# Geometrical description of amplitudes correlators in N=4


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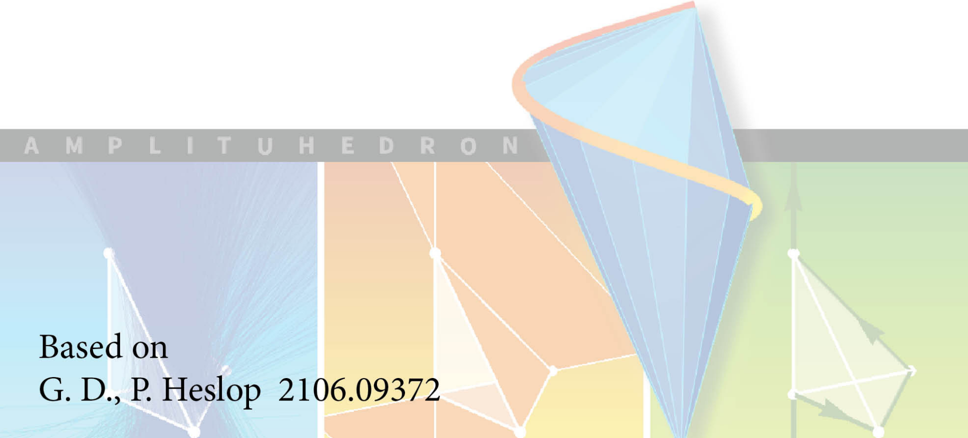
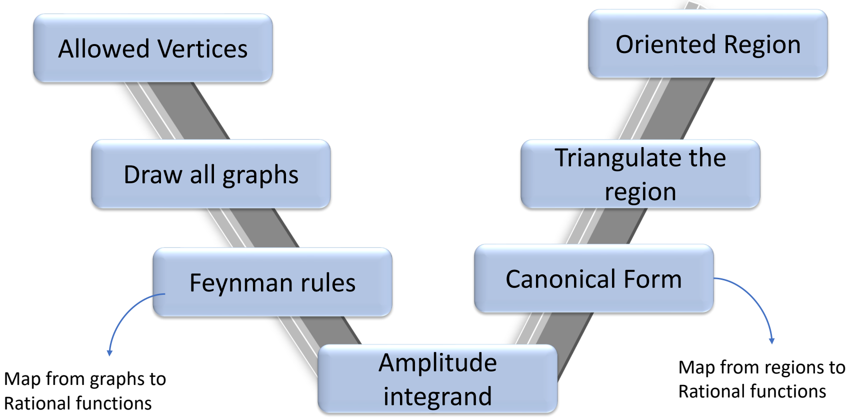



Based on
G. D., P. Heslop 2106.09372

# Amplitudes from geometry



# Topics I'd like to present

- The canonical form: a map between geometries and rational differential forms.
- The superamplitude as a differential form
- The supercorreleator as a differential form and the correlator/amplitude duality
- The amplituhedron, the squared amplituhedron and the correlahedron

# When is it possible?

- $N = 4$ SYM:
  Amplituhedron [N. Arkani-Hamed, J. Trnka] (2013)
  Loop amplituhedron canonical form [G.D., P. Heslop, A. Stewart](2022)
  Momentum amplituhedron [D. Damgaard, L. Ferro, T. Lukowski, M. Parisi](2019)
  Loop momentum amplituhedron [ L. Ferro, T. Lukowski, M. Parisi](2022)
  Correlahedron, Squared Amplituhedron [B. Eden, P. Heslop, L. Mason] (2017)
  Amplituhedron-like geometries [G. D., P. Heslop] (2021)

- Bi-adjoint $\Phi^3$, Associahedron/Halohedron/Cluster polytopes

  [N. Arkani-Hamed, Y. Bai, S. He, G. Yan ( 2017)]

- Scalars in FRW background, Cosmological Polytope

  [N. Arkani-Hamed, P. Benincasa, A. Postnikov] (2017)

- SYM and ABJM, momentum amplituhedron

  [S. He, C.-K Kuo,Y.-Q Zhanga] (2021)

  [Y.-T. Huang, R. Kojima, C. Wen, and S.-Q. Zhang] (2021)

# Why is it interesting?

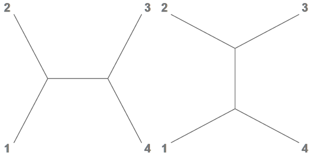
- Completely different perspective $\rightarrow$ new questions.
- Generator of new elegant formulas.
- Manifest hidden symmetries/properties, i.e. Yangian.
- Relations between amplitudes of different theories.

# Planar color ordered amplitudes

Color decomposition in the adjoint rep

$$\hat{M}_{n,l}^{a_1,\cdots,a_n} = g^{n+2(l-1)} \left( N^l \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) M_{n,l} + \mathcal{O}\left(\frac{1}{N}\right) \right) ,$$

The color ordered amplitude $M_{n,l}$ is obtained by summing over planar diagrams

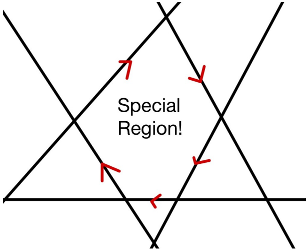



$M_{n,l}$ is a rational function of the kinematic.

# General Philosophy

In the examples where this construction is possible:

- There exists a region in the domain where the amplitude is finite and whose boundaries coincides with all the singularities of the function.




The pentagon is defined as the points to the right of the lines.

Geometry $\Leftrightarrow$ inequalities

- The amplitude is characterized as the canonical form of the region.

