Exact Four-Point Functions: Genus Expansion, Matrix Model, and Strong Coupling

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1711.05326, 1809.09145: TB, J. Caetano, T. Fleury, S. Komatsu, P. Vieira
1904.00965, 1909.04077: TB, F. Coronado, P. Vieira
+ work in progress

HIGH-ENERGY PHYSICS THEORY SEMINAR
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General Idea

In AdS_5, string amplitudes can be cut into basic patches (rectangles, pentagons, or hexagons), which can be bootstrapped using integrability at any value of the 't Hooft coupling.

- Amplitudes are given as infinite sums and integrals over intermediate states that *glue together* these integrable patches.
- This holds at the planar level as well as for non-planar processes suppressed by $1/N_c$.
- State sums are especially efficient at weak coupling.

Today: Focus on a regime where one can go to large orders in $1/N_c$, or even re-sum the genus expansion, all the way from weak to strong coupling.

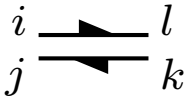
$\mathcal{N} = 4$ SYM & The Planar Limit

$\mathcal{N} = 4$ super Yang–Mills: Gauge field A_μ, scalars Φ_I, fermions $\psi_{\alpha A}$.

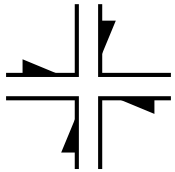
Gauge group: $\mathrm{U}(N_c)$ / $\mathrm{SU}(N_c)$.
Adjoint representation: All fields are $N_c \times N_c$ matrices.

Double-line notation:

Propagators:

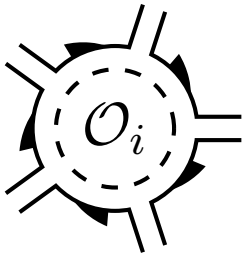
$\langle \Phi_I{}^i{}_j \Phi_J{}^k{}_l \rangle \sim g_{\mathrm{YM}}^2 \delta^i{}_l \delta^k{}_j =$ 

Vertices:

$\mathrm{Tr}(\Phi\Phi\Phi\Phi) \sim \frac{1}{g_{\mathrm{YM}}^2}$ 

- Diagrams consist of color index loops $\simeq$ oriented disks $\sim \delta^i{}_i = N_c$
- Disks are glued along propagators $\to$ oriented compact surfaces

Local operators:

$\mathcal{O}_i = \mathrm{Tr}(\Phi \dots) \sim$ 

- One fewer color loop $\to$ factor $1/N_c$
- Surface: Hole $\sim$ boundary component

Planar Limit & Genus Expansion

Every Feynman diagram is associated to an oriented compact surface.

Genus Expansion: [’t Hooft 1974]

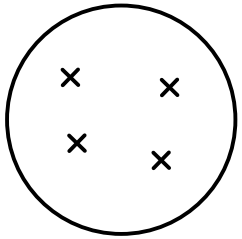
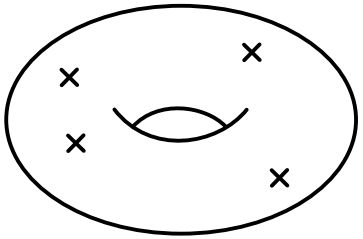
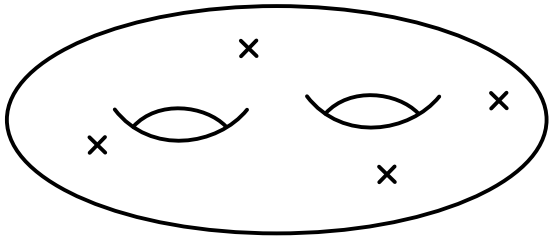
Count powers of N_c and g_{YM}^2 for
propagators ($\sim g_{YM}^2$), vertices ($\sim 1/g_{YM}^2$), and faces ($\sim N_c$)

Absorb one factor of N_c in the ’t Hooft coupling $\lambda = g_{YM}^2 N_c$

Use Euler formula $V - E + F = 2 - 2g$

$\Rightarrow$ **Correlators** of single trace operators $\mathcal{O}_i = \mathrm{Tr}(\Phi_1 \Phi_2 \dots)$:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{N_c^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_c^{2g}} G_g(\lambda)$$

$\sim \frac{1}{N_c^2}$  $+ \frac{1}{N_c^4}$  $+ \frac{1}{N_c^6}$  $+ \dots$

Spectrum: Planar Limit

Goal: Correlation functions in $\mathcal{N} = 4$ SYM

Step 1: Planar spectrum of single-trace local operators $\mathrm{Tr}(\Phi \dots)$

- Spectrum of (anomalous) scaling dimensions Δ
- Scale transformations represented by dilatation operator Γ
- Γ *mixes* single-trace (& multi-trace) operators
- Resolve mixing $\rightarrow$ Eigenstates & eigenvalues (dimensions)

Planar limit:

- Multi-trace operators suppressed by $1/N_c$
- Dilatation operator acts *locally* in color space (neighboring fields)

Organize space of single-trace operators around protected states

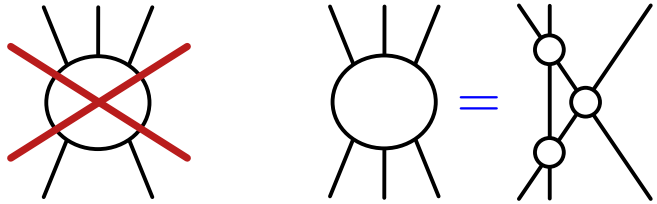
$$\mathrm{Tr}\, Z^L\,, \qquad Z = \alpha^I \Phi_I\,, \qquad \alpha^I \alpha_I = 0 \qquad \text{(half-BPS, “vacuum”)}\,.$$

Other single-trace operators: Insert impurities $\{\Phi_I, \psi_{\alpha A}, D_\mu\}$ into $\mathrm{Tr}\, Z^L$.

Planar Spectrum: Integrability

Initial observation: One-loop dilatation operator for scalar single-trace operators is integrable. Diagonalization by Bethe Ansatz. [Minahan Zarembo]

- Impurities are *magnons* in color space, characterized by *rapidity* (momentum) u and $\mathfrak{su}(2|2)^2$ *flavor index*.
  $[\mathfrak{su}(2|2)^2 \subset \mathfrak{psu}(2,2|4)$ preserves the vacuum $\mathrm{Tr}\, Z^L]$
- Dynamics of magnons: integrability:
  $\rightarrow$ No particle production
  $\rightarrow$ Individual momenta preserved
  $\rightarrow$ Factorized scattering



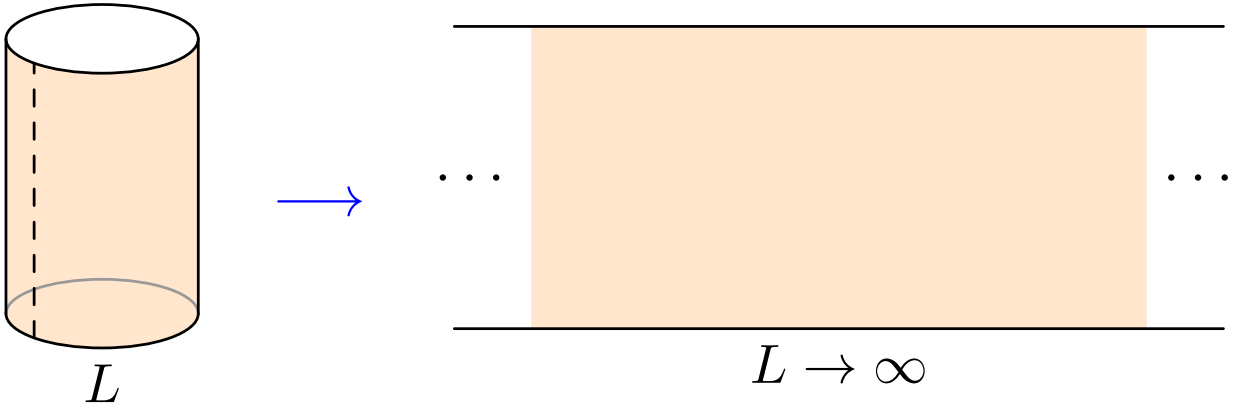
- Two-body ($\rightarrow n$-body) S-matrix completely fixed *to all loops*

[Beisert 2005] [Janik 2006] [Beisert,Hernandez Lopez 2006]

$\Rightarrow$ Asymptotic spectrum (for $L \rightarrow \infty$) solved to all loops / exactly.

