Origin limits of $\mathcal{N} = 4$ SYM amplitudes at finite coupling Part I

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DESY String Journal Club
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PRL 124, 161603 (2020) with Benjamin Basso and Lance Dixon
2211.12555 with Benjamin Basso, Lance Dixon and Yu-Ting Liu
JHEP 08 (2020) 005, JHEP 10 (2021) 007 with Niklas Henke

S-matrix of gauge theory beyond Feynman diagrams?

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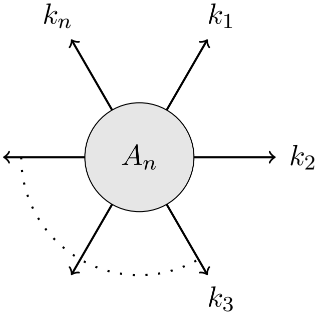

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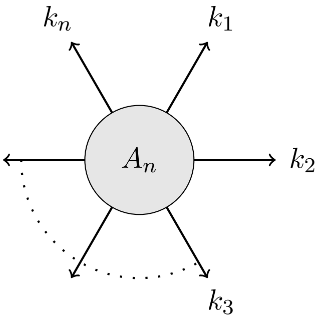

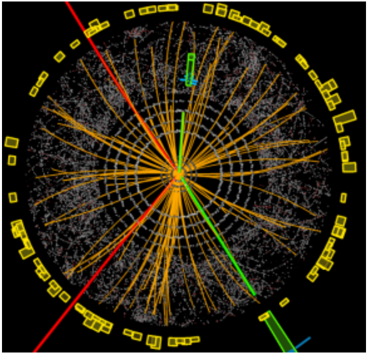

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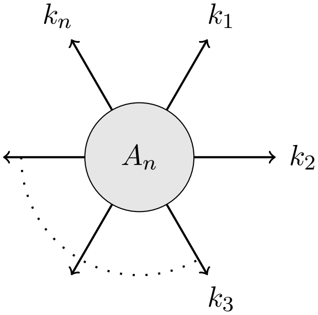

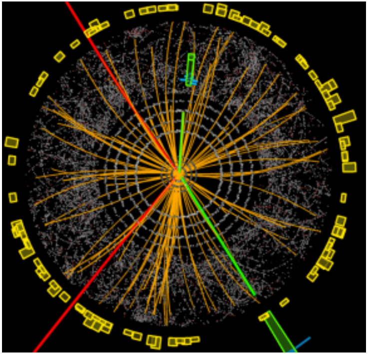

Scattering amplitudes $A = \langle \mathrm{IN}|S|\mathrm{OUT}\rangle$: $d\sigma \propto |A|^2$

- Computing efficiently necessary in practice
- Understanding beyond perturbation theory mathematically important [Millenium Prize]

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Can we hope for similar progress with amplitudes?

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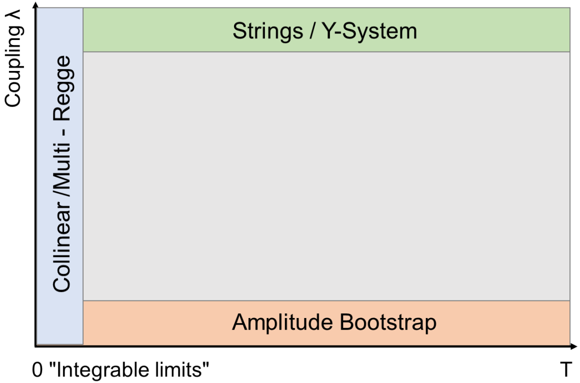

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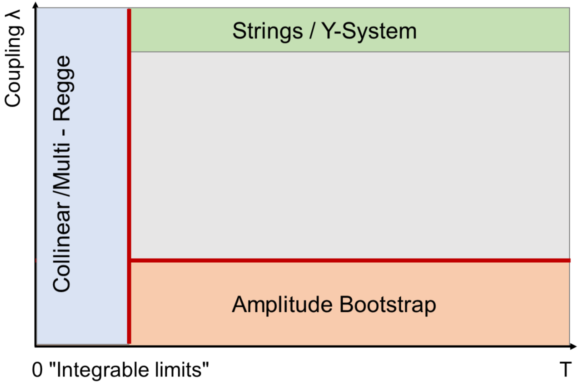

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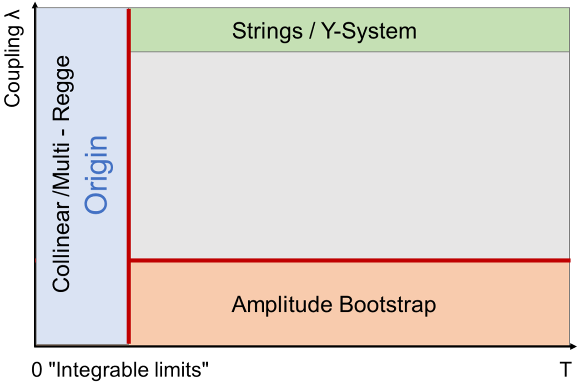