# Multivariate residues

We are interested in the so-called Leary residues [J. Leray] (1959)

[F. Cachazo, D. Kosower, K. J. Larse, S. Abreu, R. Britto, C. Duhr ]

Consider a $d$-dimensional space $X$ and a subspace $C$ of $X$, defined by the equation $f(x_1, \cdots, x_d) = 0$ where $f$ is an irreducible polynomial. If $\omega$ is a differential $k$-form defined on the complement $X - C$, then we say that $\Omega$ has a simple pole on $C$ if

$$\Omega = \frac{\mathrm{d}f}{f}\omega + \theta \ ,$$

where $\omega$ and $\theta$ are regular and non-vanishing on $C$.

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Example

$$\mathrm{Res}_{x=0} \frac{dxdy}{xy} = \frac{dy}{y}$$
$$\mathrm{Res}_{y=0} \frac{dxdy}{xy} = -\frac{dx}{x}$$

# Induced orientation

An orientation form $O$ is a non-vanishing top differential form defined up to positive rescaling

$$O' \sim \lambda O, \qquad \lambda > 0$$

Orientation in $\mathbb{R}^d$

$$\mathcal{O} = O(x_1, \cdots, x_d) \mathrm{d}x_1 \cdots \mathrm{d}x_d \; ,$$

Let the boundary be defined by $f(x) = 0$ with $f(x) > 0$ inside and $f(x) < 0$ outside the region. Then $\mathcal{O}_\partial$ is defined simply as

$$\mathrm{d}f \wedge \mathcal{O}_\partial = \mathcal{O}|_{f=0} \; .$$

Example $f(x) = x_1 = 0$

$$\begin{aligned} \mathrm{d}x_1 \wedge \mathcal{O}_\partial &= \mathcal{O}|_{f=0} O(0, x_2, \cdots, x_d) \mathrm{d}x_1 \cdots \mathrm{d}x_d \\ \mathcal{O}_\partial &= O(0, x_2, \cdots, x_d) \mathrm{d}x_2 \cdots \mathrm{d}x_d \end{aligned}$$

# The canonical form

The canonical form of an oriented region $\mathcal{X}_{\geq}$ is a differential top form recursively defined by the following property

- Simple poles on and only on codimension 1 boundary
- its residue on a boundary of $\mathcal{X}_{\geq}$ is the canonical form of the boundary
- the canonical form of a point is $\pm 1$ depending on the orientation.

A region possessing a canonical form is called a **positive geometry**. [N. Arkani-Hamed, Y. Bai, T. Lam](2017)

# Examples

Consider an oriented segment $X_{\geq} = a < x < b$, with orientation $dx$. It's canonical form is given by

$$\Omega(X_{\geq}) = \frac{dx}{x-a} - \frac{dx}{x-b}$$

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The boundary of $X_{\geq} = a < x < b$ are $x = a$ with $O|_{x=a} = 1$ and $x = b$ with $O|_{x=b} = -1$.

Then we check the definition

$$\mathrm{Res}_{x=a}\Omega(X_{\geq}) = O|_{x=a} = 1$$
$$\mathrm{Res}_{x=b}\Omega(X_{\geq}) = O|_{x=b} = -1$$

The canonical form of a segment $a < x < b$ is

$$dx(\frac{1}{x-a} - \frac{1}{x-b})$$

For two variables we have $a_1 < x_1 < b_1$ and $a_2(x_1) < x_2 < b_2(x_1)$

$$dx_1(\frac{1}{x_1-a_1} - \frac{1}{x_1-b_1}) \wedge dx_2(\frac{1}{x_2-a_2(x_1)} - \frac{1}{x_2-b_2(x_1)})$$

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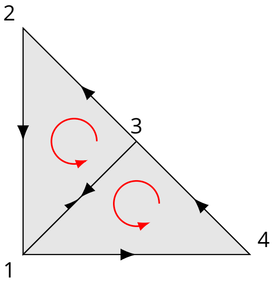
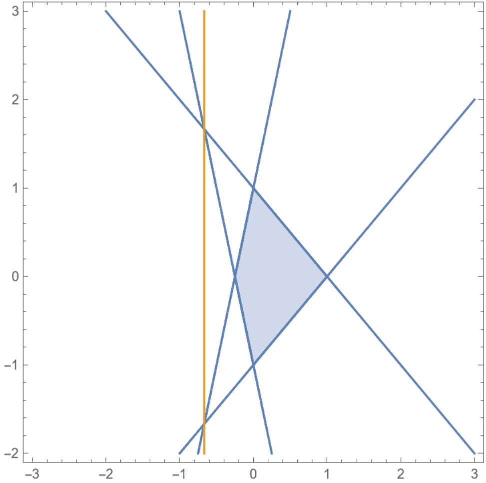
$$dx_1\left(\frac{1}{x_1-a_1} - \frac{1}{x_1-b_1}\right) \wedge dx_2\left(\frac{1}{x_2-a_2(x_1)} - \frac{1}{x_2-b_2(x_1)}\right)$$

If a **positive geometry** is given by the oriented union of positive geometries then the canonical form of the union is the sum of the canonical forms.



$$\Omega(x,y) = \frac{(10(2+3x))dxdy}{(-1+x-y)(1+4x-y)(-1+x+y)(1+4x+y)}$$



# $N = 4$ Super Yang Mills

The field content of the theory correspond to:

- 1 gluon $g = (g^+, g^-)$, helicity $h = 1$
- 4 fermions $\lambda = (\lambda^a, \lambda^{abc})$ $h = 1/2$
- 6 scalars $S^{[ab]}$, with $a, b = 1, 2, 3, 4$

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The N$^K$MHV amplitude $\rightarrow k + 2$ negative helicity gluon.