Finite-Size Effects

Asymptotic spectrum solved by Bethe Ansatz.
Resums ∞ Feynman diagrams that govern dynamics of ∞ strip:



Re-compactify: Finite-size effects.
Leading effect: Momentum quantization constraint $\equiv$ Bethe equations

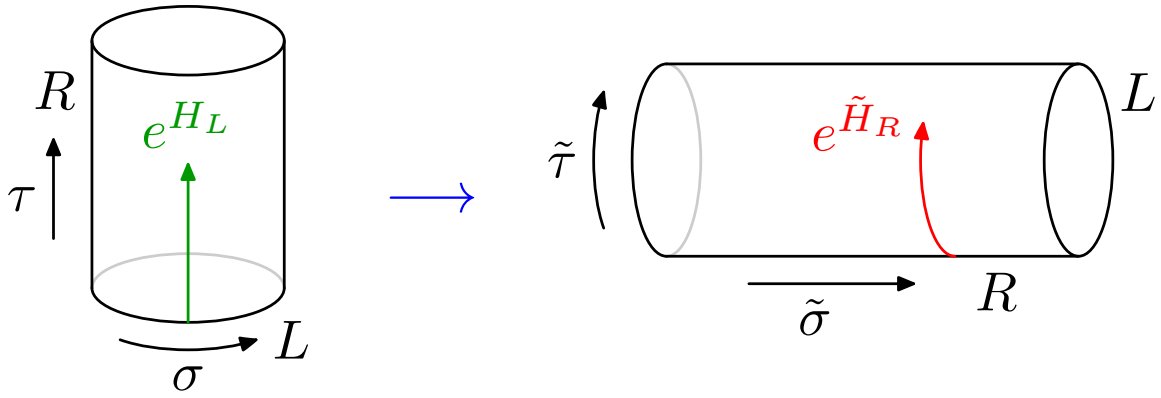
$$1 = e^{ip_jL} \prod_{j \neq k} S(p_k, p_j)$$

Moreover: Wrapping interactions.

- No notion of locality for dilatation operator
- Previous techniques (Bethe ansatz) no longer apply

Mirror Theory

Key to all-loop finite-size spectrum: Mirror map [Arutyunov Frolov]
Double Wick rotation: $(\sigma, \tau) \to (i\tilde{\tau}, i\tilde{\sigma})$ — exchanges space and time



Magnon states: Energy and momentum interchange: $\tilde{E} = ip$, $\tilde{p} = iE$

Finite size L becomes finite, periodic (discrete) time.
Energy $\sim$ Partition function at finite temperature $1/L$, with $R \to \infty$.
$\to$ *Thermodynamic Bethe ansatz*. [Bombardelli '09 Fioravanti,Tateo][Gromov,Kazakov Kozak,Vieira '09][Arutyunov Frolov '09]

Simplifications and refinements:

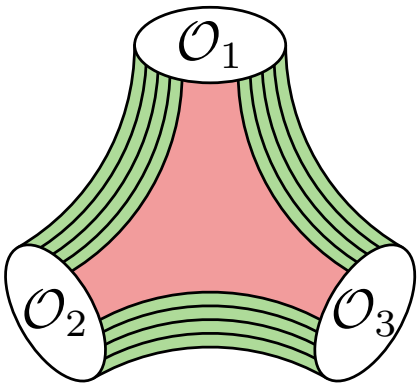
- Y-system (T-system, Q-system) [Gromov,Kazakov Kozak,Vieira '09][Arutyunov Frolov '09]
- Quantum Spectral Curve [Gromov,Kazakov Leurent,Volin '13][Gromov,Kazakov Leurent,Volin '14]

$\Rightarrow$ Scaling dimensions computable at finite coupling.

Three-Point Functions: Hexagons

Differences: Topology: Pair of pants instead of cylinder
Non-vanishing for three generic operators (two-point: diagonal)
$\Rightarrow$ Previous techniques not directly applicable

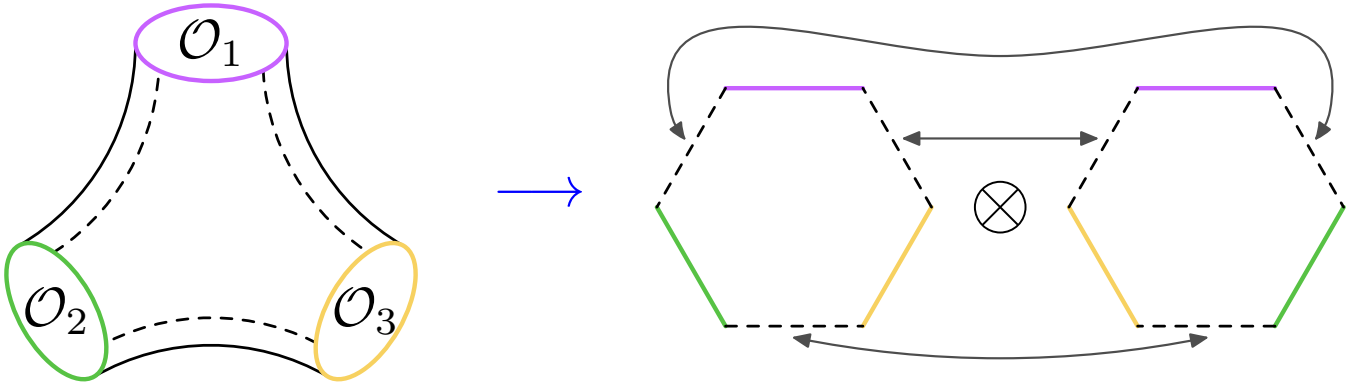
Observation:



The green parts are similar to two-point functions: Two segments of physical operators joined by parallel propagators ("bridges", $\ell_{ij} = (L_i + L_j - L_k)/2$).

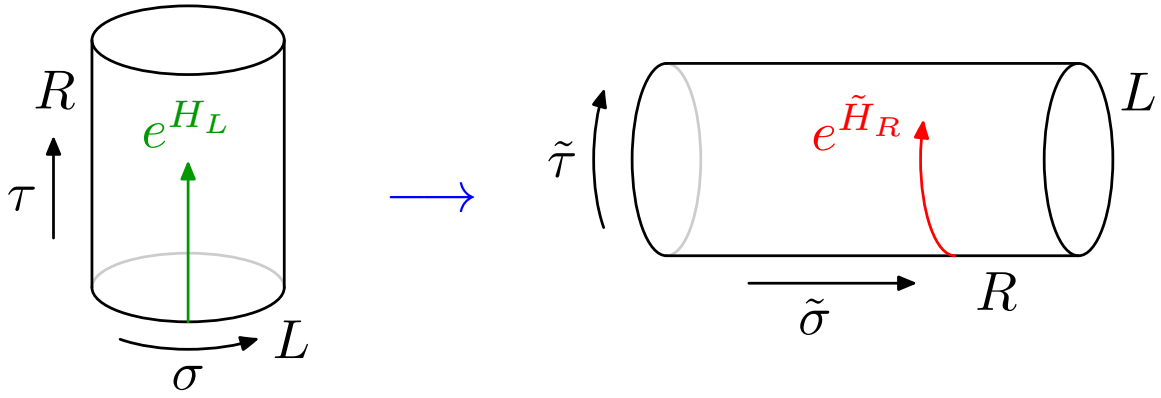
The red part is new: "Worldsheet splitting", "three-point vertex" (open strings)

Take this serious $\rightarrow$ cut worldsheet along "bridges": [Basso,Komatsu Vieira '15]



The Cut State Sum (Mirror Theory)

On each bridge lives a mirror theory:
Double-Wick (90 degree) rotation $(\sigma, \tau) \to (i\tilde{\tau}, i\tilde{\sigma})$



In all computations, the volume R can be treated as infinite.

