The asymptotic weak gravity conjecture in F-theory & M-theory compactifications

String theory and mathematical physics journal club

Cesar Fierro Cota, Universität Hamburg

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Based on: [2208.00009] & [2212.09758] by CFC, A. Mininno, T. Weigand, Max Wiesner

Outline

1. Introduction
2. The asymptotic weak gravity conjecture
3. A geometrical dictionary for the emergence string conjecture
4. Counting of non-BPS states in five dimensions

The Swampland program

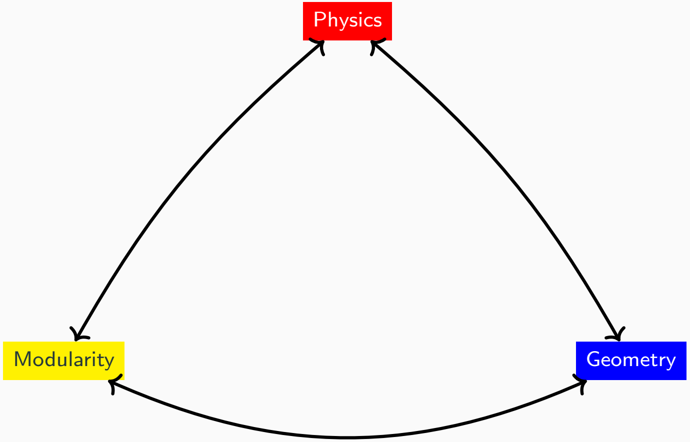
The swampland

The set of anomaly free quantum effective field theories that cannot be completed into quantum gravity in the ultraviolet.

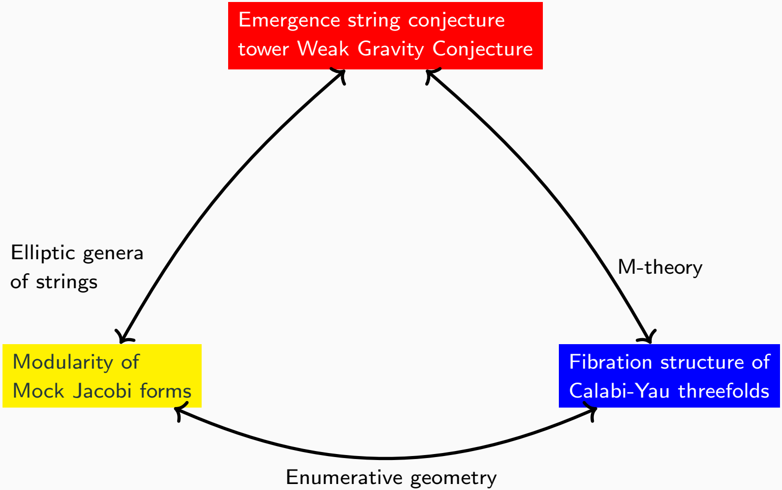
Motivation:

Find underlying principles for a theory of quantum gravity.

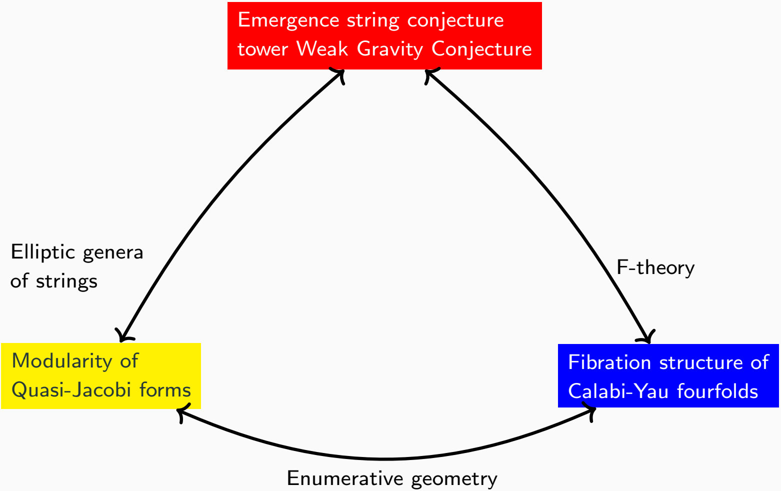
The quest for mathematical principles for quantum gravity



What is the asymptotic weak gravity conjecture about?

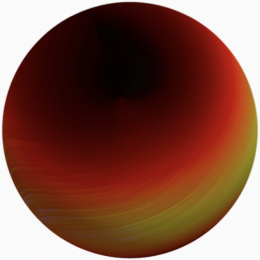


What is the asymptotic weak gravity conjecture about?



