Handling Handles: Non-Planar AdS/CFT Integrability

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1711.05326: TB, J. Caetano, T. Fleury, S. Komatsu, P. Vieira
1809.09145: TB, J. Caetano, T. Fleury, S. Komatsu, P. Vieira
18xx.xxxxx: TB, F. Coronado, P. Vieira
18xx.xxxxx: TB, F. Coronado, V. Gonçalves, P. Vieira
+ further work in progress

ENS/SACLAY INTEGRABILITY MEETING
SACLAY, OCTOBER 2018

Invitation

String Theory: String amplitudes are integrals over the moduli space of Riemann surfaces of various genus.

Large-N_c Gauge Theory: Correlation functions are sums over double-line Feynman (ribbon) graphs of various genus.

AdS/CFT: These two quantities/concepts should be the same.

Question: How does the continuous worldsheet moduli space integration emerge from the discrete sum over Feynman graphs? Answering this questions is crucial for understanding the nature of holography.

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Question: How does the continuous worldsheet moduli space integration emerge from the discrete sum over Feynman graphs? Answering this questions is crucial for understanding the nature of holography.

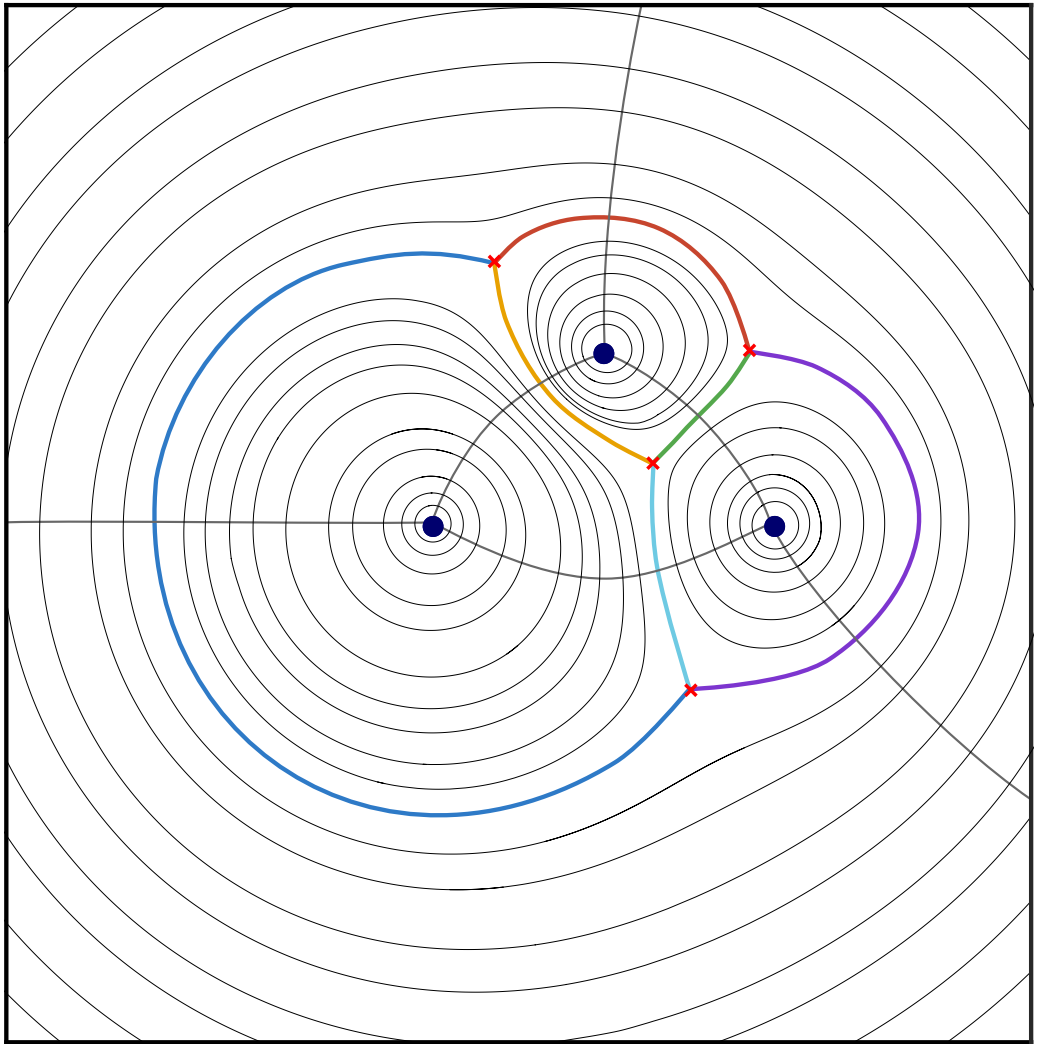
This Talk: Provide one concrete realization. [TB, Caetano, Fleury Komatsu, Vieira '17][TB, Caetano, Fleury Komatsu, Vieira '18]

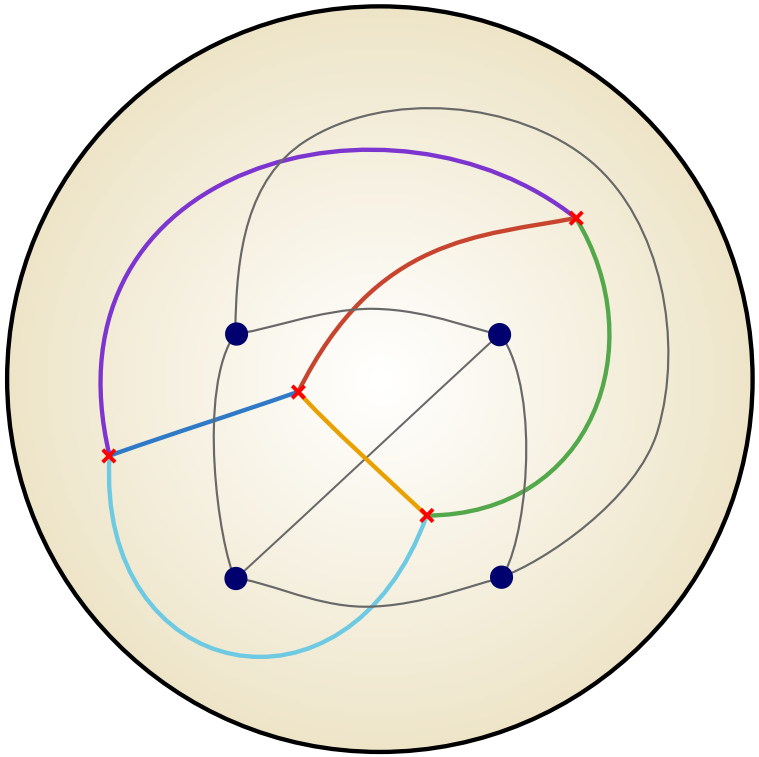
Initial motivation: Compute non-planar corrections to correlators using hexagon form factors, building on planar methods/results. [Basso,Komatsu Vieira '15][Fleury '16 Komatsu]

Along the way understood that the necessary sum over worldsheet tessellations quantizes the string moduli space integration.