The **N$^\mathbf{K}$MHV superamplitude** is $4k$ polynomial in the $\eta_1^a, ..., \eta_n^a$, whose coefficient are ordinary amplitudes.

$$
\begin{aligned}
A_4^{\text{MHV}} &= A_4[g^- g^- g^+ g^+](\eta_1)^4(\eta_2^4) + \\
&+ A_4[g^- \lambda^{123} \lambda^4 g^+](\eta_1)^4(\eta_{21}\eta_{22}\eta_{23})(\eta_{34}) + \cdots,
\end{aligned}
$$

# Momentum Twistors

After dividing by the tree level MHV superamplitudes, N=4 SYM amplitudes has conformal and dual conformal symmetry.

New variables: **momentum twistors**

$Z = (Z^1, Z^2, Z^3, Z^4)$,
$Z \equiv \lambda Z$ that is $Z \in \mathbb{P}^3$.
Dual conformal symmetry acts on complex momentum twistors as $SL(4)$. The only invariant of $SL(4)$ is

$$\langle ijkl \rangle = \mathrm{Det}(Z_i Z_j Z_l Z_k)$$

Nice expressions for planar propagators

$$\langle i i+1 j j+1 \rangle \sim (p_i + \cdots + p_{j-1})^2$$

# Supermomentum Twistors

Momentum twistor can be generalized to supermomentum twistors

$$\mathcal{Z}_i = \begin{pmatrix} z_i \\ \chi_i \end{pmatrix} \in \mathbb{C}^{4|4}, \qquad i = 1, .., n$$

The 5 point NMHV superamplitude $\mathcal{A}_{5,1}$ has the form

$$\mathcal{A}_{5,1}(\mathcal{Z}_i) = \frac{\prod_{\alpha=1}^{4}(\chi_1^\alpha \langle 2345 \rangle + \text{cyclic})}{\langle 1234 \rangle \langle 2345 \rangle \langle 3451 \rangle \langle 4512 \rangle \langle 5123 \rangle}$$

# Bosonized Supermomentum Twistors

We attach $4$ additional Grassmann odd variables $\phi_A, A = 1, .., 4$ to each $\chi$, thus obtain commuting variables $\chi_i^A \phi_A$ [Hodges(2009)]

$$\mathcal{Z}_i = \begin{pmatrix} z_i \\ \chi_i \end{pmatrix} \quad \rightarrow \quad Z_i = \begin{pmatrix} z_i \\ \chi_i \phi \end{pmatrix}$$

For example

$$\int d^4\phi \langle 12345 \rangle^4 = \prod_{I=A}^{4} (\langle 1234 \rangle \chi_5^A + \text{cyclic})$$

It's possible to promote the 4-brackets to $(k+4)$-brackets though the identity

$$\langle ijkl\rangle = \langle Y_0 ijkl\rangle = \det\begin{pmatrix} 0 & Z_i^1 & Z_j^1 & Z_k^1 & Z_l^1 \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & Z_i^4 & Z_j^4 & Z_k^4 & Z_l^4 \\ 1 & \phi.\chi_i & \phi.\chi_j & \phi.\chi_k & \phi.\chi_l \end{pmatrix}$$

The amplitude $\mathcal{A}_{5,1}$ will read

$$\mathcal{A}_{5,1} = \int d^4\phi \frac{\langle 12345\rangle^4}{\langle Y_0 1234\rangle\langle Y_0 2345\rangle\langle Y_0 3451\rangle\langle Y_0 4512\rangle\langle Y_0 5123\rangle},$$

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$$A_{5,1} = \frac{\langle 12345\rangle^4 \langle Y d^4 Y\rangle}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

where $\langle Y\mathrm{d}^4 Y\rangle := \langle Y \overbrace{\mathrm{d}Y\cdots\mathrm{d}Y}^{4}\rangle/4!$

# Parametrization

$$Y = Z_1 + x_2 Z_2 + x_3 Z_3 + x_4 Z_4 + x_5 Z_5$$

$$\langle Y1234 \rangle = x_5 \langle 51234 \rangle = -x_5 \langle 12345 \rangle$$
$$\langle Y\mathrm{d}^4 Y \rangle = \langle Y\mathrm{d}_{x_2} Y\mathrm{d}_{x_3} Y\mathrm{d}_{x_4} Y\mathrm{d}_{x_5} Y \rangle = \langle 12345 \rangle \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5$$

So $A_{5,1}$ can we written as

$$A_{5,1} = \frac{\langle 12345 \rangle^4 \langle Yd^4 Y \rangle}{\langle Y1234 \rangle \langle Y2345 \rangle \langle Y3451 \rangle \langle Y4512 \rangle \langle Y5123 \rangle} = \frac{\mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5}{x_2 x_3 x_4 x_5}$$

# Bosonized super amplitudes

The $\overline{MHV}$ reads

$$A_{n,n-4} = \frac{\langle 12 \cdots n \rangle^4 \prod_i^k \langle Y \mathrm{d}^4 Y_i \rangle}{\langle Y1234 \rangle \langle Y2345 \rangle \cdots \langle Yn123 \rangle},$$

where $Y$ is a $k$-plane $Y^{I_1 \cdots I_k} = Y_1^{I_1} \cdots Y_n^{I_k}$, that is a $k \times (k+4)$ matrix mod $GL(k)$.
Then N$^k$MHV bosonized amplitude is a top dimensional form on the space of $k$ planes in $k+4$ dimensions $Gr(k, k+4)$

A generic $n$-point dual-conformal invariant can be written as

$$\langle I_1 \rangle \langle I_2 \rangle \langle I_3 \rangle \langle I_4 \rangle \ ,$$

where here and in the following we will use a short-hand notation $I, J$ etc to represent an ordered set of particle numbers. We define $[n] := \{1, 2, ..., n\}$ and then $\binom{[n]}{k}$ to be the set of all ordered sets of $k$ elements in $[n]$.