The weak gravity conjecture

Consider a $U(1)$ gauge theory coupled to gravity

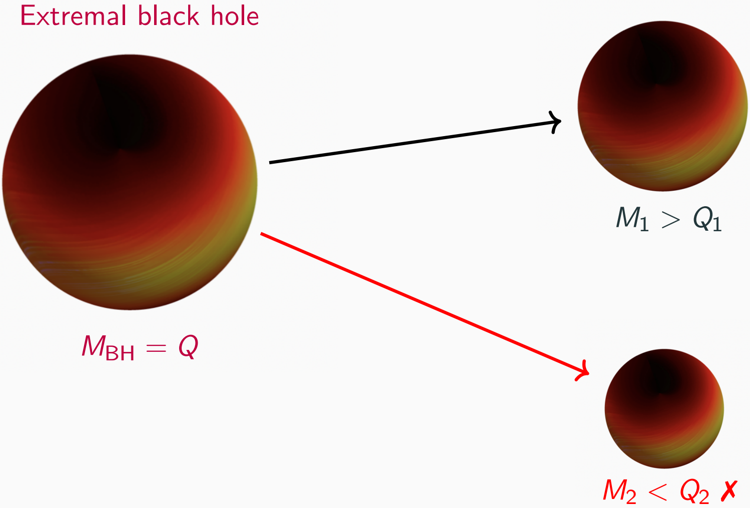


→ Black hole solutions of charge Q and mass M_{BH} subjected to:

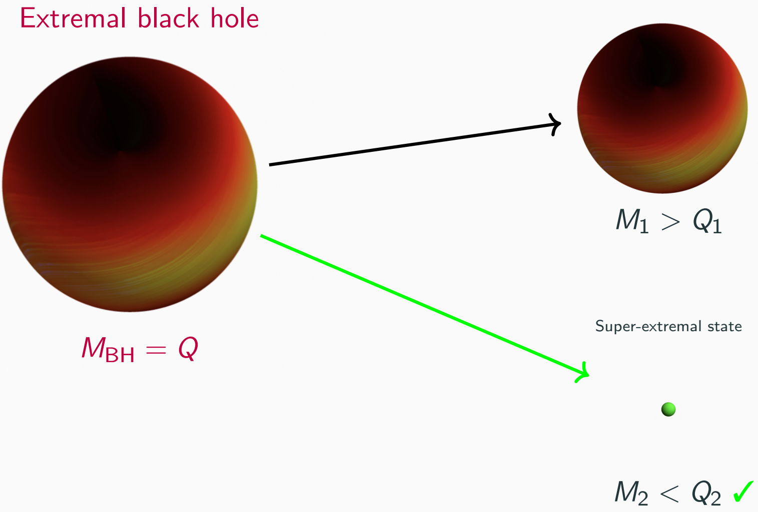
Extremality bound:

$M_{\text{BH}} \geq Q\,.$

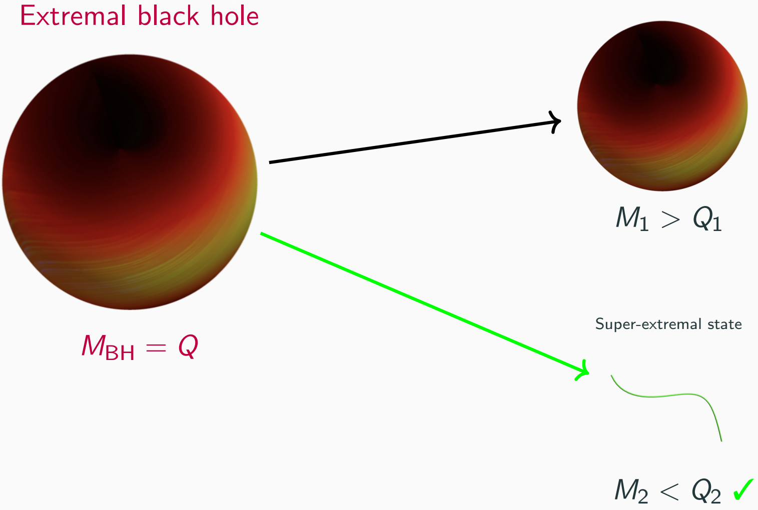
The weak gravity conjecture



The weak gravity conjecture



The weak gravity conjecture



The weak gravity conjecture (WGC)

There must exist a super-extremal state of charge q and mass m, such that

$$\frac{q}{m} \geq \frac{Q}{M_{\mathrm{BH}}}\bigg|_{\mathrm{ext}} . \tag{1}$$

Q: Black hole charge

M_{BH}: Black hole mass

[Arkani-Hamed,Motl,Nicolis,Vafa'06]

Principle: Extremal black holes should be allowed to decay.

The tower weak gravity conjecture

The WGC can be argued by the finiteness of entropy/absence of *remnants*, which imply a tower WGC version.

[Susskind'95][Arkani-Hamed,Motl,Nicolis,Vafa'06] [Hamada,Montero,Vafa,Valenzuela'21]

Need a WGC version for multiple $U(1)^r$ gauge factors, with $r > 1$.