In this respect, a finite-coupling extension of ideas by Gopakumar, Razamat et al. [Gopakumar '03 '04 '04 '05][Aharony, Komargodski Razamat '06][Aharony, David, Gopakumar Komargodski, Razamat '07][Razamat '08]

Illustration





Four strings scattering: Moduli space $\leftrightarrow$ Strebel graphs. Discretization.

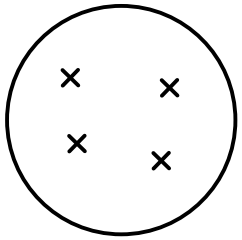
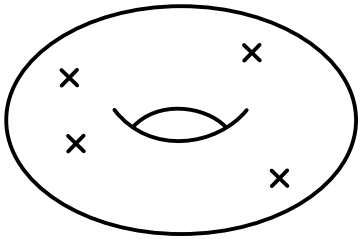
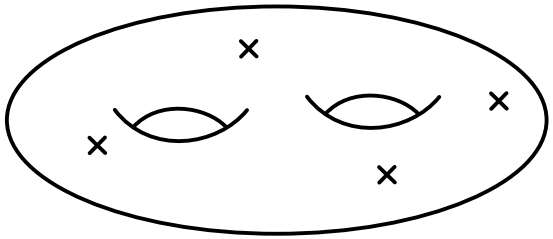
Planar Limit & Genus Expansion

Gauge theory with adjoint matter in the large N_c limit: ['t Hooft 1974]

- Feynman diagrams are double-line diagrams.
- All color lines (propagators and traces) form closed oriented loops.
- Can unambiguously assign an oriented disk (face) to each loop.
- Obtain a compact oriented surface (operators form boundaries).
- The genus of the surface defines the genus of the diagram.

Correlators of single-trace operators:
Count powers of N_c and g_{YM}^2 for propagators ($\sim g_{\mathrm{YM}}^2$), vertices ($\sim 1/g_{\mathrm{YM}}^2$), and faces ($\sim N_c$), define $\lambda = g_{\mathrm{YM}}^2 N_c$, use Euler formula:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{N_c^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_c^{2g}} G_n^{(g)}(\lambda) \qquad \mathcal{O}_i = \mathrm{Tr}(\Phi_1 \Phi_2 \dots)$$

$\sim \frac{1}{N_c^2}$  $+ \frac{1}{N_c^4}$  $+ \frac{1}{N_c^6}$  $+ \dots$

Proposal

Concrete and explicit realization
of the general genus expansion:

[TB, Caetano, Fleury Komatsu, Vieira '17] [TB, Caetano, Fleury Komatsu, Vieira '18]

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^{n} \sqrt{k_i}}{N_c^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_c^{2g(\Gamma)}} \times \times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right]^{2n+4g(\Gamma)-4} \prod_{a=1} \mathcal{H}_a \, .$$

Remarkable fact:
For $\mathcal{N} = 4$ SYM, all ingredients of the formula are well-defined and explicitly known as functions of the 't Hooft coupling λ.

This talk:
- Explain all ingredients of the formula.
- Demonstrate match with known data.
- Show (moderate) predictions.

Proposal: Ingredients I

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^n \sqrt{k_i}}{N_c^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_c^{2g(\Gamma)}} \times \\ \times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right]^{2n+4g(\Gamma)-4} \prod_{a=1} \mathcal{H}_a \, .$$

Half-BPS operators: $Q_i = Q(\alpha_i, x_i, k_i) = \mathrm{Tr}\big[(\alpha_i \cdot \Phi(x_i))^{k_i}\big] \, , \quad \alpha_i^2 = 0 \, .$
Internal polarizations α_i, positions x_i, weights (charges) k_i.

Set of all Wick contractions $\Gamma \in \boldsymbol{\Gamma}$ of the free theory, genus $g(\Gamma)$.

Promote each Γ to a triangulation $\Gamma_\triangle$ of the surface in two steps:

- Collect homotopically equivalent lines in "bridges" $\rightarrow$ skeleton graph.
  The number of lines in a bridge b is the bridge length (width) ℓ_b.
- All faces are triangles or higher polygons.
  Subdivide all faces into triangles by inserting zero-length $\ell_b = 0$ bridges.

Set of all bridges: $\boldsymbol{b}(\Gamma_\triangle)$.

On each bridge: Propagator factor $d_b^{\ell_b}$.

Proposal: Ingredients II

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^{n} \sqrt{k_i}}{N_c^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_c^{2g(\Gamma)}} \times \\ \times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right]^{2n+4g(\Gamma)-4} \prod_{a=1} \mathcal{H}_a \, .$$

On each bridge $b \in \boldsymbol{b}(\Gamma_\triangle)$:
Sum/integrate over states $\psi_b \in \boldsymbol{M}_b$ of the *mirror theory* on the bridge b, with a kinematical weight factor $\mathcal{W}(\psi_b)$ that depends on the cross ratios defined by the surrounding four operators.

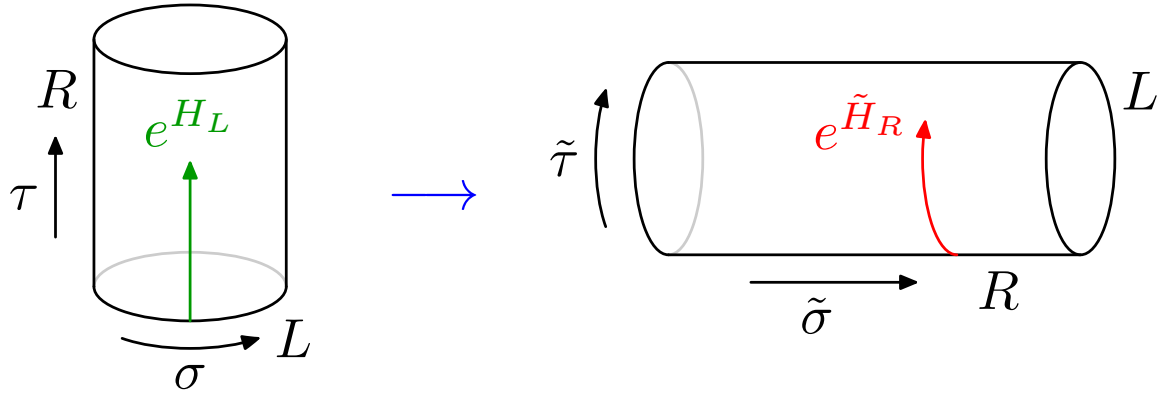
By Euler, the triangulation $\Gamma_\triangle$ contains $2n + 4g(\Gamma) - 4$ faces.