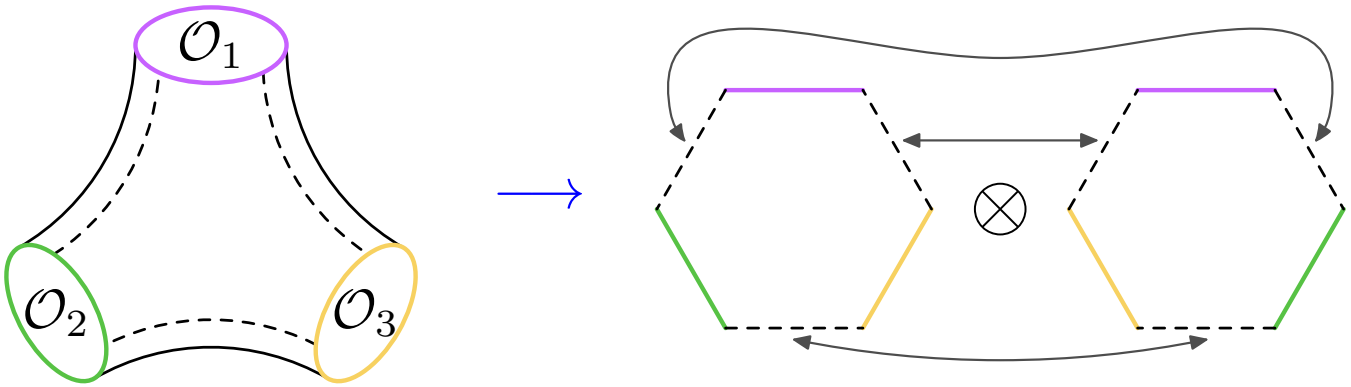
$\Rightarrow$ Mirror states are free multi-magnon Bethe states, characterized by rapidities u_i, bound state indices a_i, and flavor indices $(A_i, \dot{A}_i)$.

The mirror integration therefore expands to

$$\int_{\boldsymbol{M}_b} \mathrm{d}\psi_b = \sum_{m=0}^{\infty} \prod_{i=1}^{m} \sum_{a_i=1}^{\infty} \sum_{A_i, \dot{A}_i} \int_{u_i=-\infty}^{\infty} \mathrm{d}u_i \, \mu_{a_i}(u_i) \, e^{-\tilde{E}_{a_i}(u_i)\, \ell_b} \, .$$

μ_{a_i}: measure factor, $\tilde{E} = ip$: mirror energy,
ℓ_b: length of bridge b (discrete "time").

Hexagons & Gluing



$\otimes$ **Glue** hexagons along three *mirror channels*: [Basso,Komatsu Vieira '15][Basso,Goncalves Komatsu,Vieira '15]

- Sum over complete state basis (magnons) in the mirror theory
- Mirror magnons: Boltzmann weight $\exp(-\tilde{E}_{ij}\ell_{ij})$, $\tilde{E}_{ij} = \mathcal{O}(g^2)$
  $\rightarrow$ mirror excitations are strongly suppressed.

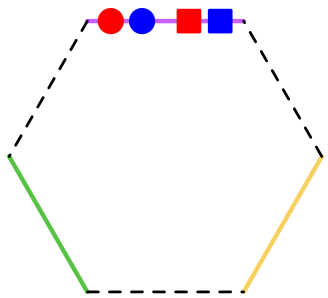
Hexagonal worldsheet patches (form factors):

- Function of rapidities u and $\mathfrak{su}(2|2)^2$ labels $(A, \dot{A})$ of all magnons.
- Conjectured exact expression, based on diagonal $\mathfrak{su}(2|2)$ symmetry as well as form factor axioms. [Basso,Komatsu Vieira '15]

Finite-coupling hexagon proposal: Supported by very non-trivial matches.

The Hexagon Form Factors

Hexagon = Amplitude that measures the overlap between three mirror and three physical off-shell Bethe states. Worldsheet branching operator that creates an excess angle of π. [Basso, Komatsu Vieira '15]

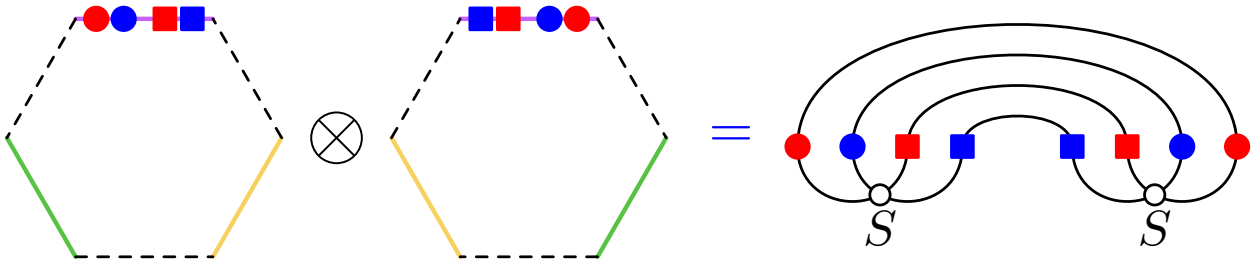


Explicitly: $\mathcal{H}(\chi^{A_1}\chi^{\dot{A}_1}\chi^{A_2}\chi^{\dot{A}_2}\dots\chi^{A_n}\chi^{\dot{A}_n})$

$$= (-1)^{\mathfrak{F}}\left(\prod_{i<j} h_{ij}\right)\langle\chi^{A_1}\chi^{A_2}\dots\chi^{A_n}|S|\chi^{\dot{A}_n}\dots\chi^{\dot{A}_2}\chi^{\dot{A}_1}\rangle$$

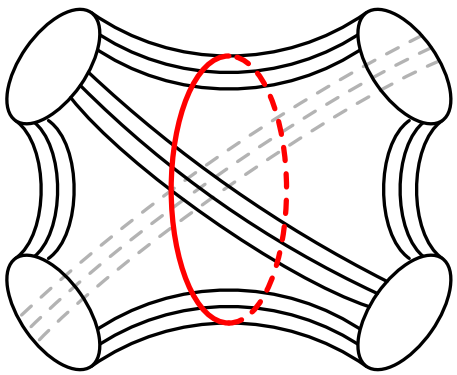
- $\chi^A, \chi^{\dot{A}}$: Left/Right $\mathfrak{su}(2|2)$ fundamental magnons
- $\mathfrak{F}$: Fermion number operator
- S: Beisert S-matrix
- $h_{ij} = \dfrac{x_i^- - x_j^-}{x_i^- - x_j^+}\dfrac{x_j^+ - 1/x_i^-}{x_2^+ - 1/x_1^+}\dfrac{1}{\sigma_{ij}}$, $\quad x^{\pm}(u) = x(u \pm \frac{i}{2})$, $\quad \frac{u}{g} = x + \frac{1}{x}$
  σ_{ij}: BES dressing phase

Example:
Two magnons
(●■, ●■)



Planar Four-Point Functions: Hexagonalization

Move on to planar four-point functions:
One way to cut (now that three-point is understood): **OPE cut**

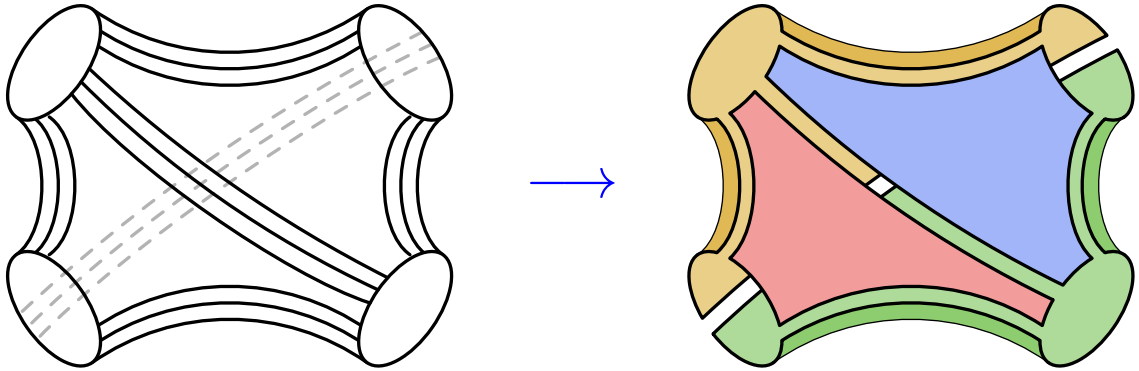


Problem: Sum over physical states!