[Cheung,Remmen'14]

Consistency required for a Kaluza-Klein reduction:

[Heidenreich,Reece,Rudelius'15]

S^1 ↘ $(d+1)$-dimensional $U(1)^r$ gauge theory : WGC ✓

d-dimensional $U(1)^{r+1}$ gauge theory : WGC ✗

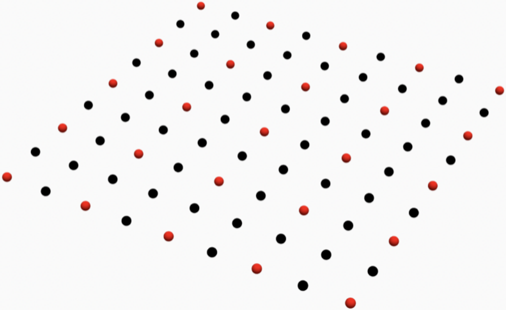
The tower weak gravity conjecture

The tower weak gravity conjecture (tWGC)

For every site Q of the charge lattice Λ_Q, there is a positive integer n such that there is a super-extremal state with charge nQ satisfying the WGC.

[Montero,Shiu,Soler'16][Heidenreich,Reece,Rudelius'16'17] [Andriolo,Junghans,Noumi,Shiu'18]

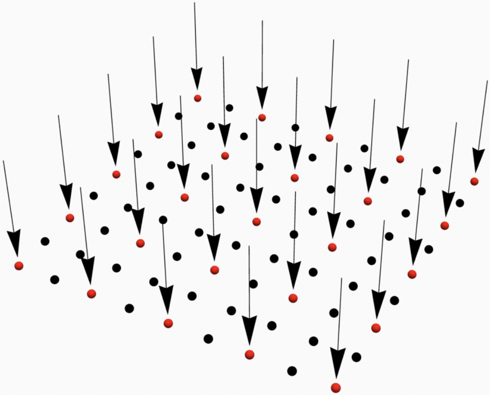
The tower weak gravity conjecture



Charge lattice: $\Lambda_Q = \{\bullet\} \cup \{\bullet\}$

WGC Sublattice: $\Lambda_{\text{ext}} = \{\bullet\}$

The tower weak gravity conjecture



Charge lattice: $\Lambda_Q = \{\textcolor{red}{\bullet}\} \cup \{\bullet\}$

WGC Sublattice: $\Lambda_{\text{ext}} = \{\textcolor{red}{\bullet}\}$

The tower weak gravity conjecture

Charge lattice: $\Lambda_Q = \{\textcolor{red}{\bullet}\} \cup \{\bullet\}$

WGC Sublattice: $\Lambda_{\text{ext}} = \{\textcolor{red}{\bullet}\}$

The tower weak gravity conjecture in M-theory

Consider M-theory on a Calabi-Yau threefold X_3.
$\Rightarrow$ five-dimensional effective theory with eight supercharges.

$U(1)^\alpha$s induced by reducing the M-theory 3-form C_3 into gauge potentials A^α that pair with harmonic forms $[\omega_\alpha]$ in $H^{1,1}(X_3, \mathbb{Z})$.

Charged objects:

- M2-branes wrapping curves $C \subset X_3 \Rightarrow$ BPS particles
- M5-branes wrapping divisors $D \subset X_3 \Rightarrow$ strings (non-BPS)

The tower weak gravity conjecture in M-theory

A tower for BPS particles was proposed by considering curve classes $[C]$ inside the movable cone $\mathrm{Mov}(X_3, \mathbb{Z})$, which is dual to the cone of effective divisors $\mathrm{Eff}^1(X_3, \mathbb{Z})$.

[Alim,Heidenreich,Rudelius'21], [Boucksom,Demaily,Paun,Peternell'13]

Non-trivial checks for the tWGC performed via explicit computation of Gopakumar-Vafa invariants of several geometries.

[Alim,Heidenreich,Rudelius'21], [Gendler,Heidenreich,McAllister,Moritz,Rudelius'22]

However:

There exist ray charges where extremal black holes and such a BPS tower do not coincide.

$\Rightarrow$ Potential room for tWGC violation!

The asymptotic weak gravity conjecture in M-theory I

Claim 1

Suppose there exists a primitive charge vector $Q^0 \in \Lambda_Q$ such that

$$\{\lambda Q^0\}_{\lambda \in \mathbb{R}} \cap \Lambda_Q$$

is not populated by a BPS tower of super-extremal states. Then:

1. There exists no limit in moduli space in which

$$\text{weak coupling limit:} \quad \frac{\Lambda_{\text{WGC}}^2}{\Lambda_{\text{QG}}^2} \to 0 \,. \tag{2}$$

2. There exists a non-BPS tower of states along $\{\lambda Q^0\}_{\lambda \in \mathbb{R}} \cap \Lambda_Q$ that is part of the tower of excitations of weakly coupled critical string in the limit (2). This tower satisfies the tWGC.