For each face a, insert one hexagon form factor $\mathcal{H}_a$: Accounts for interactions among three operators and three mirror states ψ_b.

Finally, $\mathcal{S}$: Stratification. Sum over graphs quantizes the integration over the string moduli space. $\mathcal{S}$ accounts for contributions at the boundaries.

The Sum over Mirror States

On each bridge lives a mirror theory, which is obtained from the physical worldsheet theory by an analytic continuation, a double-Wick (90 degree) rotation $(\sigma, \tau) \to (i\tilde{\tau}, i\tilde{\sigma})$ that exchanges space and time:



In all computations, the volume R can be treated as infinite.

$\Rightarrow$ Mirror states are free multi-magnon Bethe states, characterized by rapidities u_i, bound state indices a_i, and flavor indices $(A_i, \dot{A}_i)$.

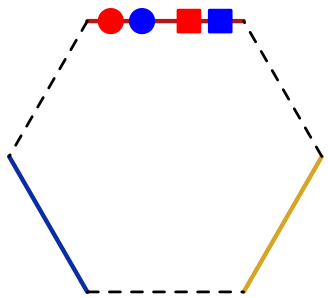
The mirror integration therefore expands to

$$\int_{\boldsymbol{M}_b} \mathrm{d}\psi_b = \sum_{m=0}^{\infty} \prod_{i=1}^{m} \sum_{a_i=1}^{\infty} \sum_{A_i, \dot{A}_i} \int_{u_i=-\infty}^{\infty} \mathrm{d}u_i \, \mu_{a_i}(u_i) \, e^{-\tilde{E}_{a_i}(u_i)\, \ell_b} \, .$$

μ_{a_i}: measure factor, $\tilde{E}$: mirror energy, ℓ_b: length of bridge b (discrete "time").

The Hexagon Form Factors

Hexagon = Amplitude that measures the overlap between three mirror and three physical off-shell Bethe states. Worldsheet branching operator that creates an excess angle of π. [Basso,Komatsu Vieira '15]

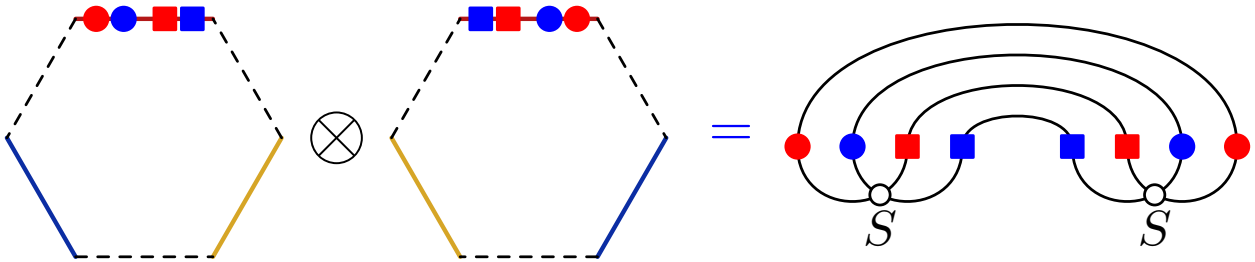


Explicitly: $\mathcal{H}(\chi^{A_1}\chi^{\dot{A}_1}\chi^{A_2}\chi^{\dot{A}_2}\dots\chi^{A_n}\chi^{\dot{A}_n})$

$$= (-1)^{\mathfrak{F}}\left(\prod_{i<j} h_{ij}\right)\langle\chi^{A_1}\chi^{A_2}\dots\chi^{A_n}|S|\chi^{\dot{A}_n}\dots\chi^{\dot{A}_2}\chi^{\dot{A}_1}\rangle$$

- $\chi^A, \chi^{\dot{A}}$: Left/Right $\mathfrak{su}(2|2)$ fundamental magnons
- $\mathfrak{F}$: Fermion number operator
- S: Beisert S-matrix
- $h_{ij} = \dfrac{x_i^- - x_j^-}{x_i^- - x_j^+}\dfrac{x_j^+ - 1/x_i^-}{x_2^+ - 1/x_1^+}\dfrac{1}{\sigma_{ij}}$, $\quad x^{\pm}(u) = x(u \pm \frac{i}{2})$, $\quad \frac{u}{g} = x + \frac{1}{x}$
  σ_{ij}: BES dressing phase

Example:
Two magnons
(●■, ●■)

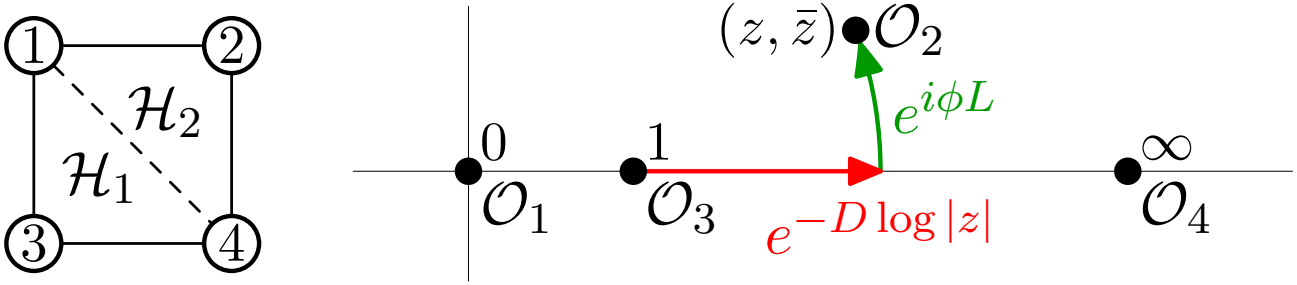


Frames & Weight Factors

Hexagon depends on positions x_i and polarizations α_i of the three half-BPS operators $\mathcal{O}_i = \mathrm{Tr}[(\alpha_i \cdot \Phi(x_i))^k]$. These preserve a diagonal $\mathfrak{su}(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

Two neighboring hexagons share two operators, but the third/fourth operator may not be identical. $\Rightarrow$ The two hexagon frames are misaligned.

In order to consistently sum over mirror states, need to align the two frames by a $\mathrm{PSU}(2,2|4)$ transformation g that maps $\mathcal{O}_3$ onto $\mathcal{O}_2$:



$$g = e^{-D\log|z|} e^{i\phi L} \cdot e^{J\log|\alpha|} e^{i\theta R}\,,$$
$$e^{2i\phi} = z/\bar{z}\,,$$
$$e^{2i\theta} = \alpha/\bar{\alpha}\,.$$

Hexagon $\mathcal{H}_1 = \hat{\mathcal{H}}$ is canonical, and $\mathcal{H}_2 = g^{-1}\hat{\mathcal{H}}g$.