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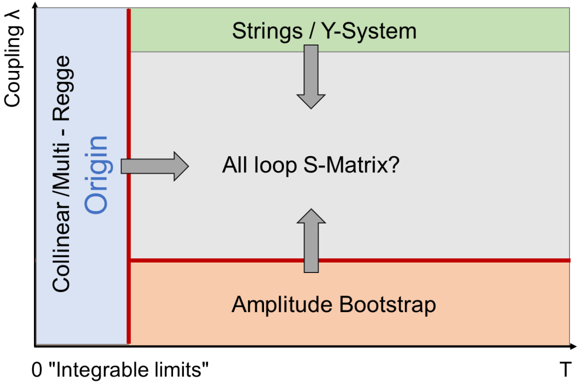

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Outline

Intro: $\mathcal{N} = 4$ SYM Amplitudes

The (Six-particle) Origin of Intriguing Observations

Higher-point Origins

Origin limits: Classification with *cluster algebras*

Maximally Hellicity Violating (MHV) Gluon Amplitudes

Gluons are massless → helicity $h = \vec{S} \cdot \hat{p} = \pm 1$ good quantum number.

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- remarkably, dual to null polygonal Wilson loops.
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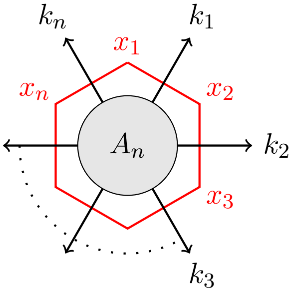

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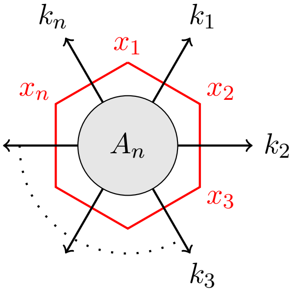

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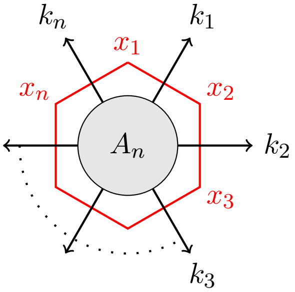

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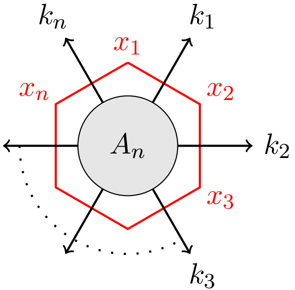

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- hence *dual conformal invariant* (in appropriate normalization)

The Origin of the Six-Gluon Amplitude

$\mathcal{E}_6$ (and $\mathcal{E}_7$) computed most efficiently in general kinematics & at fixed order in the coupling via *Amplitude Bootstrap*.
[Recent SAGEX Review, Chapter 5: GP]

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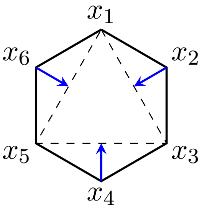
Natural to scan space of kinematics for all-loop patterns and simplifications. Here: Focus on limit when $u_i \to 0$: "origin"



$$u_1 \equiv u_{1,4} = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2}\,, \quad x_{13}^2 = s_{12} \to 0 \quad \text{plus } i \to i+2$$

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In the origin limit $u_i \to 0$, from perturbative results up to 7 loops, observed that six-particle amplitude takes the form, [Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP]

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Why? How about $\Gamma_{\rm hex}, C_0$?

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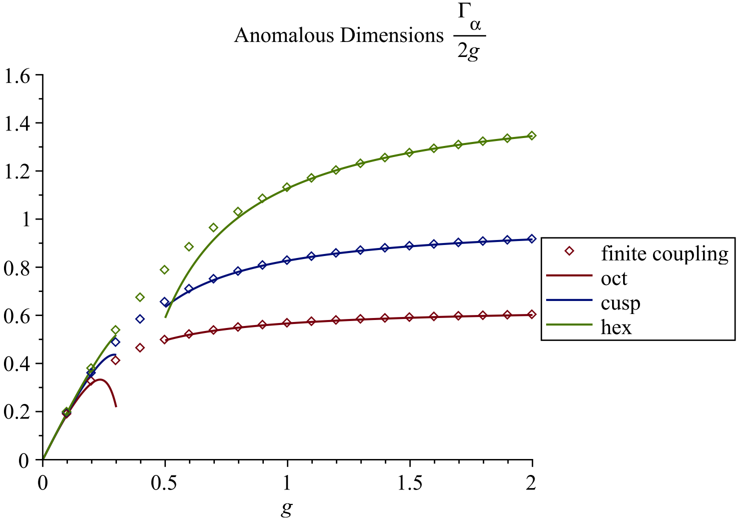
$$C_0 = \frac{\zeta_2}{4}\Gamma_{\pi/4} + D(\pi/4) - D(\pi/3) - \frac{1}{2}D(0) \,, \qquad D(\alpha) \equiv \ln\det\left[1 + \mathbb{K}(\alpha)\right].$$