Here: $\Lambda_{\text{WGC}}^2 = g_{\text{YM}}^2 M_{\text{Pl}}^3$

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3. A geometrical dictionary for the emergence string conjecture
4. Counting of non-BPS states in five dimensions

Emergence string conjecture & tower weak gravity conjecture ⟷ Fibration structure of Calabi-Yau threefolds

The emergence string conjecture

Emergence string conjecture

Any infinite distance limit is either a decompatification limit or a limit in which a weakly coupled critical string becomes tensionless.

[Lee,Lerche,Weigand'19]

Geometrisation of emergent string limits in M-theory

Infinite distance limits in classical Kähler moduli space of a Calabi-Yau threefold X_3, subjected to a finite volume, are

[Lee,Lerche,Weigand'19]

1. **Limits of type T^2 :** X_3 admits a T^2-fibration

$$\pi : X_3 \to B_2 \, .$$

   The weak coupling limit corresponds to the volume of the generic fiber T^2 shrinking.

2. **Limits of type K3/T^4 :** X_3 allows for a surface fibration

$$\rho : X_3 \to \mathbb{P}^1 \, .$$

   The weak coupling limit corresponds to the volume of the generic surface fiber **S** shrinking.

The asymptotic weak gravity conjecture in M-theory II

Claim 2

In M-theory on a Calabi-Yau threefold X_3, the only $U(1)$s that admit a weak coupling limit are obtained from the M-theory 3-form C_3 reduced on a curve:

- A generic T^2
- Curves in a generic K3/T^4 fiber
- Curves in degenerate fibers at finite distance in the fiber moduli space

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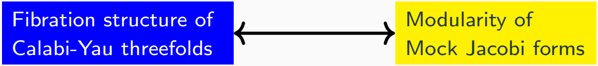


Limits of type T^2

In a T^2-fibered Calabi-Yau threefold $\pi : X_3 \to B_2$, the Shioda-Tate-Wazir-theorem states:

$$h^{1,1}(X_3) = h^{1,1}(B_2) + 1 + n - 1$$

- $h^{1,1}(B_2)$: pullback of base divisors
- 1: zero-N-section
- $n - 1$: additional sections + kodaira resolution divisors

Limits of type T^2

In a T^2-fibered Calabi-Yau threefold $\pi : X_3 \to B_2$, the Shioda-Tate-Wazir-theorem states:

Role in six-dimensional F-theory dual:

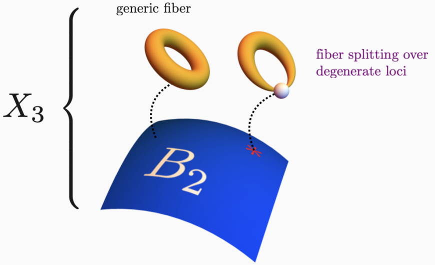
$$h^{1,1}(X_3) \;=\; h^{1,1}(B_2) \;+\; 1 \;+\; n-1$$

- $h^{1,1}(B_2)$ → tensor multiplet $U(1)$s
- 1 → Kaluza-Klein $U(1)_{KK}$
- $n-1$ → abelian $U(1)$s + Cartan $U(1)$s

Limits of type T^2

A basis for effective curves in X_3 is $\{\mathcal{C}^a, \mathcal{C}^i_f\}$, where:

- $\{\mathcal{C}^a\}_{a=1,\ldots,h^{1,1}(B_2)}$: base curves
- $\{C^i_f\}_{i=1,\ldots,n}$: fibral curves



The generic T^2-fiber is a linear combination

$$T^2 = \sum_{i=1}^{n} c_i C^i_f \,.$$

Limits of type T^2

- We proved that the only ray charge direction that allows for a weakly coupled limit $\Lambda^2_{\text{WGC}}(U(1))/\Lambda^2_{QG} \to 0$ is

$$U(1)_{KK} = \sum_{i=1}^{n} c_i U(1)^i \quad \text{with} \quad T^2 = \sum_{i=1}^{n} c_i \mathcal{C}_f^i \,.$$

- Such a limit corresponds to a decompactification limit into the six-dimensional F-theory dual counterpart.
- Charge lattice Λ_Q in tWGC populated by non-trivial BPS states counted by genus zero Gopakumar-Vafa invariants

$$n^0_{kT^2} = -\chi(X_3)\,, \quad \text{with} \quad k \in \mathbb{N}\,,$$

which contribute ot the base degree zero coefficient of the top. string partition function, which transforms as a Jacobi form of weight $k = -2$.

[Huang,Katz,Klemm'15][CFC,Klemm,Schimannek'19][Oehlmann,Schimannek'19][Kashani-Poor'19]

Limits of type K3

One can take a similar approach to separate base and curves in the fiber of $\rho : X_3 \to \mathbb{P}^1$.