Sum over states in mirror channel: [Fleury '16, Komatsu]

$$\sum_\psi \mu(\psi)\langle\mathcal{H}_2|\psi\rangle\langle\psi|\mathcal{H}_1\rangle = \sum_\psi \mu(\psi)\langle g^{-1}\hat{\mathcal{H}}|\psi\rangle\langle\psi|g|\psi\rangle\langle\psi|\hat{\mathcal{H}}\rangle$$

Weight factor: $\mathcal{W}(\psi) = \langle\psi|g|\psi\rangle = e^{-2i\tilde{p}_\psi \log|z|} e^{J_\psi \varphi} e^{i\phi L_\psi} e^{i\theta R_\psi}$, $i\tilde{p} = (D-J)/2$.

$\rightarrow$ Contains all non-trivial dependence on cross ratios $z, \bar{z}$ and $\alpha, \bar{\alpha}$.

Some Remarks

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^{n} \sqrt{k_i}}{N_c^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_c^{2g(\Gamma)}} \times \\ \times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right]^{2n+4g(\Gamma)-4} \prod_{a=1} \mathcal{H}_a \, .$$

Separates sum over graphs and topologies from λ dependence:
At given genus, can construct sum over graphs once and for all.

Should in principle hold at any value of the coupling λ.

Still a sum over infinitely many mirror contributions that cannot be evaluated in general. But may admit high-loop or even exact expansions in specific limits.

Stratification operator $\mathcal{S}$ looks innocent, but is in fact non-trivial.

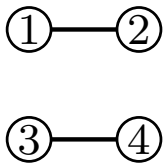
Known Non-Planar Data

Half-BPS operators: First non-trivial correlator: Four-point function.

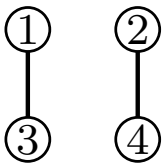
$$\mathcal{Q}_i^k \equiv \mathrm{Tr}\big[(\alpha_i \cdot \Phi(x_i))^k\big]\,, \quad \Phi = (\phi_1, \ldots, \phi_6)\,, \quad \alpha_i^2 = 0\,.$$

Specialize to equal weights $k_1 \ldots k_4 = k$,
and to specific polarizations α_i with $\alpha_1 \cdot \alpha_4 = \alpha_2 \cdot \alpha_3 = 0$.
Possible propagator structures:

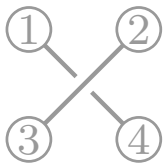
$$X \equiv \frac{\alpha_1 \cdot \alpha_2\, \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} =$$



$$,\quad Y \equiv$$



$$,\quad \Big(Z \equiv$$



$$\Big)\,.$$

Correlators for general N_c: [Arutyunov Sokatchev '03][Arutyunov,Penati '03 Santambrogio,Sokatchev]

$$G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle^{\text{loops}} = R \sum_{m=0}^{k-2} \mathcal{F}_{k,m}\, X^m Y^{k-2-m}$$

Supersymmetry factor: $R = z\bar{z}X^2 - (z+\bar{z})XY + Y^2$

Main data: Coefficients $\mathcal{F}_{k,m} = \mathcal{F}_{k,m}(\lambda, N_c; z, \bar{z})$

The Data: Coefficients

$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) =$$

$$-\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(1)}\,,$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar{z}) =$$

$$\frac{4k^2}{N_c^2}\Bigg[\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(2)}$$

$$+\Bigg\{\frac{t}{4}+\frac{1}{N_c^2}\Bigg[\left(\left[\tfrac{7}{2}r^2-\tfrac{1}{8}\right]k^2+\tfrac{5}{8}k-\tfrac{1}{4}\right)s_+-r\left(\left[\tfrac{17}{6}r^2-\tfrac{7}{8}\right]k^3+3k^2-\tfrac{13}{12}k\right)s_-$$

$$+\left(\left[\tfrac{29}{24}r^4-\tfrac{11}{16}r^2+\tfrac{15}{128}\right]k^4+\left[\tfrac{17}{8}r^2-\tfrac{21}{32}\right]k^3-\left[\tfrac{23}{24}r^2-\tfrac{39}{32}\right]k^2-\tfrac{9}{8}k+\tfrac{1}{2}\right)t\Bigg]\Bigg\}\left(F^{(1)}\right)^2$$

$$-\frac{1}{N_c^2}\left[r\left\{\left[\tfrac{7}{6}r^2-\tfrac{1}{8}\right]k^3+\tfrac{3}{2}k^2+\tfrac{10}{3}k\right\}F^{(2)}_{\mathrm{C},-}+\left\{\left[\tfrac{5}{4}r^2-\tfrac{19}{48}\right]k^3+\left[\tfrac{3}{2}r^2+\tfrac{7}{8}\right]k^2+\tfrac{1}{3}k\right\}F^{(2)}_{\mathrm{C},+}\right]$$

$$+\frac{1}{4}\left\{1+\frac{(k-1)(k^3+3k^2-46k+36)}{12N_c^2}\right\}\left(s\delta_{m,0}+\delta_{m,k-2}\right)\left(F^{(1)}\right)^2$$

$$+\left\{1+\frac{(k-2)_4}{12N_c^2}\right\}\left(\delta_{m,0}F^{(2)}_{z-1}+\delta_{m,k-2}F^{(2)}_{1-z}\right)\Bigg]\,,$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of $X^m Y^{k-2-m}$.

First Test: Genus One, Large Charges

Focus on leading order in large charges (large k) $\to$ several simplifications: Data, sum over graphs, and loop expansion (mirror states) all simplify.

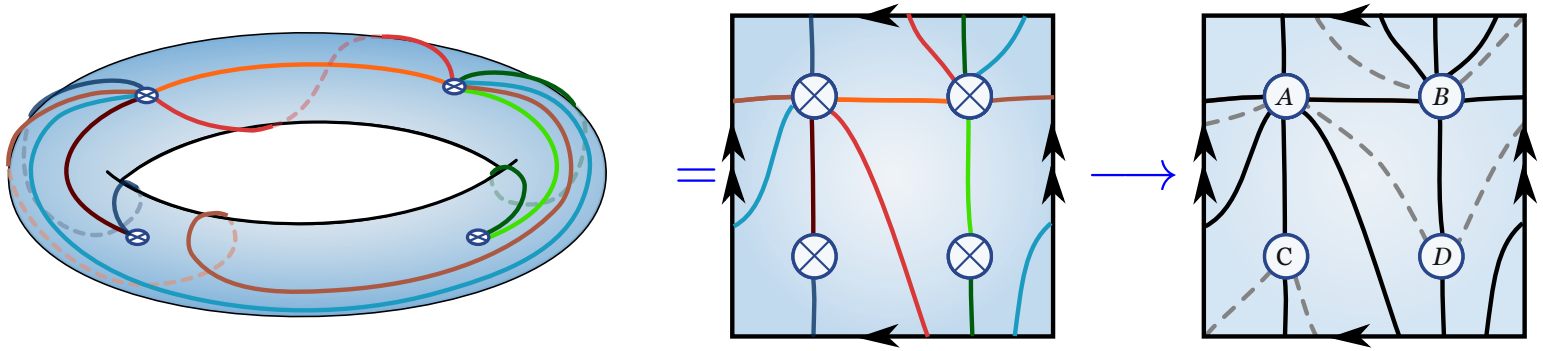
Data: $$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = -\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\} t\,F^{(1)}\,,$$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar{z}) = \frac{4k^2}{N_c^2}\left[\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\} t\,F^{(2)}\right.$$
$$\left.+\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{29}{6}r^4-\tfrac{11}{4}r^2+\tfrac{15}{32}\right]k^4+\mathcal{O}(k^3)\right]\right\}\frac{t^2}{4}\left(F^{(1)}\right)^2\right],$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of $X^m Y^{k-2-m}$.