- No loop suppression, all states contrib.
- Double-trace operators.

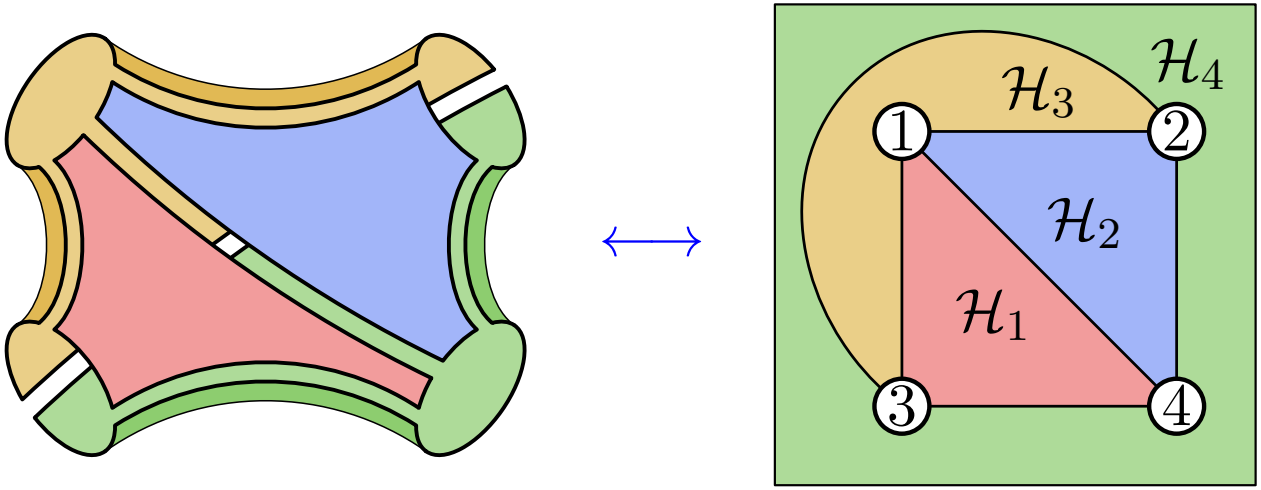
Instead: Cut along propagator bridges

[Fleury '16 Komatsu][Eden '16 Sfondrini]



Benefits:
- Mirror states highly suppressed in g.
- No double traces.

Hexagonalization: Formula



$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \left[\prod_{\substack{\text{channels} \\ c \in \{1,2,3\}}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \right] \mathcal{H}_1(\psi_1, \psi_2, \psi_3)\, \mathcal{H}_2(\psi_1, \psi_2, \psi_3)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_{\substack{\text{planar} \\ \text{prop. graphs}}} \left[\prod_{\substack{\text{channels} \\ c \in \{1,\ldots,6\}}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \right] \mathcal{H}_1\, \mathcal{H}_2\, \mathcal{H}_3\, \mathcal{H}_4$$

New Features: [Fleury '16, Komatsu]

- Bridge lengths vary, may go to zero $\Rightarrow$ Mirror corrections at one loop
- Mirror corrections may span several hexagons

Non-Planar Extension

- Fix worldsheet topology
- Cut into planar hexagons
- Sum mirror states

Simple Proposal:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle^{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\substack{\text{graphs} \\ (\text{genus } g)}} \prod_c d_c^{\ell_c} \sum_{\substack{\text{mirror} \\ \text{states}}} \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \dots \mathcal{H}_F$$

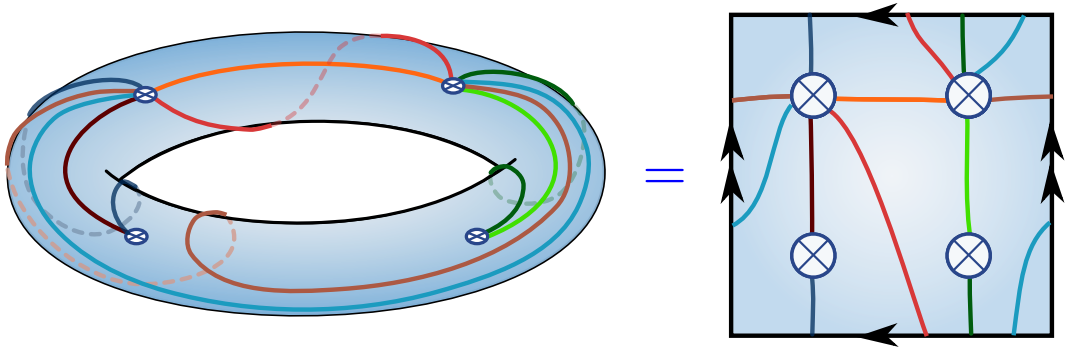
In full, concrete and explicit

[TB, Caetano, Fleury Komatsu, Vieira '17][TB, Caetano, Fleury Komatsu, Vieira '18]

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^n \sqrt{k_i}}{N_c^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_c^{2g(\Gamma)}} \times$$
$$\times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right]^{2n+4g(\Gamma)-4} \prod_{a=1} \mathcal{H}_a \, .$$

Remarkable fact:
For $\mathcal{N} = 4$ SYM, all ingredients of the formula are well-defined and explicitly known as functions of the 't Hooft coupling λ.

Features



Number of graphs (Wick contractions) grows factorially with the genus

Performing the mirror sums is very difficult in general

Mirror states may encircle operators (wrapping) or handles

Sum over graphs discretizes the string moduli space integration

Need to subtract overcounting at the moduli space boundaries

Mirror particles explore vicinity of discrete points

Need to mod out by modular transformations $\rightarrow$ difficult

Simplify: Large-Charge Limit

Everything simplifies in the large-charge limit due to combinatorics!

Assume two operator share m propagators.
Consider a graph where these two operators are connected by b bridges (propagator bundles).

For large m, summing over all ways to distribute the m propagators on the b bridges gives a combinatorial factor

$$\sum_{\substack{m_1,\ldots,m_b \\ m_1+\cdots+m_b=m}} = \frac{m^{b-1}}{(b-1)!} + \mathcal{O}(m^{b-2})$$

Consider a correlator of operators $\mathcal{O}_i$ with charges k_i.
In the large-charge limit $k_i \to \infty$:

Octopus principle:

- Only graphs with a maximal number of bridges contribute
- All bridges are occupied by a large number of propagators

Large-Charge Limit

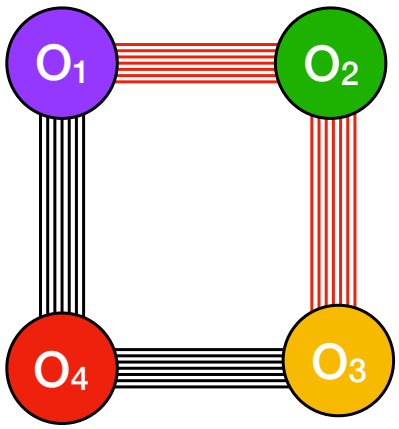
Features:

- All bridges are large
  $\rightarrow$ all mirror corrections on such bridges are strongly suppressed.
- Far away from all moduli space boundaries

For n half-BPS operators with generic polarizations, all contributing graphs will consist of hexagons separated by large bridges. Hence *all* loop corrections will be suppressed!