However: K3 fibrations can have degenerations over points in $\mathbb{P}^1$ that occur at finite/infinite distance in the K3 fiber moduli space:

Semi-stable degeneration classification of K3 surfaces

$$\Longrightarrow \begin{cases} \text{Type I Kulikov models} & \text{(at finite distance)} \\ \text{Type II/III Kulikov models} & \text{(at infinite distance)} \end{cases}$$

Limits of type K3

Consider the restriction of the K3 fibration to a disk D centered around a point $p \in \mathbb{P}^1$

$$\rho_D : X_D \to D\,,$$

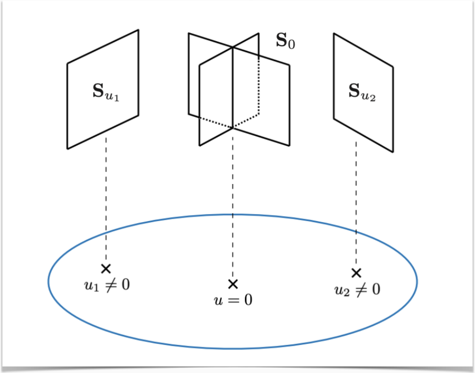
where the fiber $\mathbf{S}_u$ over a generic point $0 \neq u \in D$ is smooth, while the central fiber $\mathbf{S}_0$ degenerates into a union

$$\mathbf{S}_0 = \bigcup_{M=1}^{N} \mathbf{S}_M\,.$$

Here $\mathbf{S}_M$ has at worst normal crossing singularities.

Type I Kulikov models $\longrightarrow N = 1$ and $\mathbf{S}_0$ is smooth.
Type II/III Kulikov models $\longrightarrow N > 1\,.$

Limits of type K3



Semi-stable degeneration of K3 surfaces

Picture: [Lee,Lerche,Weigand'21]

Limits of type K3

A basis for effective curves in $\rho : X_3 \to \mathbb{P}^1$ is $\{\mathbb{P}^1, \mathcal{C}^\iota, \mathcal{C}^m, \mathcal{C}^\mu\}$, where:

- $\mathcal{C}^\iota$: curve generators located in the generic K3 fiber
- $\mathcal{C}^m$: curve generators localized in special fibers of Kulikov Type I
- $\mathcal{C}^\mu$: curve generators localized in special fibers of Kulikov Type II/III

We proved that allowed $U(1)$ charges with weakly coupled limit $\Lambda^2_{\text{WGC}}(U(1))/\Lambda^2_{QG} \to 0$ are of the form

$$U(1)_C = \sum_\mu c_\mu U(1)^\mu + U_{C_{\text{rest}}} \,,$$

where $C = \sum_\mu c_\mu \mathcal{C}^\mu + C_{\text{rest}}$ lies inside the generic K3 fiber.

Counting towers over limits of type K3

- **Emergent string conjecture:** The U(1)s arise from the perturbative gauge sector of the heterotic string.
- Heterotic strings in five-dimensional M-theory arise by wrapping M5-branes along the K3 fiber.
- The counting of heterotic strings excitations can be determined via the modified elliptic genus of MSW strings:

[Maldacena,Strominger,Witten'97][Gaiotto,Strominger,Yin'06]

$$Z_{\mathbf{S}}^{(r)}(\tau,\bar{\tau},z) = \mathrm{Tr}_{\mathrm{RR}} F_{\mathrm{R}}^2 (-1)^{F_{\mathrm{R}}} q^{L_0-\frac{c_{\mathrm{L}}}{24}} \bar{q}^{\bar{L}_0-\frac{c_{\mathrm{R}}}{24}} e^{2\pi i z^i Q_i} \,. \qquad (3)$$

Here $\mathbf{S}$ is the divisor wrapped by the M5-brane r times.
τ is the T^2 modulus in the MSW-CFT and $q = \exp(2\pi i\tau)$.
z^i are $U(1)$s worldsheet fugacity parameters and $Q_i \in \mathbb{Z}$.

Counting towers over limits of type K3 — Kulikov Type I

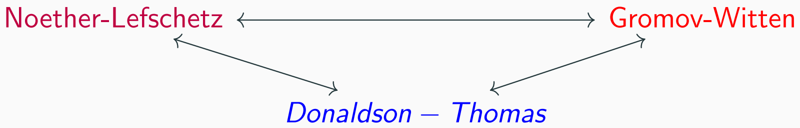
Consider K3 fibrations $\rho : X_3 \to \mathbb{P}^1$ of Type I Kulikov.
$\Rightarrow$ the fibers admits a Λ-polarization with $\Lambda \subset U^3 \oplus E_8(-1)^2$.

This setup allows for a Noether-Lefschetz theory counting, which counts intersection of special divisor loci in the moduli space of K3 surfaces $\mathcal{M}_\Lambda$ with the base image in $\mathbb{P}^1 \hookrightarrow \mathcal{M}_\Lambda$.

Modularity:

Noether-Lefschetz numbers determined by a vector-valued-modular form Φ specified by the lattice information of Λ.