Step 1: Sum over propagator graphs: Split in two steps:

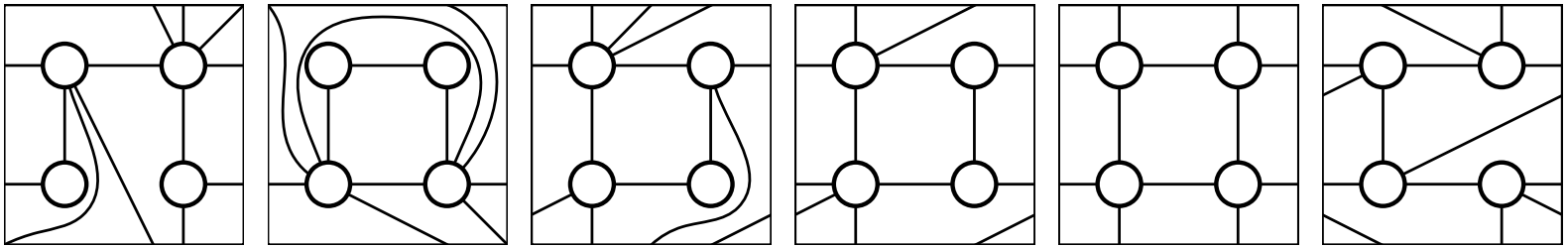
- Sum over torus "skeleton graphs" with non-parallel edges ($\equiv$ bridges).
- Sum over distributions of parallel propagators on bridges.



First Test: Large Charges: Graphs

Large k: Combinatorics of distributing propagators on bridges:
Two operators typically connected by $j > 1$ bridges.
Sum over distributions of m propagators on $j+1$ bridges $\to m^j/j!$
$\Rightarrow$ Only graphs with a maximum number of bridges contribute.

Genus-one four-point graphs with the maximal number of bridges:



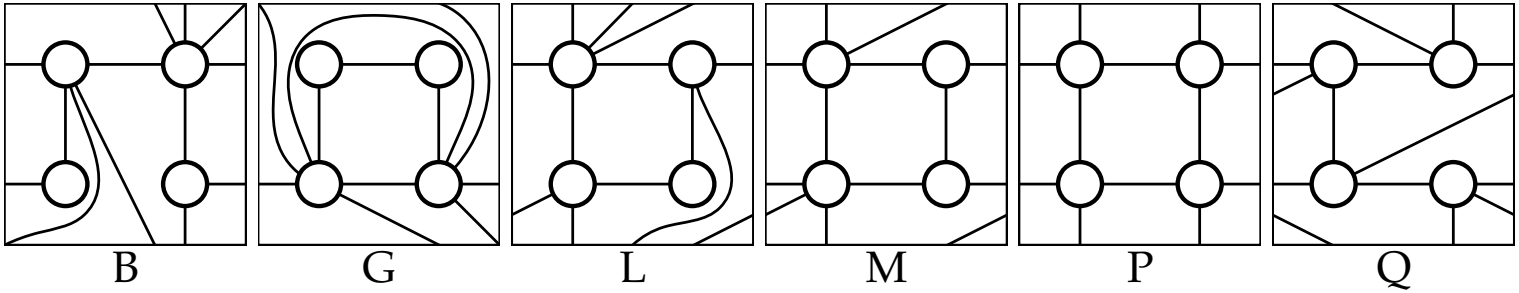
B G L M P Q

Sum over labelings:

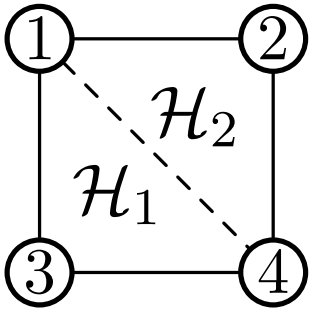
Case	Inequivalent Labelings (clockwise)	Combinatorial Factor
B	$(1,2,4,3),(2,1,3,4),(3,4,2,1),(4,3,1,2)$	$m^3(k-m)/6$
B	$(1,3,4,2),(3,1,2,4),(2,4,3,1),(4,2,1,3)$	$m(k-m)^3/6$
G	$(1,2,4,3),(3,4,2,1)$	$m^4/24$
G	$(1,3,4,2),(2,4,3,1)$	$(k-m)^4/24$
L	$(1,2,4,3),(3,4,2,1),(2,1,3,4),(4,3,1,2)$	$m^2/2\cdot(k-m)^2/2$
M	$(1,2,4,3),(2,1,3,4),(1,3,4,2),(3,1,2,4)$	$m^2(k-m)^2/2$
P	$(1,2,4,3)$	$m^2(k-m)^2/2$
Q	$(1,2,4,3)$	$m^2(k-m)^2$