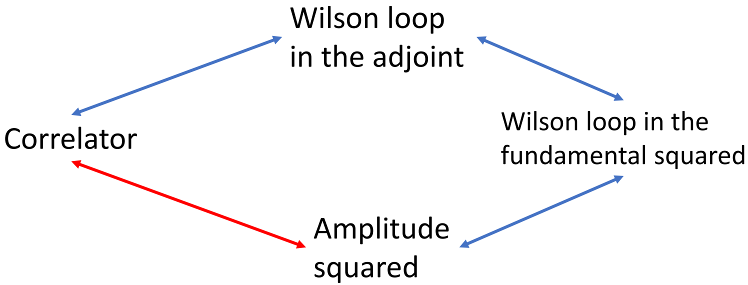
# The supercorrelator

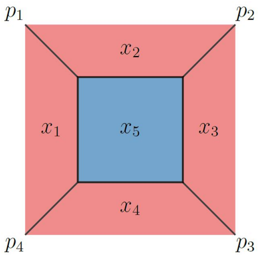
Simplest half BPS operators

$$\mathcal{O}_2^{ABCD} := \mathrm{Tr}(\phi^{AB}\phi^{CD}) - \frac{1}{4!}\epsilon^{ABCD}\mathrm{Tr}(\phi^{EF}\phi_{EF}) \ ,$$

This operator can be embedded the so called stress-energy tensor supermultiplet $\mathcal{T}^{ABCD}(x,\theta,\bar{\theta})$.

Lightlike limit $(x_{i+1} - x_i)^2 = 0$






$$p_i = x_{i+1} - x_i\,, \qquad q = x_i - x_j$$

$$\frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

external momentum conservation is trivialized

$$\sum_{i=0}^{n} p_i = \sum_{i=0}^{n} x_{i+1} - x_i = -x_1 + x_{n+1} = 0\,.$$

The potential $\mathcal{G}$ is implicitly defined by the relation

$$G_n(x, y, \theta) = \left( \prod_{i=1}^{n} D_4 \right) \mathcal{G}_n(x, \theta) \ ,$$

Hidden permutation symmetry of the correlator

$$\mathcal{G}^{(l)}_{n,n-4} = \frac{2(N_c^2 - 1)}{(-4\pi^2)^{n+l}} \mathcal{I}_n f^{(n+l)} \ ,$$

where $\mathcal{I}_n$ is the unique maximal $n$-point superconformal invariant.

# Twistors

Twistors lines are lines in $\mathbb{P}^3$

$$X^{IJ} = X^I_a X^J_b \epsilon^{ab} \ ,$$

$$X = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

The conformal group acts linearly on twistors.

$$x^2_{ij} \sim \langle X_i X_j \rangle \ ,$$

Super-twistors

$$\mathcal{X}_{s,i} = \begin{pmatrix} X_{s,i} \\ \chi_{s,i} \end{pmatrix} \in \mathbb{C}^{(4|4)}/GL(1) \, ,$$

$$\mathcal{X}^{IJ}_i = \mathcal{X}^I_{a,i} \mathcal{X}^J_{a,i} \epsilon^{ab} \, , \qquad \mathcal{X}^{IJ}_i \in \mathbb{C}^{2\times(4|4)}/GL(2) \, , \qquad I,J = 1,..,8 \, .$$

# Lightlike limit in twistor space

$$(x_i - x_{i+1})^2 \to 0 \quad \Rightarrow \quad \langle X_i X_{i+1} \rangle$$

Two lines on $\mathbb{P}^2$ always intersects

$$Z_i = X_i \cap X_{i+1}$$

Same relation in superspace

$$\mathcal{Z}_i = \mathcal{X}_i \cap \mathcal{X}_{i+1}$$

**Correlator/ amplitude duality**

$$\lim_{x^2_{i,i+1} \to 0} \mathcal{G}_{n,k} / \mathcal{G}^{(0)}_{n;0} = (\mathcal{A}^2)_{n,k} = \sum_{k'=0}^{k} \mathcal{A}_{n,k'} \mathcal{A}_{n,k-k'}$$

[Adamo, Bullimore, Mason, Skinner](2011)
[Eden, Heslop, Korchemsky, Sokatchev](2011)

# Bosonized supertwistors

Bosonized supertwistors

$$\mathcal{X}_{s,i} = \begin{pmatrix} X_i \\ \chi_{s,i} \end{pmatrix} \quad \to \quad X_{s,i} = \begin{pmatrix} X_i \\ \chi_i \cdot \phi_1 \\ \vdots \\ \chi_{s,i} \cdot \phi_k, \end{pmatrix} ,$$

The unique maximal superconformal invariant can be written as

$$\mathcal{I}_n = \delta^{4\times(2n-4)}(X^{\perp} \cdot \chi) ,$$

and then as

$$\mathcal{I}_n = \int \prod_{i=1}^{2n-4} d^4\phi_i \langle X_1 \cdots X_n \rangle^4 .$$