[Maulik,Pandharipande'07][Borcherds'99][Kudla-Millson'90]



Counting towers over limits of type K3 — Kulikov Type I

The elliptic genus for a heterotic MSW string decomposes as

$$Z_{\mathbf{S}}^{(r)}(\tau, \bar{\tau}, z) = \sum_{\mu \in \Lambda^*/r\Lambda} Z_\mu(\tau) \Theta^*_{\mu,r}(\tau, \bar{\tau}, z)\,, \tag{4}$$

where Λ^* is the dual lattice of Λ, $\Theta_{\mu,r}$ is a Siegel-Theta series, and Z_μ is a vector-valued-modular form encoding DT invariants that is equivalent to Noether-Lefschetz vector-valued-modular form Φ.

[Bouchard,Creutzig,Diaconescu,Doran,Quigley,Sheshmani'16]

Moreover,

$$Z_\mu(\tau) = \sum_{n=0}^{\infty} \Omega(\gamma) q^{n + \frac{Q^2}{2r} - 1}\,, \tag{5}$$

where $\Omega(\gamma)$ is a BPS (DT invariant) with D4-D2-D0 brane charges $\gamma = (r, Q, n)$ with $r, n \in \mathbb{Z}_{\geq 0}$ and $Q \in \Lambda^*$.

Counting towers over limits of type K3 — Kulikov Type I

- The lattice Λ^* decomposes as $\Lambda^* = \Lambda_+ \oplus \Lambda_-$ with $\Lambda_\pm$ being the self/anti-self dual part of Λ^*.

- Using Noether-Lefschetz constraints, we proved that there is a non-trivial tower of states with charges γ such that $n = -Q_-^2/2$ with $Q_- \in \Lambda_-$.

- Moreover, we proved that such states correspond to excitations of strings (non-BPS) in five dimensions at level $n = -Q_-^2/2$ that fulfill the tWGC.

- Using Noether-Lefschetz theory, and BPS states with charges also in Λ_+, we can populate the entire charge lattice Λ^* with superextremal states!

Counting towers over limits of type K3 — Kulikov Type II/III

For Type II/III Kulikov models, Noether-Lefschetz theory needs to be better understood.

For a general MSW string wrapped on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the MSW string elliptic genus is of the form

$$Z_{\mathcal{D}}^{(r)}(\tau, \bar{\tau}, z) = \sum_{\mu \in \Lambda^*/r\Lambda} \widehat{Z}_\mu(\tau) \Theta^*_{\mu,r}(\tau, \bar{\tau}, z) \,, \qquad (6)$$

similar definitions as before, but now

$$\widehat{Z}(\tau, \bar{\tau}) = Z_\mu(\tau) - \frac{1}{2} R_\mu(\tau, \bar{\tau}) \qquad (7)$$

is a vector-valued mock modular form of depth $n-1$.

[Alexandrov,Banerjee,Manschot,Pioline'16]

Z_μ : holomorphic part

$R_\mu(\tau, \bar{\tau})$: non-holomorphic completion

Example

6 Example

We illustrate the possible weak coupling limits and their associated super-extremal towers by means of a Calabi–Yau 3-fold X_3 which admits both a K3-fibration $\rho : X_3 \to \mathbb{P}^1$ and a compatible elliptic fibration $\pi : X_3 \to B_2$. The elliptic fibration is constructed as a generic Weierstrass model over the base $B_2 = \mathrm{Bl}(\mathbb{F}_2)$, the blowup of the Hirzebruch surface $\mathbb{F}_2$ in one point. Since B_2 is rationally fibered, X_3 admits also a compatible K3-fibration.

The resulting Calabi–Yau X_3 can be described torically via the following data:

$$\begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \end{array} \begin{array}{c} \begin{array}{cccc} & & & \end{array} \begin{array}{cccc} \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \mathcal{C}^3 \end{array} \\ \left(\begin{array}{cccc|cccc} -2 & -3 & -1 & -2 & 1 & -1 & 0 & 1 \\ -2 & -3 & -1 & 1 & 0 & 1 & 0 & -1 \\ -2 & -3 & 0 & -1 & -2 & 1 & 0 & 0 \\ -2 & -3 & 0 & 1 & 0 & 0 & 0 & 1 \\ -2 & -3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\ -2 & -3 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right) \end{array} . \tag{6.1}$$

Assigning projective coordinates $[s : t : u : v : w : x : y : z]$ to the toric divisors $\{D_i\}_{i=1,\dots,8}$ in the same corresponding ordering, we obtain the Stanley–Reissner ideal

$$\mathrm{SR} = \{tu, uv, sw, tw, sv, xyz\}\,. \tag{6.2}$$

The Euler number of X_3 is $\chi(X_3) = -420$. The Mori cone is simplicial and generated by the curves $\mathcal{C}^i$, $i = 0, 1, 2, 3$. The dual Kähler cone generators J_i are expressed in terms of the toric

Example

divisors, for instance, as

$$J_0 = D_1 + D_2\,, \quad J_1 = D_2 + D_4\,, \quad J_2 = \frac{1}{2}D_6\,, \quad J_3 = D_4\,. \tag{6.3}$$

In particular, D_1 and D_2 are among the generators of the cone of effective divisors. Furthermore, $c_2(X_3) \cdot J_\alpha = (24, 48, 82, 36)$. This identifies J_0 as the divisor associated with the K3-fiber of ρ, and $J_2 = S_0 + \pi^* c_1(B_2)$ with S_0 being the zero-section of the elliptic fibration. Its dual Mori cone generator $\mathcal{C}^2$ therefore corresponds to the class generic elliptic fiber.