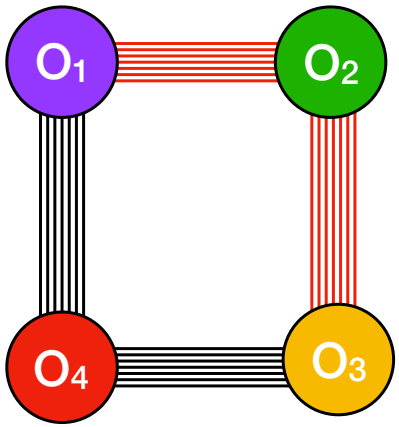
Want something more interesting: Consider operators $\mathcal{O}_1, \ldots, \mathcal{O}_4$ that are polarized such that each operator can only contract with its direct neighbors.

$$\mathcal{O}_1 = \mathrm{Tr}(\bar{Z}^k \, \bar{X}^k)(0) + \texttt{perm.}$$
$$\mathcal{O}_2 = \mathrm{Tr}(X^{2k})(z)$$
$$\mathcal{O}_3 = \mathrm{Tr}(\bar{Z}^k \, \bar{X}^k)(1) + \texttt{perm.}$$
$$\mathcal{O}_4 = \mathrm{Tr}(Z^{2k})(\infty)$$



The "Simplest" Correlator

$$\mathcal{O}_1 = \mathrm{Tr}(\bar{Z}^k\,\bar{X}^k)(0) + \texttt{perm.}$$
$$\mathcal{O}_2 = \mathrm{Tr}(X^{2k})(z)$$
$$\mathcal{O}_3 = \mathrm{Tr}(\bar{Z}^k\,\bar{X}^k)(1) + \texttt{perm.}$$
$$\mathcal{O}_4 = \mathrm{Tr}(Z^{2k})(\infty)$$



For large k, all interactions are confined to the inside and the outside of the square propagator frame.

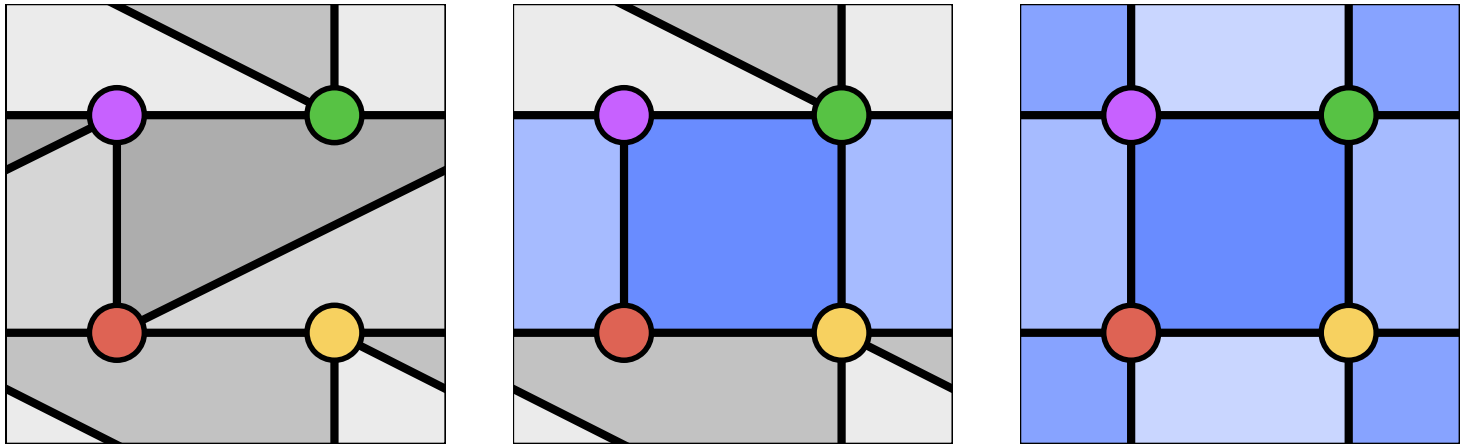
Each of the two faces constitute an *octagon* $\mathbb{O}$ that consists of two hexagons which are glued along a bridge of *zero length*.

This octagon $\mathbb{O}$ (mirror sum) has been computed to 24 loops [Coronado 2018]

It is a polynomial in ladder integrals,
has been bootstrapped to all loops (Steinmann basis, lightcone limit) [Coronado 2018]
and has a determinant representation [Kostov, Petkova Serban 2019]

Regardless of the value of the octagon $\mathbb{O}$, let us consider the correlator at higher genus.

Higher Genus Graphs



(a) $\mathbb{O}^0$ (b) $\mathbb{O}^2$ (c) $\mathbb{O}^4$

Example graphs on the torus

Each line = a large number of propagators

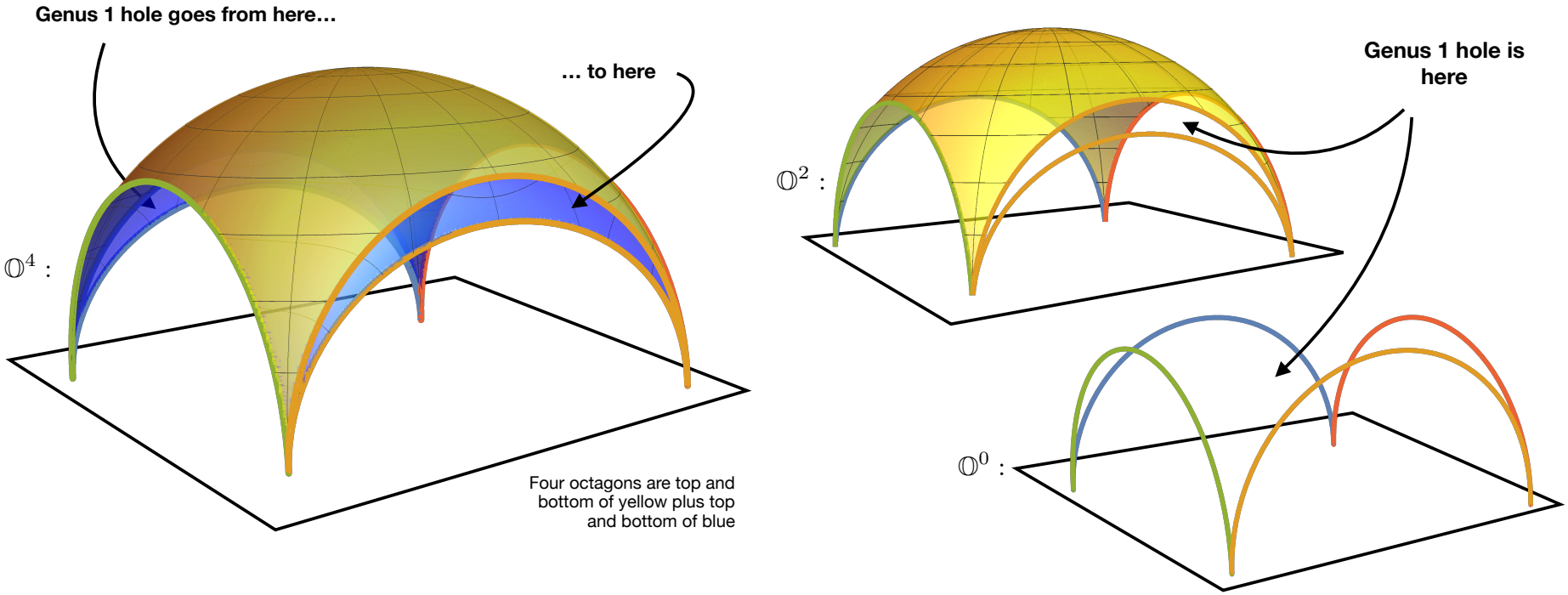
All loop corrections are confined to individual faces

Gray faces: Touch at most three operators $\rightarrow$ protected

Blue faces: Touch four operators, each face = one factor $\mathbb{O}$

At higher genus:
All faces are octagons (some protected, some unprotected)
All bigger faces could be split into octagons

String picture



Each edge is a bundle of $\mathcal{O}(\sqrt{N_c})$ propagators, therefore becomes a heavy BMN geodesic connecting two operators.