First Test: Large Charges: Hexagons

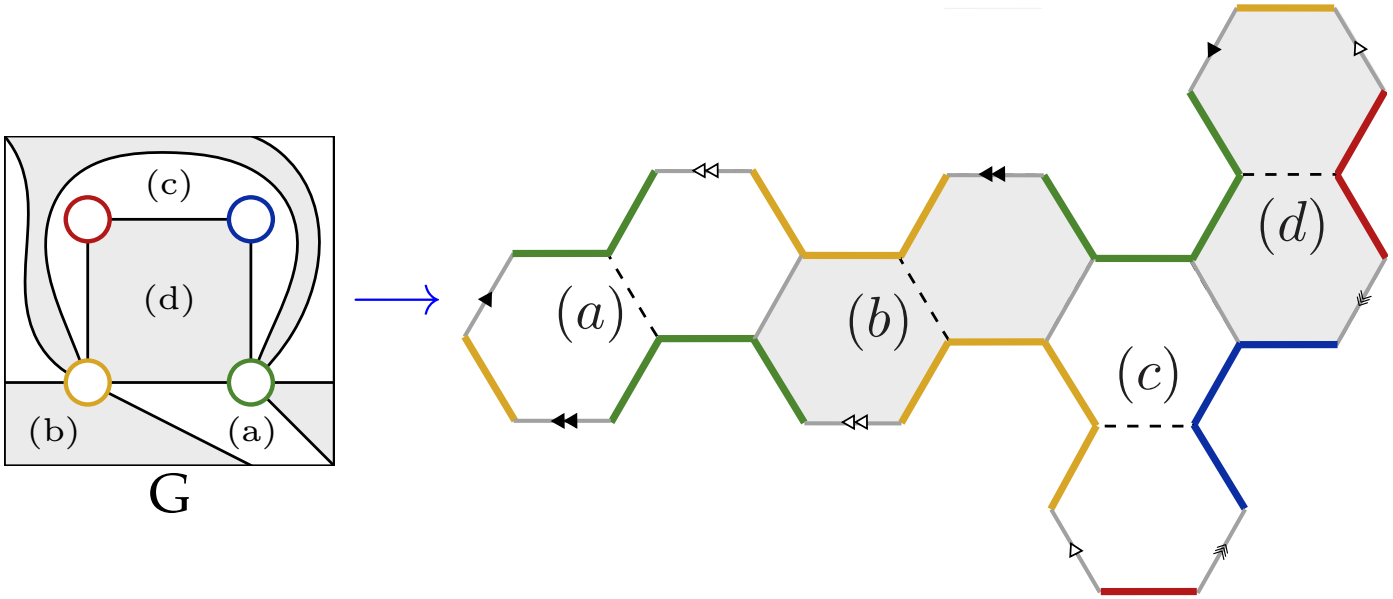
Graphs:



All graphs consist of only octagons!
Split each octagon into two hexagons with a zero-length bridge.



Example:



First Test: Large Charges: Mirror Particles

Loop Counting:

Expand mirror propagation $\mu(u)\, e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling g.

$\rightarrow$ n particles on bridge of size ℓ: $\mathcal{O}(g^{2(n\ell+n^2)})$

All graphs consist of octagons framed by parametrically large bridges.

$\rightarrow$ Only excitations on zero-length bridges inside octagons survive.

Excited Octagons:

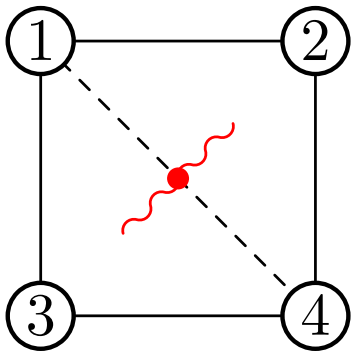
n particles on a zero-length bridge $\rightarrow$ $\mathcal{O}(g^{2n^2})$

$\rightarrow$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops.

Octagon 1–2–4–3 with 1 particle:

[Fleury '16 Komatsu][TB,Caetano,Fleury Komatsu,Vieira '18]

$$\mathcal{M}(z,\alpha) = \left[z + \bar{z} - (\alpha + \bar{\alpha})\frac{\alpha\bar{\alpha} + z\bar{z}}{2\alpha\bar{\alpha}}\right] \cdot \left(g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \dots\right)$$



For $Z = 0$: R-charge cross ratios

$\alpha = z\bar{z}\, X/Y$ and $\bar{\alpha} = 1$.

First Test: Large Charges: Match & Prediction

We are Done:
Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
$\Rightarrow$ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

$$\mathcal{F}^{\mathrm{U}}_{k,m}(z,\bar{z})\big|_{\text{torus}} = -\frac{2k^6}{N_c^4}\Big\{ g^2\big[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\big]\, t\, F^{(1)} \quad \checkmark \text{ match}$$

$$- 2g^4\Big[\big[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\big]\, t\, F^{(2)} + \big[\tfrac{29}{6}r^4 - \tfrac{11}{4}r^2 + \tfrac{15}{32}\big]\frac{t^2}{4}\big(F^{(1)}\big)^2\Big] \quad \checkmark \text{ match}$$

$$+ 3g^6\Big[\big[\tfrac{17r^4}{6} - \tfrac{7r^2}{4} + \tfrac{11}{32}\big]\, t\, F^{(3)} + \big[\tfrac{29r^4}{18} - \tfrac{11r^2}{12} + \tfrac{5}{32}\big]\, t^2\, F^{(2)}F^{(1)} + \big[\tfrac{(1-4r^2)^2}{96}\big]\big(F^{(1)}\big)^3\Big]$$

$$+ \mathcal{O}(g^8) + \mathcal{O}(1/k)\Big\}. \qquad \text{prediction!}$$

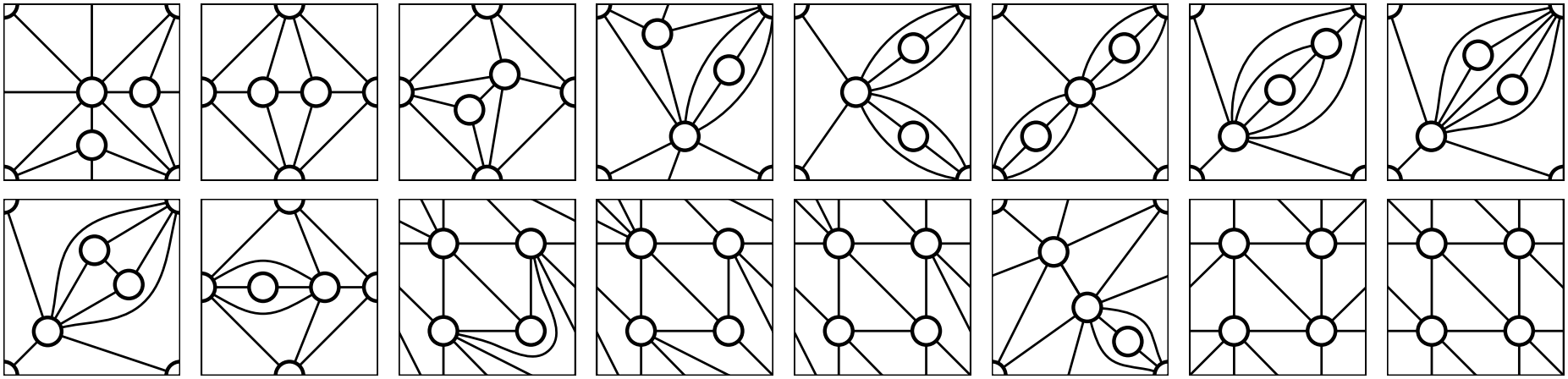
In fact, the octagon can be evaluated to much higher loop orders,
is a polynomial in ladder integrals. [Coronado to appear]

$\Rightarrow$ Immediate high-loop prediction for the four-point function.

More Tests: Finite Charges

Finite k: Need to include *all* four-point graphs on the torus.