The generic rational fiber of B_2 lies in the class $\mathcal{C}^1 + \mathcal{C}^3$. The base of this rational fibration is the base $\mathbb{P}^1$ of the K3-fibration ρ; its class coincides with $\mathcal{C}^0$. Over a special point on the base $\mathbb{P}^1$, the rational fiber of B_2 splits into two rational curves in class $\mathcal{C}^1$ and $\mathcal{C}^3$, each of self-intersection -1 on B_2. The elliptic fibration over each of these two curves defines a rational elliptic surface, or dP_9, of Euler characteristic 12. As a result, the K3-fibration undergoes a Kulikov Type II degeneration, in which the generic K3 fiber class splits as

$$\mathbf{S}_0 = \mathbf{S}_1 \cup \mathbf{S}_2\,. \tag{6.4}$$

We identify the class of $\mathbf{S}_1$ and $\mathbf{S}_2$ with the toric divisor classes D_1 and D_2.

To the given basis $\{\mathcal{C}^\alpha\}_{\alpha=0,\ldots,3}$ of the Mori we can now associate a basis $\{\mathrm{U}(1)^\alpha\}$ of the Abelian gauge factors and hence a basis of charges $\{Q_\alpha\}$ that parametrize the charge lattice. We notice that $\mathcal{C}^2$ is the only Mori cone generator that is also a movable curve. Hence the results of [20] imply that there is an infinite tower of BPS states with charge

$$\boldsymbol{Q} = (Q_0, Q_1, Q_2, Q_3) = (0, 0, n, 0)\,. \tag{6.5}$$

In fact, since $\mathcal{C}^2$ is the elliptic fiber class, the genus-zero BPS invariants along this direction are

$$N^0_{(0,0,n,0)} = -\chi(X_3) = 420\,. \tag{6.6}$$

On the other hand, the rays in the charge lattice with $Q_2 = 0$ do not support towers of BPS states and hence invite an application of Claim 1.

Example

To this end we should first consider which linear combinations of $\mathrm{U}(1)^0$, $\mathrm{U}(1)^1$, and $\mathrm{U}(1)^3$ admit weak coupling limits. Let us begin with the K3-fibration ρ and its associated weak coupling limit of Type K3. The dual of the polarization lattice is spanned by the generators of the ρ-relative Mori cone that lie in the generic K3-fiber. This identifies

$$\Lambda^* = \langle \mathcal{C}^2, \mathcal{C}^1 + \mathcal{C}^3 \rangle \simeq U \,, \tag{6.7}$$

where U is the hyperbolic lattice of signature $(1,1)$. According to the general discussion of Section 4.2, the two Kähler cone generators J_1 and J_3 dual to the curves $\mathcal{C}^1$ and $\mathcal{C}^3$ in the generic rational fiber must satisfy a homological relation of the form (4.48). Indeed, from the intersection form

$$\begin{aligned} \mathcal{I}(X_3) = & 7J_2^3 + 2J_2^2 \cdot J_0 + 4J_2^2 \cdot J_1 + 3J_2^2 \cdot J_3 + 2J_1^2 \cdot J_2 + J_3^2 \cdot J_2 + J_0 \cdot J_1 \cdot J_2 \\ & + J_0 \cdot J_2 \cdot J_3 + 2J_1 \cdot J_2 \cdot J_3 \end{aligned} \tag{6.8}$$

Example

it follows that

$$J_3 \cdot J_0 \cdot J_\alpha = J_1 \cdot J_0 \cdot J_\alpha \,, \quad \forall \alpha = 0, \dots, 3 \,. \tag{6.9}$$

An infinite distance limit of Type K3 is parametrised as

$$J = \lambda \tilde{v}^0 J_0 + \frac{1}{\sqrt{\lambda}} \tilde{v}^i J_i \,, \quad \lambda \to \infty \,. \tag{6.10}$$