# Bosonized correlator

$$G_{n,n-4} = \prod_{\alpha=1}^{2n-4} \langle Y d^4 Y_\alpha \rangle \langle X_1 \cdots X_n \rangle^4 f^{(n-4)} \ ,$$

$$G_{5,1} = \frac{\langle X_1 X_2 X_3 X_4 X_5 \rangle^4 \prod_{\alpha=1}^{6} \langle Y d^4 Y_\alpha \rangle}{\prod_{\text{permutations}} \langle Y X_i X_j \rangle}$$

The light-like residue of the correlator can be written as the projection of a multiresidue

$$T(Y_1, \ldots, Y_n) \wedge \mathcal{A}^2_{n,n-4} = \text{Res}_{\langle Y X_i X_{i+1} \rangle = 0} \left( G_{n,n-4} \right) \Big|_{(Y_1 \cdots Y_n)^\perp} =$$

$$= \langle Z_1 \cdots Z_n \rangle^4 \prod_{\alpha=1}^{n-4} \langle \hat{Y} d^4 \hat{Y}_\alpha \rangle f^{(n-4)} (\langle \hat{Y} Z_i Z_{i+1} Z_j Z_{j+1} \rangle) \ ,$$

where $\hat{Y}_i = Y_{i+n}$ and $Z_i = X_i \cap X_{i+1}$

# The Amplituhedron

The amplituhedron must be a region that has the planar amplitude singularities as boundaries. A natural guess could be

$$(p_i \cdots p_{j+1})^2 \sim \langle Y i i+1 j j+1 \rangle > 0$$

# The Amplituhedron intrinsic definition

*Unwinding the Amplituhedron in Binary* [N. Arkani-Hamed, H. Thomas, J. Trnka] (2017)

The $\mathscr{A}_{n,k,l}$ is defined as the set of planes $Y \in Gr(k, k+4)$ st

$$\langle Yii+1jj+1\rangle > 0, \qquad (-1)^k\langle Y1nii+1\rangle > 0,$$

The string

$$S_Y := \{\langle Y1234\rangle, \cdots, (-1)^k\langle Y123n\rangle\},$$

must have exactly $f_Y = k$ sign flips.

Moreover $Z$ must be an element of $Gr_>(k+4, n)$.

# Computing NMHV 6-point with the amplituhedron

Let's consider the geometry of the NMHV 6-point amplitude. The condition $\langle Y i i+1 j j+1 \rangle > 0$ reads

$$
\begin{aligned}
\langle Y1234\rangle,\ \langle Y1245\rangle, \langle Y1256\rangle &> 0 \\
\langle Y2345\rangle,\ \langle Y2356\rangle &> 0, \\
\langle Y3456\rangle &> 0, \\
\langle Y1623\rangle,\ \langle Y1634\rangle,\ \langle Y1645\rangle &< 0
\end{aligned}
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The sign flip condition instead reads

$$\{\langle 1234\rangle, \langle 1235\rangle, \langle 1236\rangle\} = \{+,-,-\} \text{ or } \{+,+,-\}$$

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The sign flip condition for $n = 7$ reads

$$\{\langle 1234\rangle, \langle 1235\rangle, \langle 1236\rangle, \langle 1237\rangle\} = \{+,-,-,-\} \text{ or } \{+,+,-,-\} \text{ or } \{+,+,+,-$$

We fix $Z$ as an element in $Gr_{>}0(k+4,n)$

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and parametrize $Y$ as a point in $\mathbb{P}^4$

$$Y = (1, c_1, c_2, c_3, c_4)$$

For example $\langle Y1235 \rangle = c_3$. The amplituhedron is described by

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$$c_4 > 0 \wedge c_2 > 0 \wedge c_2 + c_3 > 0 \wedge c_3 + 1 > 0 \wedge c_3 + c_4 > 0 \wedge c_1 + 1 > 0$$
$$\wedge c_1 + c_4 > 0 \wedge c_1 + c_2 > 0 \wedge (c_3 > 0 \vee -c_3 > 0)$$

If we decompose these inequalities using cylindrical decomposition we obtain

$$c_1 > 0 \wedge c_2 > 1 \wedge c_3 > 0 \wedge c_4 > 0$$
$$c_1 > 0 \wedge c_2 > 1 \wedge c_4 > -c_3 \wedge -1 < c_3 \wedge c_3 < 0$$
$$c_1 > 0 \wedge c_3 > 0 \wedge c_4 > 0 \wedge 0 < c_2 \wedge c_2 < 1$$
$$c_2 > 1 \wedge c_3 > c_1 \wedge c_4 > -c_1 \wedge -1 < c_1 \wedge c_1 < 0$$
$$c_1 > 0 \wedge c_4 > -c_3 \wedge 0 < c_2 \wedge -c_2 < c_3 \wedge c_2 < 1 \wedge c_3 < 0$$
$$c_2 > 1 \wedge c_4 > -c_3 \wedge -1 < c_1 \wedge -1 < c_3 \wedge c_1 < 0 \wedge c_3 < c_1$$
$$c_3 > c_1 \wedge c_4 > -c_1 \wedge -1 < c_1 \wedge -c_1 < c_2 \wedge c_1 < 0 \wedge c_2 < 1$$
$$c_4 > -c_3 \wedge -1 < c_1 \wedge -c_1 < c_2 \wedge c_1 < 0 \wedge -c_2 < c_3 \wedge c_2 < 1 \wedge c_3 < c_1$$