Complete list of "maximal" graphs:



All other graphs can be obtained from maximal graphs by deleting bridges.
Number of genus-one graphs by number of bridges:

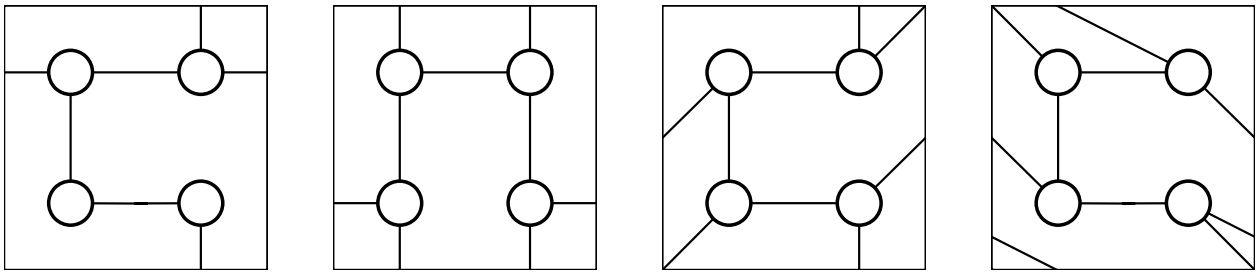
#bridges:	12	11	10	9	8	7	6	5	≤ 4
#graphs:	7	28	117	254	323	222	79	11	0

Even for our $Z = 0$ correlators, graphs with Z propagators need to be included: Mirror contributions depend on R-charges and can effectively cancel Z-propagators contained in free-theory graphs.

Finite-Charge Tests

Small and finite k:
Few propagators $\rightarrow$ Fewer bridges $\rightarrow$ Graphs with fewer edges
$\Rightarrow$ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:



Hexagonalization:
Each $2n$-gon: Split into $n - 2$ hexagons by $n - 3$ zero-length briges.

Loop Expansion: Much more complicated!
All kinds of excitation patterns already at low loop orders

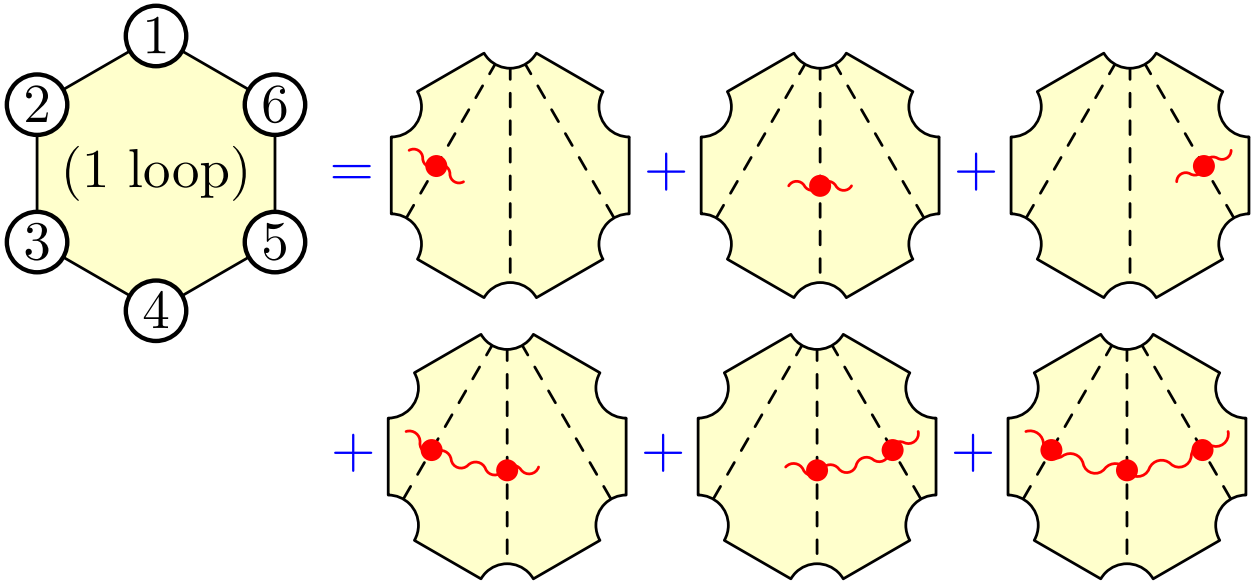
- Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- Strings of excitations wrapping around operators

Finite k: One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent *zero-length* bridges contribute.

$\Rightarrow$ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:



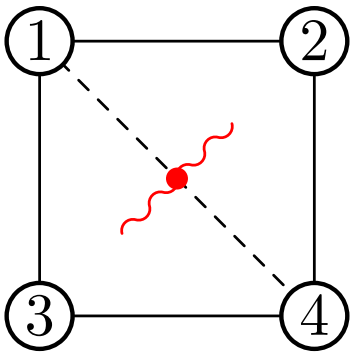
One-strings: understood ✓ [Fleury '16 Komatsu]
Two-strings: understood ✓ [Fleury '17 Komatsu]
Longer strings: need to compute!

One-String and Two-String

One-String: Can be written as

$$\mathcal{M}^{(1)}(z,\alpha) = m(z) + m(z^{-1})\,,$$

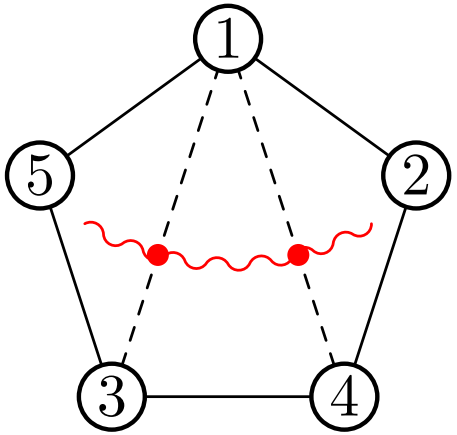
with building block



$$m(z) = m(z,\alpha) = g^2\,\frac{(z+\bar{z})-(\alpha+\bar{\alpha})}{2}\,F^{(1)}(z,\bar{z})$$

Two-string: Despite complicated computation, simplifies to [Fleury '17 Komatsu]