In terms of the rescaled Mori cone volumes $\hat{\tilde{v}}^\alpha = \frac{\tilde{v}^\alpha}{\mathcal{V}^{1/3}}$, the gauge kinetic matrix $f_{\alpha\beta}$ at leading order in λ takes the form

$$f_{\alpha\beta} = \lambda \frac{\hat{\tilde{v}}_0^2 \hat{\tilde{v}}_2^2}{\left(\hat{\tilde{v}}_0 \hat{\tilde{v}}_2 \left(\hat{\tilde{v}}_1 + \hat{\tilde{v}}_2 + \hat{\tilde{v}}_3\right)\right)^{4/3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \frac{\hat{\tilde{v}}_1^2 + 2\left(\hat{\tilde{v}}_2 + \hat{\tilde{v}}_3\right)\hat{\tilde{v}}_1 + 2\hat{\tilde{v}}_2^2 + \hat{\tilde{v}}_3^2 + 2\hat{\tilde{v}}_2\hat{\tilde{v}}_3}{\hat{\tilde{v}}_2^2} & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} + \mathcal{O}\left(1/\sqrt{\lambda}\right) \,. \tag{6.11}$$

We notice that the second and the fourth rows (or columns), associated to the divisors J_1 and J_3 satisfying (6.9), are identical. At leading order, the rank of the matrix f_{ij} is therefore reduced, as expected from the discussion in Section 4.2. In particular, the space of asymptotically weakly coupled abelian gauge symmetries is spanned by the combination

$$\mathrm{U}(1)_+ = \mathrm{U}(1)^1 + \mathrm{U}(1)^3 \,, \tag{6.12}$$

together with $\mathrm{U}(1)^2$, while any U(1) involving the orthogonal combination $\mathrm{U}(1)^1 - \mathrm{U}(1)^3$ as well as $\mathrm{U}(1)^0$ cannot become asymptotically weakly coupled in the limit of Type K3. In particular, $\mathrm{U}(1)^1$ and $\mathrm{U}(1)^3$ individually do not admit weak coupling limits as in (3.8). Hence, we do not expect to find any super-extremal non-BPS string excitations with charge $\boldsymbol{Q} = (0, n, 0, 0)$ or $\boldsymbol{Q} = (0, 0, 0, n)$ for $n > 1$. Instead a tower of super-extremal excitations charged under $\mathrm{U}(1)^1$ or $\mathrm{U}(1)^3$ must have $Q_1 = Q_3$. And indeed, $\mathrm{U}(1)_+$ and $\mathrm{U}(1)^2$ are precisely the abelian gauge symmetries under which the curve classes in the dual polarization lattice Λ^* are charged. From the heterotic perspective, these are the U(1)s associated to winding and momentum along the heterotic S^1. For these U(1)s the existence of states satisfying (5.5) can be established from the elliptic genus as in Section 5.2.1.

Example

string is

$$Z_{\text{het}}(\tau) = -\frac{23}{12}\frac{E_4E_6}{\eta^{24}} - \frac{1}{12}\frac{E_4^2E_2}{\eta^{24}} = -2q^{-1} - \chi(X_3) + \mathcal{O}(q) = q^{-1}\sum_d N^0_{\mathcal{C}^1+\mathcal{C}^3+d\mathcal{C}^2} q^d \,. \tag{6.14}$$

As discussed at the end of Section 5.2.1, from the latter expression we derive the holomorphic piece for the heterotic MSW string from (5.22). Since $\Lambda^* = U$, the discriminant group Λ^*/Λ only contains the trivial class. Hence, the non-holomorphic completion constrains the five-dimensional heterotic elliptic genus $Z^{(1)}_{J_0}(\tau,\bar\tau,z,\mathcal{B}) = \widehat{Z}_{\mathbf{0}}(\tau,\bar\tau)\Theta^*_{\mathbf{0},1}(\tau,\bar\tau,z,\mathcal{B})$ to take the form

$$\widehat{Z}_{\mathbf{0}}(\tau,\bar\tau) = -\frac{23}{12}\frac{E_4E_6}{\eta^{24}} - \frac{1}{12}\left(\frac{E_4}{\eta^{12}}\right)^2\widehat{E}_2\,, \tag{6.15}$$

where $\widehat{E}_2 = E_2 - 3/\pi\text{Im}(\tau)$ is the non-holomorphic second Eisenstein series, which is also a mock modular form. Notice that the quadratic factors E_4/η^{12} in (6.15) are meromorphic modular forms corresponding to the MSW strings deriving from the dP_9 surfaces [62] given by D_1 and D_2; their quadratic product is expected to be present in the non-holomorphic contribution since $J_0 = D_1 + D_2$ [72]. Using (6.15), similar arguments as discussed in Section 5.2.1 can be repeated to argue for the existence of a non-trivial tower of states satisfying (5.5).

Outlook

- **In five-dimensional theories realized by M-theory:** We proved the existence of towers of non-BPS objects & BPS objects that satisfy the tWGC in the charge lattice directions allowing for an asymptotic weakly coupled limit.

- A similar story holds for four-dimensional $\mathcal{N} = 1$ theories realized by F-theory compactifications. See [2208.00009].

- **Work in progress:** Type-T^4 limits

- A generalization for Noether-Lefschetz theory for Type II/III Kulikov models is desirable to argue Type-K3 limits in full generality.