Using iteratively the canonical form of the segment we obtain

$$\begin{aligned}
\mathrm{d}c_1\mathrm{d}c_2\mathrm{d}c_3\mathrm{d}c_4\Big( &-\frac{1}{c_1(c_1+1)(c_2-1)(c_3-c_1)(c_1+c_4)} - \frac{1}{c_1(c_1+1)(1-c_2)(c_1+c_2)(c_3-c_1)(c_1+c_4)} \\
&-\frac{1}{c_1(c_1+1)(c_2-1)(c_1-c_3)(c_3+1)(c_3+c_4)} - \frac{1}{c_1(c_2-1)c_3(c_3+1)(c_3+c_4)} \\
&-\frac{1}{c_1(c_1+1)(1-c_2)(c_1+c_2)(c_1-c_3)(c_2+c_3)(c_3+c_4)} + \frac{1}{c_1(c_2-1)c_3c_4} \\
&-\frac{1}{c_1(1-c_2)c_2c_3(c_2+c_3)(c_3+c_4)} + \frac{1}{c_1(1-c_2)c_2c_3c_4}\Big)
\end{aligned}$$

After simplifying a bit we get

$$\mathrm{d}c_1\mathrm{d}c_2\mathrm{d}c_3\mathrm{d}c_4\Big(\frac{1}{c_1c_2c_3c_4} - \frac{1}{c_3\,(c_3+1)\,(c_3-c_1)\,(c_2+c_3)\,(-c_3-c_4)}$$
$$-\frac{1}{c_1\,(c_1+1)\,(-c_1-c_2)\,c_3\,(-c_1-c_4)} - \frac{1}{(c_1+1)\,(-c_1-c_2)\,c_3\,(c_3-c_1)\,(-c_1-c_4)}\Big)$$

and finally, by covariantizing the result, we obtain

$$\mathrm{NMHV}(6) =$$
$$\langle Yd^4Y\rangle\Big(\frac{\langle 12345\rangle^4}{\langle 1234\rangle\langle 1235\rangle\langle 1245\rangle\langle 1345\rangle\langle 2345\rangle} - \frac{\langle 13456\rangle^3\langle 12345\rangle}{\langle 1235\rangle\langle 1345\rangle\langle 1346\rangle\langle 1456\rangle\langle 3456\rangle}$$
$$+\frac{\langle 12356\rangle^4}{\langle 1235\rangle\langle 1236\rangle\langle 1256\rangle\langle 1356\rangle\langle 2356\rangle} + \frac{\langle 12356\rangle\langle 13456\rangle^3}{\langle 1235\rangle\langle 1346\rangle\langle 1356\rangle\langle 1456\rangle\langle 3456\rangle}\Big)$$

[R. Kojima, C. Langer] (2020)

# Amplituhedron-Like Geometries

The $\mathscr{H}_{n,k}^{(f_Y)}$ amplituhedron-like geometry is defined as the plane $Y$ satisfying

$$\langle Yii+1jj+1\rangle > 0, \qquad (-1)^{f_Y}\langle Y1nii+1\rangle > 0,$$

Moreover the string

$$S_Y := \{\langle Y1234\rangle, \cdots, (-1)^{f_Y}\langle Y123n\rangle\},$$

where $S_Y$ have $f_Y$ flips.

| | geometry | bosonised superspace | superspace |
|---|---|---|---|
| Amplituhedron | $\mathscr{A}_{n,k}$ | $A_{n,k}$ | $\mathcal{A}_{n,k}$ |
| Amplituhedron-like | $\mathscr{H}^{(f)}_{n,k}$ | $H^{(f)}_{n,k}$ | $\mathcal{H}^{(f)}_{n,k}$ |

We proved that

$$\boxed{H^{(f)}_{n,n-4} = A_{n,f} * A_{n,n-f-4}},$$

where $*$ represent the product in bosonized space.

Thus amplituhedron-like geometries with general winding number give products of amplitudes.

# Correlators without Feynman diagrams

*The Correlahedron* [B. Eden, P. Heslop, L. Mason] (2017)

The Correlahedron $\mathscr{G}_{n,n-4}$ is defined as the set of planes $Y \in Gr(n+k, n+k+4)$ st

$$\langle Y X_i X_j \rangle > 0 \qquad \forall\, i, j$$

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$$\langle YX_iX_j \rangle > 0 \qquad \forall\, i,j$$

In the light like limit $\langle YX_iX_{i+1} \rangle \to 0$ and $X_i \to Z_iZ_{i+1}$ so

$$\langle YX_iX_j \rangle > 0 \ \to \ \pm\langle Yii+1jj+1 \rangle > 0$$

# The squared amplituhedron

The geometrical light like limit of the Correlahedron is called the squared amplituhedron

$$\langle Yii+1jj+1\rangle > 0, \qquad \langle Y1nii+1\rangle > 0,$$

or

$$\langle Yii+1jj+1\rangle > 0, \qquad -\langle Y1nii+1\rangle > 0$$

but this region can be written as the union over all possible flipping number!

$$(\mathscr{A}^2)_{n,n-4} = \bigcup_f \mathscr{H}^{(f)}_{n,n-4}$$

which imply the relation between the oriented canonical forms

$$(A^2)_{n,n-4} = \sum_f^{n-4} H^{(f)}_{n,n-4} = \sum_{f=0}^{n-4} A_{n,f} * A_{n,n-4-f}$$

# Conclusions

We generalized (for $k = n - 4$) the amplituhedron correspondence to

Amplituhedron-like geometries $\leftrightarrow$ product of amplitudes

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**Outlooks**

- Extend to non-maximal amplitudes, i.e. $k < n - m$
- Bootstrap correlators from the geometry (work in progress with A. Stewart and P. Heslop)
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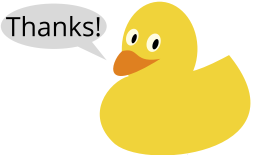