Comparison: Finite-coupling numerics & weak/strong coupling analytics



$$\Gamma_{\rm oct} = \Gamma_0\,, \quad \Gamma_{\rm cusp} = \Gamma_{\pi/4}\,, \quad \Gamma_{\rm hex} = \Gamma_{\pi/3}$$

Questions seeking answers

- Physical significance of α?
- Other physical quantities Γ_α describes for various values of α?

$$R_n = \sum_\alpha \tilde{\Gamma}_\alpha(g) \times P_\alpha^{\Sigma_n} ,$$

with $\tilde{\Gamma}_\alpha = \Gamma_\alpha - \Gamma_{\pi/4}$ and with the sum running over

$$\alpha = \frac{\pi}{2} - \frac{\pi p}{3} - \frac{\pi k}{3(n-4)} ,$$

with $k = 1, \ldots, n-5$ and $p = 0, 1, 2$.

[Basso,Dixon,Liu,GP]

1. n-gluon generalizations of origin limits → *cluster algebras*
2. Amplitude kinematic dependence, $P_\alpha^{\Sigma_n}$ → *pert. data & bootstrap*
3. Values of α → *thermodynamic Bethe ansatz* (TBA)

Classifying n-gluon origin limits $\mathrm{O}^{(n)}$

$$\mathrm{O}^{(6)}: \quad u_i \equiv u_{i+1,i+4} \to 0\,, \quad i = 1,2,3.$$

⛔ However, in general $n(n-5)/2$ dual conformal cross ratios, but only $3(n-5)$ independent kinematic variables ⇒ Cannot set all $u_{i,j} \to 0$!

💡 ∃ well-defined notion of region of *positive kinematics*, where amplitudes believed to be singularity-free.
[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [Arkani-Hamed,Lam,Spradlin]

⇒ Look at *boundary* of this region, as first place for potential origin-type divergent behavior! Completely captured by *cluster algebras*.

Cluster Algebras [Fomin,Zelevinsky'01A][Fock,Goncharov]

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- Grouped into overlapping subsets $\{x_1, \ldots, x_d\}$ of *rank* d, the *clusters*

Cluster Algebras [Fomin,Zelevinsky'01A][Fock,Goncharov]

They consist of

- A set of variables x_i, the *cluster ($\mathcal{X}$-)coordinates*
- Grouped into overlapping subsets $\{x_1, \ldots, x_d\}$ of *rank* d, the *clusters*
- Constructed recursively from initial cluster via *mutations*, encoded d-dimensional matrix B with elements b_{ij}.

Mutation associated to coordinate x_k:

$$x_i \to x_i' = \begin{cases} 1/x_i & k = i\,, \\ x_i\left(1 + x_k^{-\mathrm{sgn}(b_{ki})}\right)^{-b_{ki}} & k \neq i\,, \end{cases}$$

In new cluster, $B \to B'$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{for } i = k \text{ or } j = k \\ b_{ij} + \max\left(0, -b_{ik}\right) b_{kj} + b_{ik} \max\left(0, b_{kj}\right) & \text{otherwise}\,. \end{cases} ,$$

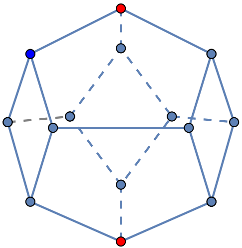
Exchange graph: Clusters=vertices, mutations=edges

Example: The six-particle positive region

Described by $Gr(4,6) \simeq A_3$ cluster algebra

Initial cluster $\{x_1, x_2, x_3\}$

Origin limit clusters



Positive region maps to interior of exchange graph/polytope, described by $\infty > x_i > 0$.

$$u_1 = \frac{x_2 x_3}{(1+x_1+x_1x_2)(1+x_2+x_2x_3)}, \quad u_2 = \frac{x_1 x_2}{1+x_1+x_1x_2}, \quad u_3 = \frac{1}{1+x_2+x_2x_3}.$$

In initial cluster, $b_{12} = b_{23} = -b_{21} = -b_{32} = 1 \Rightarrow$

$$x_1' = x_1\,(1+x_2), \quad x_2' = \frac{1}{x_2}, \quad x_3' = \frac{x_2 x_3}{1+x_2}.$$