The geodesics are fold lines that connect adjacent octagons.

The non-trivial non-BPS octagon $\mathbb{O}$ extends in AdS and touches all four operators.

The BPS octagons have no extent, they curl up along the geodesics.

Higher Genus Systematics

Have seen: In the large charge limit, all faces are octagons

Euler: $2 - 2g = (V = 4) + (F = n) - (E = 4n/2) = 4 - n$
$\Rightarrow n = 2g + 2$ octagons, $4n/2 = 4g + 4$ edges

In every graph, $\mathbb{O}$ appears an even number of times
The four-point correlator therefore is

$$G(N_c, \lambda) = G(N_c = \infty, \lambda = 0) \cdot \left(\mathbb{O}^2 + \frac{k^4}{N_c^2} P_2(\mathbb{O}^2) + \frac{k^8}{N_c^4} P_3(\mathbb{O}^2) + \dots \right)$$

The P_n are polynomials of degree n in $\mathbb{O}^2$.

Finding the P_{g+1} amounts to counting graphs of genus g,
more specifically counting quadrangulations.

Consider the double-scaling limit $\quad k \sim \sqrt{N_c}\,, \quad N_c \to \infty\,, \quad \zeta \equiv \frac{k}{\sqrt{N_c}}$ fixed.

The correlator becomes:

$$\frac{G(N_c, \lambda)}{G(\infty, 0)} \longrightarrow \sum_{g=0}^{\infty} \zeta^{4g} P_{g+1}(\mathbb{O}^2) \equiv \mathcal{A}(\zeta, \mathbb{O})$$

Matrix Model I

Counting graphs (here: quadrangulations) is what matrix models do.

Pass to dual graphs (exchange faces and vertices).

There are 4 types of bridges $\mathcal{O}_i$—$\mathcal{O}_{i+1}$,
these become the 4 complex matrices A, B, C, D of the matrix model.

There are 10 original faces:
1–2–3–4 ($= \mathbb{O}$), 1–4–3–2 ($= \mathbb{O}$),
1–2–4–2 (and 3 cyclic), 1–2–1–2 (and 3 cyclic).
These are the vertices of the matrix model.

$$S_{\text{kin}} = \operatorname{Tr}\left[\frac{A\bar{A}}{k_1} + \frac{B\bar{B}}{k_2} + \frac{C\bar{C}}{k_3} + \frac{D\bar{D}}{k_4}\right]$$

$$S_{\text{int}} = \mathbb{O}\operatorname{Tr}(ABCD) + \mathbb{O}\operatorname{Tr}(\bar{D}\bar{C}\bar{B}\bar{A}) + \operatorname{Tr}\left[\frac{(A\bar{A})^2 + (B\bar{B})^2 + (C\bar{C})^2 + (D\bar{D})^2}{2} + AB\bar{B}\bar{A} + BC\bar{C}\bar{B} + CD\bar{D}\bar{C} + DA\bar{A}\bar{D}\right]$$

$$Z \equiv \int [\mathcal{D}A][\mathcal{D}B][\mathcal{D}C][\mathcal{D}D] \exp\left(-S_{\text{kin}}[A,B,C,D] + S_{\text{int}}[A,B,C,D]\right)$$

Matrix Model II

$$S_{\text{kin}} = \text{Tr}\left[\frac{A\bar{A}}{k_1} + \frac{B\bar{B}}{k_2} + \frac{C\bar{C}}{k_3} + \frac{D\bar{D}}{k_4}\right]$$

$$S_{\text{int}} = \mathbb{O}\,\text{Tr}(ABCD) + \mathbb{O}\,\text{Tr}(\bar{D}\bar{C}\bar{B}\bar{A})$$
$$+ \text{Tr}\left[\frac{(A\bar{A})^2 + (B\bar{B})^2 + (C\bar{C})^2 + (D\bar{D})^2}{2} + AB\bar{B}\bar{A} + BC\bar{C}\bar{B} + CD\bar{D}\bar{C} + DA\bar{A}\bar{D}\right]$$

$$Z \equiv \int [\mathcal{D}A][\mathcal{D}B][\mathcal{D}C][\mathcal{D}D] \exp\left(-S_{\text{kin}}[A,B,C,D] + S_{\text{int}}[A,B,C,D]\right)$$

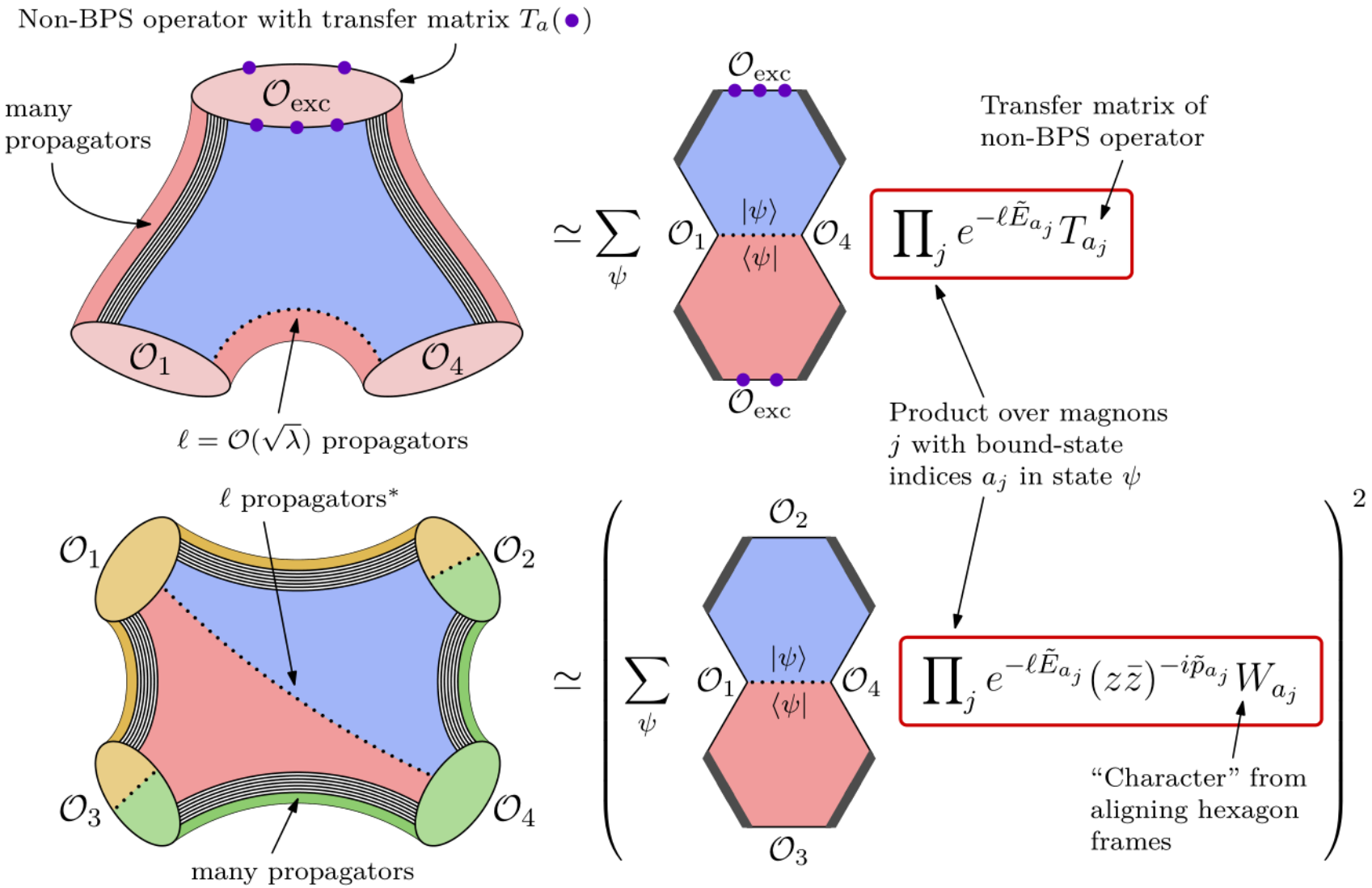
Here, we have generalized to k_i propagators between $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$

$$\mathcal{A}(\zeta_1,\zeta_2,\zeta_3,\zeta_4|\mathbb{O}) = \sum_{g=0}^{\infty} \frac{P_{4g|g+1}(k_1,k_2,k_3,k_4|\mathbb{O})}{N_c^{2g}}$$

The polynomials $P_{4g|g+1}$ are of degree $g+1$ in $\mathbb{O}$,
and of homogeneous degree $4g$ in the k_i.

$$P_{4g|g+1} \xrightarrow{k_i=k} k^{4g} P_{g+1}$$

Octagon at Strong Coupling



*either $\ell \simeq 0$ (most of this work), or $\ell = \mathcal{O}(\sqrt{\lambda})$, see Appendix B.