# Full result

$$H^{(k',l')}_{n,n-4,l} = \binom{l}{l'} A_{n,k',l'} * A_{n,n-k'-4,l-l'}$$

The loop amplituhedron like geometry

$$\mathscr{H}^{(f;f_1,..,f_l)}_{n,k,l} := \left\{ Y,(AB)_1,..,(AB)_l \;\middle|\; \begin{array}{ll} Y \in \mathscr{H}^{(f)}_{n,k} & \\ \langle Y(AB)_j i i{+}1\rangle > 0, & \forall j,\ \forall i = 1,..,n{-}1 \\ \langle Y(AB)_j 1 n\rangle\,(-1)^{f_j} > 0 & \forall j \\ \{\langle Y(AB)_j 1 i\rangle\} & \text{has } f_j \text{ flips as } i = 2,..,n, \forall j \\ \langle Y(AB)_i (AB)_j\rangle > 0 & \forall i \neq j \end{array} \right\}$$

for $Z \in Gr_{>}(k+4,n)$ .

The loop flipping number can only assume the values $f, f+2$. So we define

$$\mathscr{H}^{(f;l')}_{n,n-4,l} := \bigcup_{\sigma \in S_l/(S_{l'} \times S_{l-l'})} \mathscr{H}^{(f;\sigma(\overbrace{f+2,..,f+2}^{l'},\overbrace{f,..,f}^{l-l'}))}_{n,n-4,l}$$

## On-shell diagrams and bosonized amplitudes

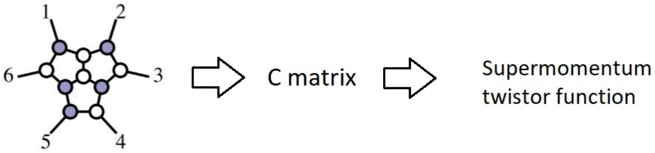
On shell diagrams are trivalent planar diagrams with white and black vertices. To each on-shell diagram in $N = 4$ you can associate a $C$ matrix

$$f^{(k)} = \int \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k}}{\alpha_{4k}} \delta(C(\alpha) \cdot z) \delta^{(4\times k)}(C(\alpha).\chi),$$

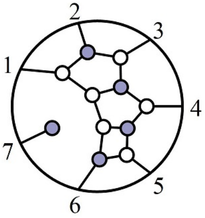
where,

$$\delta^{k\times 4}(C \cdot \chi) := \prod_{i=1}^{4} \prod_{\alpha=1}^{k} C_{\alpha a} \chi^i_a.$$

Solving for $\int \delta(C(\alpha) \cdot z)$ simply corresponds to a map from the $\alpha$ to momentum twistors determinants $\langle ijkl \rangle$.



From on shell diagrams we can get the bosonized expression directly by imposing $Y = C \cdot Z$




$$C = \begin{pmatrix} 1 & \alpha_2+\alpha_4+\alpha_6+\alpha_8 & (\alpha_2+\alpha_4+\alpha_6)\,\alpha_7 & (\alpha_2+\alpha_4)\,\alpha_5 & \alpha_2\,\alpha_3 & 0 \\ 0 & 1 & \alpha_7 & \alpha_5 & \alpha_3 & \alpha_1 \end{pmatrix}$$

$$\frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 d\alpha_8}{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8} =$$

$$\frac{(\langle 1267\rangle\langle 134567\rangle - \langle 1367\rangle\langle 124567\rangle)^4 \langle Yd^4Y_1\rangle\langle Yd^4Y_2\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 1367\rangle\langle 1467\rangle\langle 1567\rangle(\langle 1235\rangle\langle 1467\rangle - \langle 1234\rangle\langle 1567\rangle)(\langle 1567\rangle\langle 2346\rangle - \langle 1467\rangle\langle 2356\rangle)\langle 4567\rangle}$$

Consider two on shell diagrams $f_1^{(k_1)}$ and $f_2^{(k_2)}$ which are associated to matrices $C_1(\alpha)$ and $C_2(\beta)$ . Their product is given by

$$f_1^{(k_1)} f_2^{(k_2)} = \int \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{4k_1}}{\alpha_{4k_1}} \frac{d\beta_1}{\alpha_1} \cdots \frac{d\beta_{4(k_2)}}{\beta_{4(k_2)}} \delta(C_{1,2}(\alpha,\beta)\cdot Z)\delta(C_{1,2}(\alpha,\beta)\cdot\chi).$$

where $C_{1,2} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

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where $C_{1,2} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

Does $Y = C_{1,2}\cdot Z$ for $\alpha, \beta > 0$ belong to the amplituhedron like geometry $\mathscr{H}_{n,k_1+k_2}^{(k1)}$ ?

## Positroids

It exists a $C$ **special parametrization** such that for $\alpha_i > 0$ all minors of $C$ are positive, that is $C \subset \mathrm{Gr}_{\geq}(k, n)$.

$C(\alpha)$ for $\alpha_i > 0$ identifies a region in the oriented Grassmannian $\widetilde{\mathrm{Gr}}(k, n)$ and taking mod $GL(1)$ also a region in $\mathrm{Gr}(k, n)$. These regions are called a positroids.

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$Y(\alpha) = C(\alpha) \cdot Z$ for $\alpha_i > 0$ identifies a point in the amplituhedron $A_{n,k}$.

