauged theory, it is convenient to consider its *dualised* version $\mathcal{L} = -i\frac{1}{2\pi}F^{(2)} \wedge (\mathrm{d}\widehat{\mathcal{A}}^{(1)} - NB^{(2)}) - \frac{ipN}{4\pi} B \wedge B$ for a field $\widehat{A}$ having 1-form symmetry: $\mathbb{Z}_N$ $\widehat{A}^{(1)} \to \widehat{A}^{(1)} + N\lambda^{(1)}$. This theory by the virtue of the "*Lagrange multiplier*" $\mathcal{F}^{(2)} = \mathrm{d}\mathcal{A}^{(1)}$ reduces to the BF theory upon using the Equation of Motions. To couple it to the SU(N) theory, we take $\widehat{\mathcal{A}}^{(1)}$ to be a SU(N) connection (in general it is a U(N) connection but it reduces to a SU(N) one as a consequence of its gauge freedom) that we identify with $\widehat{\mathcal{A}}^{(1)} = A^{(1)}$ the gauge connection of the YM theory . Hence, the full gauge invariant action for the SU(N) theory coupled to the BF term has the action:

$$S = \frac{1}{4} \int (F^{(2)} - B^{(2)}) \wedge *(F^{(2)} - B^{(2)}) + \frac{i}{2\pi} \int \mathcal{F}^{(2)} \wedge (\mathrm{d}A^{(1)} - NB^{(2)}) - \frac{ipN}{4\pi} \int B^{(2)} \wedge B^{(2)} + S_\theta$$

As promised this theory has a $\mathbb{Z}_N^{(1)}$ magnetic symmetry corresponding to the symmetry of $\widehat{\mathcal{A}}^{(1)}$ and coincides with the PSU(N) theory as thoroughly explained in section 7 of [23]. Furthermore the θ term:

$$S_\theta = \frac{i\theta}{8\pi^2} \int (F^{(2)} - B^{(2)}) \wedge (F^{(2)} - B^{(2)}) \tag{6.15}$$

under the shift $\theta \to \theta + 2\pi$, using the equation of motion for $\mathcal{F}^{(2)}$ changes as:

$$\Delta S_\theta = \frac{i}{4\pi} \int F^{(2)} \wedge F^{(2)} - \frac{iN}{4\pi} \int B^{(2)} \wedge B^{(2)} \tag{6.16}$$

That is equivalent to $p \to p+1$. Furthermore, as a result of the dA-flux quantization ($\mathbb{Z}_N^{(1)}$magnetic symmetry) and the equation of motion of $\mathcal{F}^{(2)}$: $\frac{1}{4\pi}\int B \wedge B \in (2\pi\mathbb{Z})/N^2$. Hence, when $p \to p+N$, $\frac{iN}{4\pi}\int B^{(2)} \wedge B^{(2)} \to 2\pi N\mathbb{Z}$. Putting all together, this means that $S_\theta - \frac{iN}{4\pi}\int B^{(2)} \wedge B^{(2)}$ define a fractional theta angle $\theta_{\mathrm{PSU}(N)} = 2\pi p + \theta_{\mathrm{SU}(N)}$ with periodicity $2\pi N$: exactly as we expected from Chapter 1. In the *old* language used in that chapter, magnetic lines had values in the gauged subgroup which is nothing but the magnetic 1-form symmetry of the gauged theory. We started in Chapter 1 by studying the topology of gauge bundles and we have now completed the task of linking it to the genuine (higher dimensional) symmetries of the theory.

6.3.2 Time reversal anomaly

A very nice example Mixed 't Hooft anomaly has been studied in [12]: in pure SU(N) YM in 4 dimensions there is a mixed anomaly between the electric 1-form symmetry and the time reversal (equivalently CP as CPT is always a symmetry) $\mathcal{T} : t \to -t$. The YM coupling is invariant under $\mathcal{T}$ as the change in sign of t is reabsorbed by the change of sing of the oriented volume form inside $*$. Instead, the θ-term is not invariant as $F \wedge F \mapsto -F \wedge F$, i.e. the action of time reversal is $\mathcal{T} : \theta \to -\theta$. Therefore, for SU($N$) YM theory $\mathcal{T}$ is a symmetry at the specific values $\theta = 0$ and $\theta = \pi$. As follows from the discussion above, once we couple background gauge fields, the theta angle becomes $\theta' = [2\pi p + \theta] \in [0, 2\pi N], p = 0, \cdots, N-1$. However, given that the background is non-dynamical, we have the freedom of fixing a particular value of p since we are still describing a SU(N) bundle that strictly has a 2π periodic θ-angle. Now, at $\theta = 0$ the $\mathcal{T}$ symmetry is preserved as we can fix $p = 0$ and therefore the theory is not-anomalous at $\theta = 0$. Instead at $\theta = \pi$ the situation is different. In order to preserve $\mathcal{T}$, we must satisfy $\theta' = 0$ or $\theta' = N\pi$ (respectively equivalent to $\theta = 0$ and $\theta = \pi$). Given that $\theta'(\theta = \pi) = (2p+1)\pi$, we readily conclude that the former option is not achievable. For N even neither the latter is possible and therefore the theory has a mixed 't Hooft anomaly at $\theta = \pi$ for N even. The situation for N odd is trickier: in this case fixing $p = (N-1)/2$ makes the theory anomaly free at $\theta = \pi$. However, such a value of p is not consistent with $p = 0$ at $\theta = 0$ for $N > 1$. Hence an anomaly free theory at $\theta = \pi$ would imply a 't Hooft anomaly at $\theta = 0$: for N odd there is no consistent choice of p that preserves the time reversal at either places simultaneously. In [12] it is argued that lattice simulations push toward assuming that at $\theta = 0$ $\mathcal{T}$ is indeed preserved. Then, we must have a mixed 't Hooft anomaly at π even for N odd. Summarising, there is a mixed 't Hooft anomaly between the electric 1-form symmetry and time reversal at the value of $\theta = \pi$. More recently, by employing an anomaly-matching argument, this result has been used in [10] to show that the IR theory vacuum at $\theta = \pi$ must be either $\mathcal{T}$-breaking **or** confining **or** gapped, illustrating the power of 't Hooft anomaly matching.

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