Summary and Outlook

Hexagons compute correlators, in principle at any λ and any genus.

Details for higher g not fully sorted out. Need more data!

Simplest regime: Large-charge limit! Can re-sum large-N_c expansion in a double-scaling limit, via a matrix model.

Next: Reduce charges in controlled way.

Five/Six-point functions of similar type?

Twist the theory (fishnet) $\rightarrow$ no susy, "BPS" operators become non-trivial, still the charge-flow is very constraining.

More broadly: String-bit-like picture. How general is this?
Any worldsheet theory? Setup a bootstrap?

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Thank you!

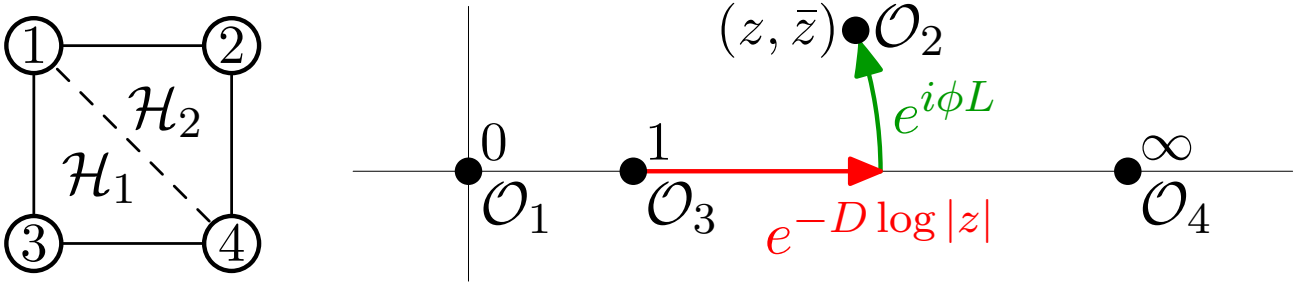
Extra slides follow.

Frames & Weight Factors

Hexagon depends on positions x_i and polarizations α_i of the three half-BPS operators $\mathcal{O}_i = \mathrm{Tr}[(\alpha_i \cdot \Phi(x_i))^k]$. These preserve a diagonal $\mathfrak{su}(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

Two neighboring hexagons share two operators, but the third/fourth operator may not be identical. $\Rightarrow$ The two hexagon frames are misaligned.

In order to consistently sum over mirror states, need to align the two frames by a $\mathrm{PSU}(2,2|4)$ transformation g that maps $\mathcal{O}_3$ onto $\mathcal{O}_2$:



$$g = e^{-D\log|z|} e^{i\phi L} \cdot e^{J\log|\alpha|} e^{i\theta R}\,,$$
$$e^{2i\phi} = z/\bar{z}\,,$$
$$e^{2i\theta} = \alpha/\bar{\alpha}\,.$$

Hexagon $\mathcal{H}_1 = \hat{\mathcal{H}}$ is canonical, and $\mathcal{H}_2 = g^{-1}\hat{\mathcal{H}}g$.

Sum over states in mirror channel: [Fleury '16, Komatsu]

$$\sum_\psi \mu(\psi)\langle\mathcal{H}_2|\psi\rangle\langle\psi|\mathcal{H}_1\rangle = \sum_\psi \mu(\psi)\langle g^{-1}\hat{\mathcal{H}}|\psi\rangle\langle\psi|g|\psi\rangle\langle\psi|\hat{\mathcal{H}}\rangle$$

Weight factor: $\mathcal{W}(\psi) = \langle\psi|g|\psi\rangle = e^{-2i\tilde{p}_\psi \log|z|} e^{J_\psi \varphi} e^{i\phi L_\psi} e^{i\theta R_\psi}$, $i\tilde{p} = (D-J)/2$.

$\rightarrow$ Contains all non-trivial dependence on cross ratios $z, \bar{z}$ and $\alpha, \bar{\alpha}$.

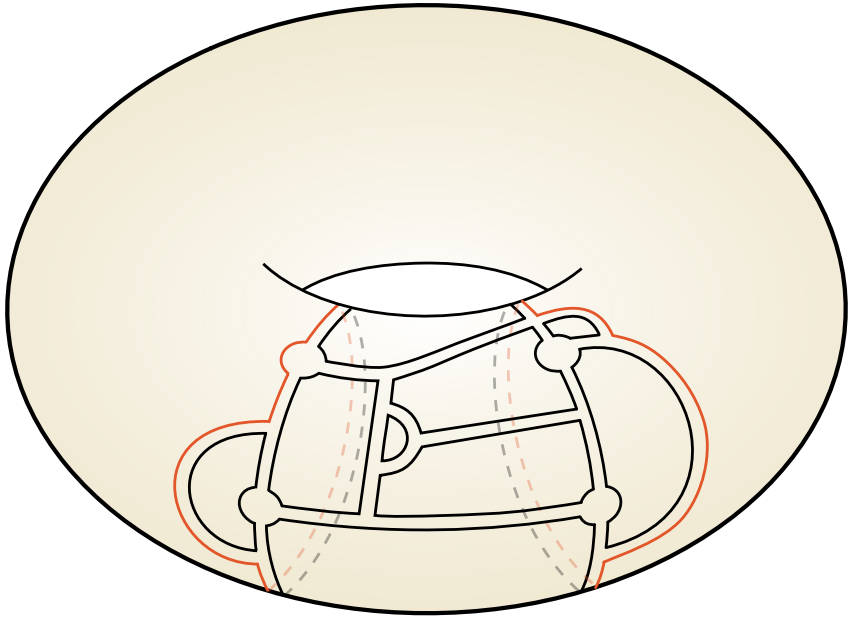
Stratification & Moduli Space

Stratification is also natural from the string theory point of view:
The sum over graphs discretizes the integration over the moduli space of worldsheet Riemann surfaces.

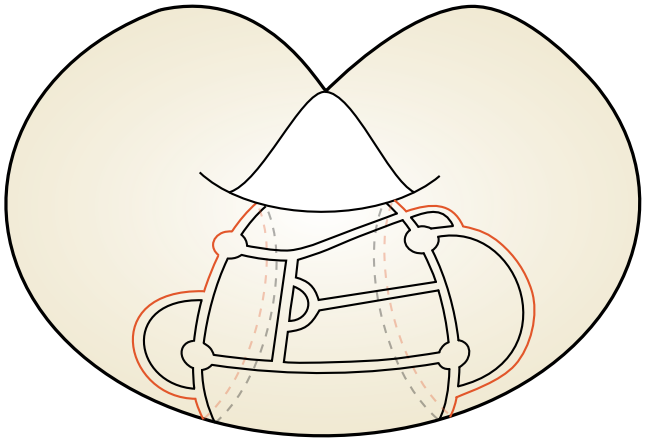
The moduli space includes boundaries. In continuous integrations, these boundaries are measure-zero sets and hence do not contribute. But in a discretized sum, it matters which terms are included or dropped.