# On-shell diagram triangulates the maximal amplituhedron like geometry

Let's consider $k = 1$ $n = 6$. The amplitude is given by 3 on-shell diagrams with $C$ matrices

$$\begin{pmatrix} 1 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha_4 & \alpha_3 & 0 & \alpha_2 & \alpha_1 \end{pmatrix}$$

$(NMHV)^2$ will be given by on-shell diagrams with matrices

$$\begin{pmatrix} 1 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ 1 & 0 & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_5 \end{pmatrix} \begin{pmatrix} 1 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ 1 & \alpha_8 & \alpha_7 & 0 & \alpha_6 & \alpha_5 \end{pmatrix} \begin{pmatrix} 1 & 0 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ 1 & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_5 & 0 \end{pmatrix}$$

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Is it true that

$$H^{(f)}_{n,n-m} \supseteq \{Y = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot Z \mid C_1 \in \mathrm{Gr}_>(f,n) \ \wedge \ C_2 \in \mathrm{Gr}_>(n-m-f,n)\} \ ?$$

Answer: No.

But instead what is true is that the image of $C = \begin{pmatrix} C_1 \\ \mathrm{alt}(C_2) \end{pmatrix}$.

$$\begin{pmatrix} 1 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ -1 & 0 & -\alpha_8 & \alpha_7 & -\alpha_6 & \alpha_5 \end{pmatrix} \begin{pmatrix} 1 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 0 \\ -1 & \alpha_8 & -\alpha_7 & 0 & -\alpha_6 & \alpha_5 \end{pmatrix}$$

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In fact we proved that

$$H^{(f)}_{n,n-m} \supseteq \{Y = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot Z \mid C_1 \in \mathrm{Gr}_{>}(f,n) \ \wedge \ C_2 \in \mathrm{alt}(\mathrm{Gr})_{>}(n-m-f,n)\}$$

for reference, the equivalent statement for the amplituhedron is

$$A_{n,k} \supseteq \{Y = C \cdot Z \mid C \in \mathrm{Gr}_{>}(k,n)\}$$

# * product explicit formula

Given $I_a \in \binom{[n]}{k_1+m}$ and $J_b \in \binom{[n]}{k_2+m}$. the $*$ product of bosonized brackets is given by the formula

$$\left(\prod_{a=1}^{m} \langle I_a \rangle_{k_1+m}\right) * \left(\prod_{b=1}^{m} \langle J_b \rangle_{k_2+m}\right) = \frac{(-1)^{(k_1 k_2+k_2)m}}{m!} \sum_{\sigma\in S_m} \prod_{a=1}^{m} \langle Y(I_a \cap J_{\sigma(a)}) \rangle_{k_1+k_2+m},$$

where $S_m$ is the set of permutations of $m$ elements and $(I \cap J)$ represents an intersection in $k_1 + k_2 + m$ dimensions, explicitly:

$$\langle Y(I \cap J) \rangle = \sum_{i \in M(I)} \langle Y i \rangle \langle \bar{i} J \rangle \operatorname{sgn}(i\bar{i}),$$

where $M(I) = \binom{I}{m}$, that is the set of ordered $m$ tuples in $I$, and $\bar{i}$ is the ordered complement of $i$ in $I$, that is $\bar{i} = I - i$.

# Feynman diagrams in twistor space

$$\mathcal{G}_\Gamma = \int \prod_{i=1}^{n} \prod_{m_i=1}^{M_i} \frac{d^2\sigma_{m_i}}{(\sigma_{m_i}, \sigma_{m_i+1})} \prod_{p=1}^{n+k} \bar{\delta}^{4|4}(r_p Z_* + \sigma_{i_p j_p} \cdot X_{i_p} + \sigma_{j_p i_p} \cdot X_{j_p}) \,.$$

# Product of bosonized super amplitudes

We define an operation isomorphic to the product on super space on the bosonized space

$$\mathcal{B}(\mathcal{A}_{n,f}\mathcal{A}_{n,n-f-4}) = A_{n,f} * A_{n,n-f-4}$$

Example:

$$\frac{\langle 12345\rangle^4 \langle Y_1 d^4 Y_1\rangle}{\langle Y_1 1234\rangle\langle Y_1 2345\rangle\langle Y_1 3451\rangle\langle Y_1 4512\rangle\langle Y_1 5123\rangle} * \frac{\langle 12356\rangle^4 \langle Y_2 d^4 Y_2\rangle}{\langle Y_2 1235\rangle\langle Y_2 2356\rangle\langle Y_2 3561\rangle\langle Y_2 5612\rangle\langle Y_2 6123\rangle} =$$

$$= \frac{(\langle 12345\rangle \cap \langle 12356\rangle)^4 \langle Yd^4 Y_1\rangle\langle Yd^4 Y_2\rangle}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y1235\rangle^2\langle Y2356\rangle\langle Y3561\rangle\langle Y5612\rangle\langle Y6123\rangle} =$$

$$\frac{\langle Y1235\rangle^2 \langle 123456\rangle^4 \langle Yd^4 Y_1\rangle\langle Yd^4 Y_2\rangle}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y2356\rangle\langle Y3561\rangle\langle Y5612\rangle\langle Y6123\rangle}$$

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General formula

$$\left(\prod_{a=1}^{m}\langle I_a\rangle_{k_1+m}\right) * \left(\prod_{b=1}^{m}\langle J_b\rangle_{k_2+m}\right) = \frac{(-1)^{(k_1k_2+k_2)m}}{m!}\sum_{\sigma\in S_m}\prod_{a=1}^{m}\langle Y(I_a\cap J_{\sigma(a)})\rangle_{k_1+k_2+m},$$