Moduli space discretizations have been considered before in the context of matrix models, and the right treatment of boundary contributions in the known cases is in line with the above prescription (stratification). [Chekhov 1995]

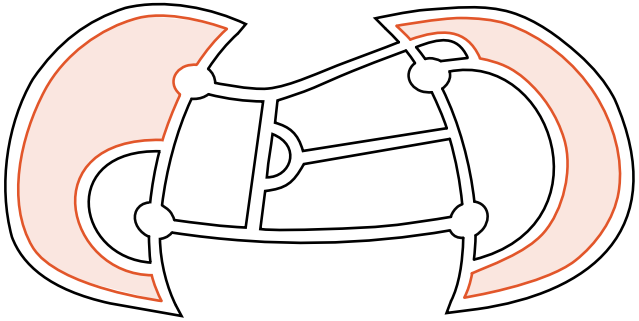
Stratification: Degeneration Type I



(a)

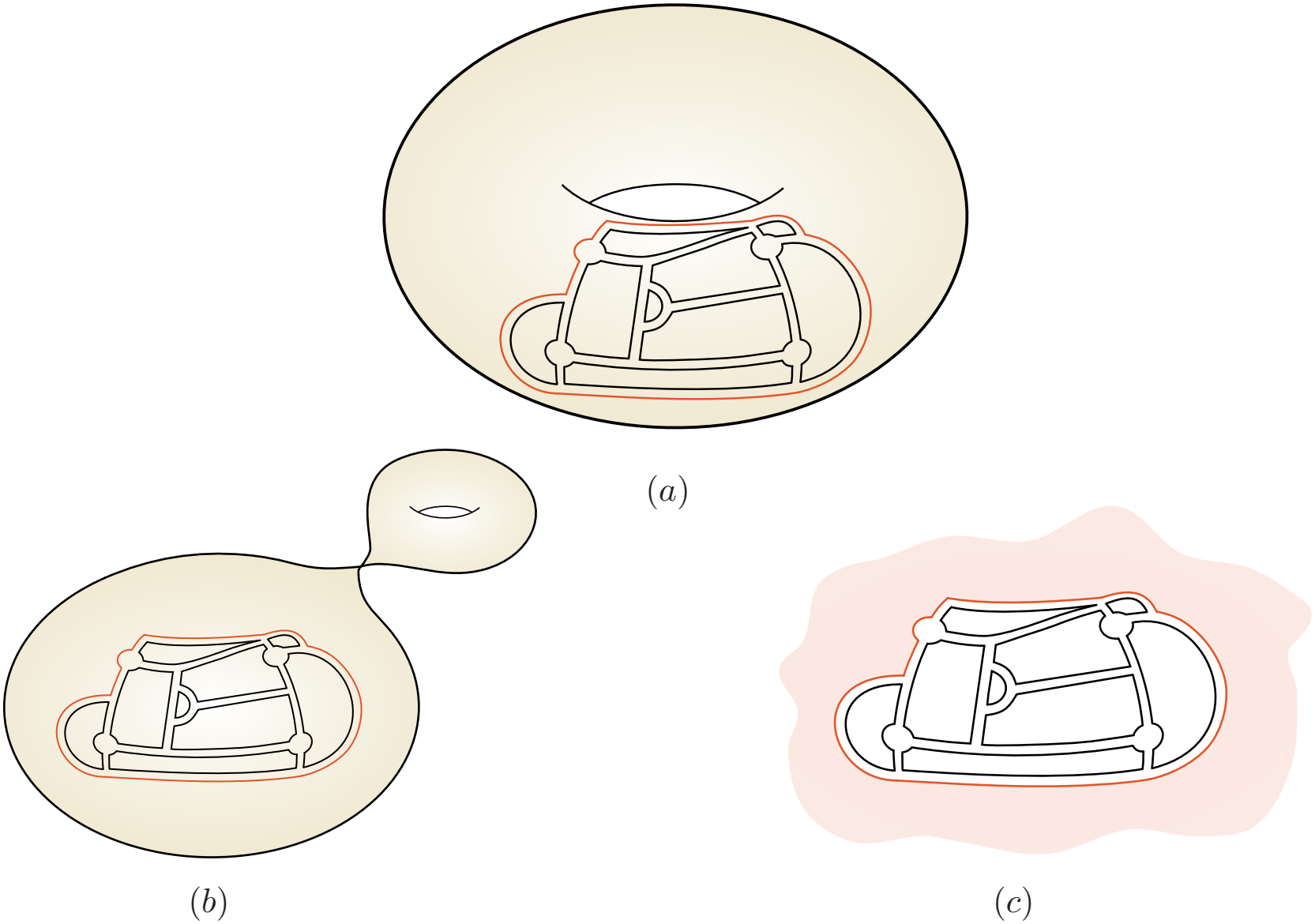


(b)



(c)

Stratification: Degeneration Type II



Stratification: Final Formula

At higher genus, simple degenerations subtract terms multiple times
$\rightarrow$ need to be compensated by *adding* double degenerations etc.
$\rightarrow$ alternating sum.
Also need to account for disconnected degenerations. Final result:

$$\mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \equiv \sum_{g=0}^{\infty} \sum_{c=1}^{2g+n-2} \sum_{\tau \in \boldsymbol{\tau}_{g,c,n}} (-1)^{\sum_i m_i/2} \sum_{\Gamma \in \Sigma_\tau} .$$

c: Number of components of the surface

τ: Genus-g topology with c components and n punctures:

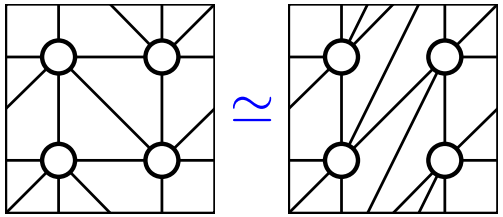
$$\boldsymbol{\tau}_{g,c,n} = \left\{ \{(g_1, n_1, m_1) \dots (g_c, n_c, m_c)\} \,\middle|\, \sum_i n_i = n,\ \sum_i (g_i + \tfrac{m_i}{2}) - c + 1 = g \right\}$$

where (g_i, n_i, m_i) labels the genus, the number of punctures, and the number of marked points on component i.

Σ_τ: Set of all graphs Γ (connected and disconnected) that are compatible with the topology τ and that are embedded in the surface defined by τ in all inequivalent possible ways (Γ may cover all or only some components of the surface).

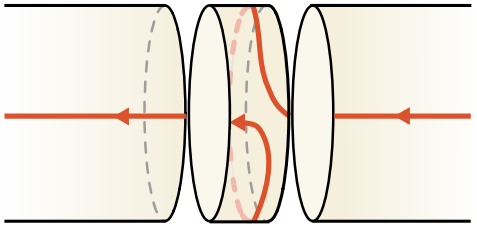
Dehn Twists and Modular Group

We implicitly identified graphs that only differ by "twists" of a handle:



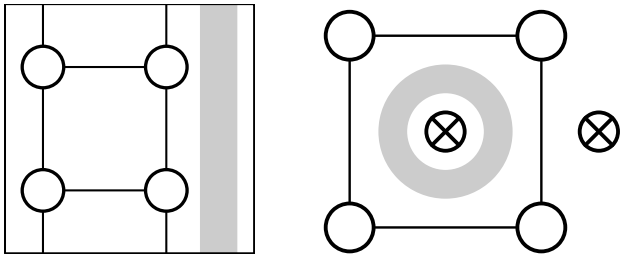
This makes sense at weak coupling: Identity at the level of Feynman graphs.

Also makes sense from string moduli space perspective: *Dehn twists* are modular transformations that leave the complex structure invariant.



Modding out by Dehn twists has non-trivial implications for the summation over mirror states, especially for stratification terms:

Dehn twists along cycles not covered by the propagator graph act trivially in the absence of mirror particles:



Once the cycle is dressed with zero-length bridges and mirror particles, Dehn twists will non-trivially map sets of mirror magnons onto each other.
$\rightarrow$ Need to mod out by this non-trivial action!