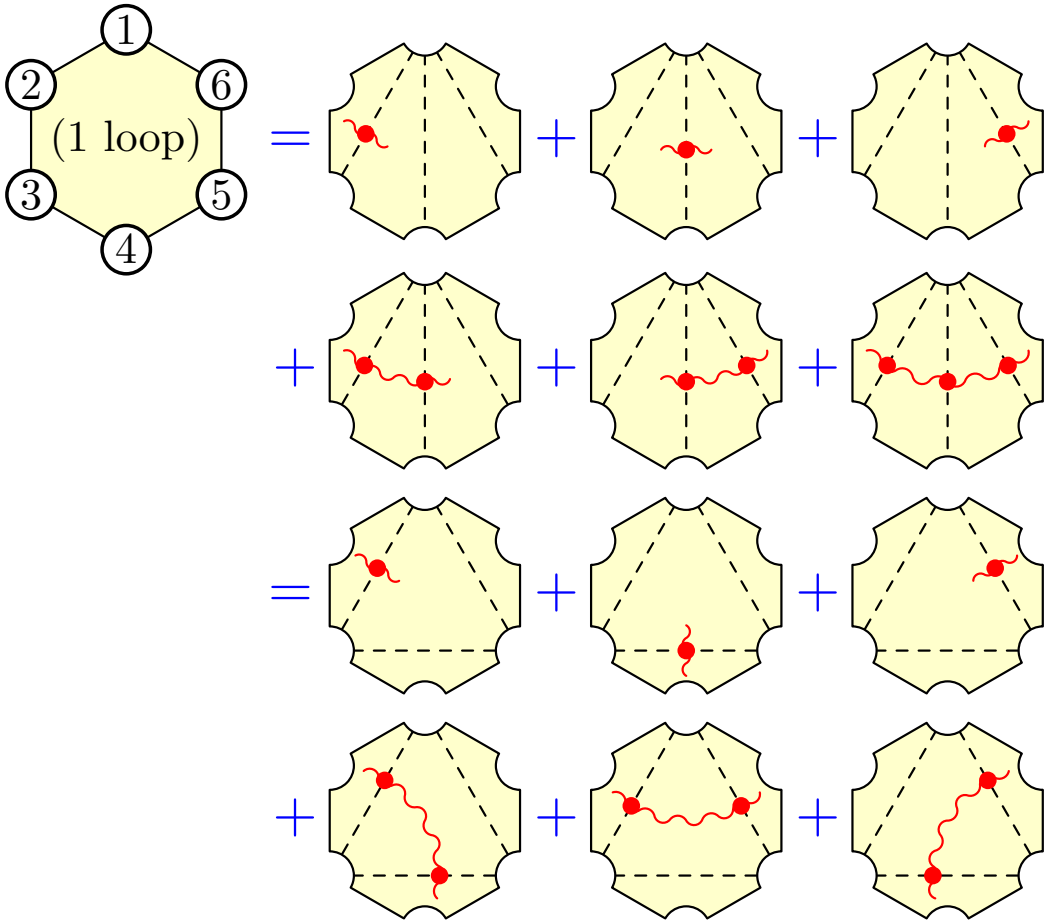
$$\begin{aligned}\mathcal{M}^{(2)}&(z_1,z_2,\alpha_1,\alpha_2)\\ &= m\left(\frac{z_1-1}{z_1 z_2}\right) + m\left(\frac{1-z_1+z_1 z_2}{z_2}\right)\\ &\quad + m\big(z_1(1-z_2)\big) - m(z_1) - m(z_2^{-1})\,,\end{aligned}$$



with the same building block $m(z)$!

Finite Charges: Larger Strings

Larger strings: Computation will be even more complicated!
But: Can in fact bootstrap all of them by using flip invariance!



Apply recursively:

- 3-string $\simeq$ 1-strings & 2-strings
- ...iterate ...
- n-string $\simeq$ 1-strings & 2-strings

$\Rightarrow$ Can write all polygons in terms of only 1-strings & 2-strings.

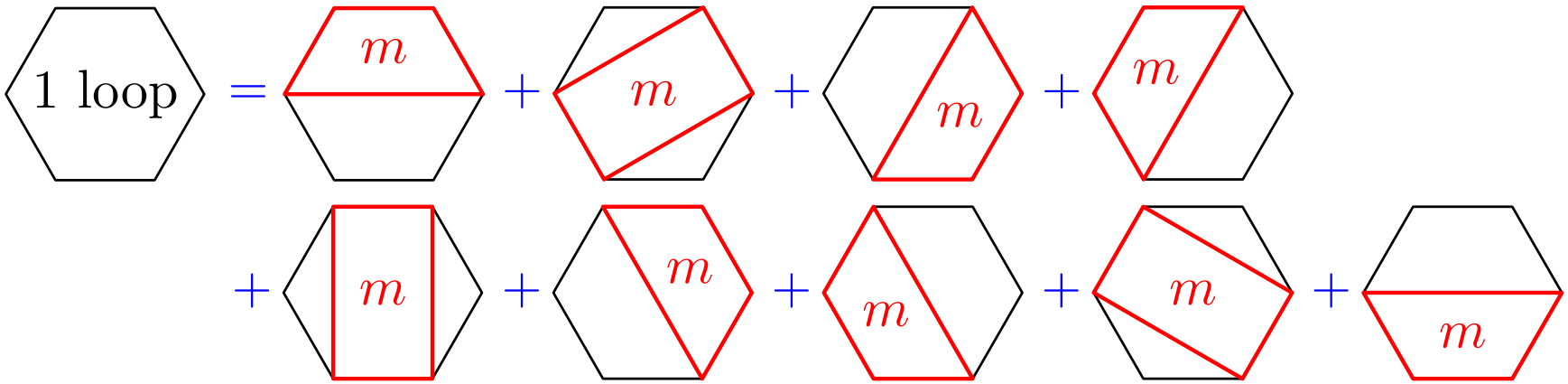
$\Rightarrow$ All n-strings can be written as linear combinations of one-string building blocks $m(z)$.

Finite k: General Polygons at One Loop

Polygon with 2n edges:
Sum over all strings inside the polygon greatly simplifies to:

$$\mathcal{P}_{2n}^{(1)} = \sum_{\substack{\{j,k\}\text{ non-}\\ \text{consecutive}}} m\left(z_{jk} \equiv \frac{x_{j,k+1}^2 x_{j+1,k}^2}{x_{jk}^2 x_{j+1,k+1}^2}\right)$$

$\rightarrow$ Sum over $m(z)$ evaluated in each subsquare:



Recall the one-loop building block:

$$m(z) = g^2 \frac{(z+\bar{z}) - (\alpha+\bar{\alpha})}{2} F^{(1)}(z,\bar{z})$$

Finite k, One Loop: Result

Done! Sum over all graphs, expand all polygons to their one-loop values.

Numbers of labeled graphs with assigned bridge sizes:

k:	2	3	4	5
$g=0$:	3	8	15	24
$g=1$:	0	32	441	2760

Data:

$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = -\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(1)},$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of $X^m Y^{k-2-m}$.

Result: For $k=2,3,4,5,\ldots$:
Matches the $\mathrm{U}(N_c)$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m}: \quad \texttt{Result} = \underbrace{(\texttt{torus data})}_{\checkmark\checkmark\checkmark} + \frac{1}{N_c^2}\underbrace{(\texttt{planar data})}_{???}$$

$\Rightarrow$ Puzzle.

Resolution of Mismatch: Stratification

We have based the computation on a sum over genus-one graphs of the free theory that cover all cycles of the torus.

We therefore miss contributions from purely virtual handles. In the language of hexagons, these come from graphs where a handle of the torus is traversed only by zero-length bridges (no propagators).

Resolution: Include graphs that are by themselves planar, but drawn on the torus, and fully tessellate the torus by zero-length bridges (as before).

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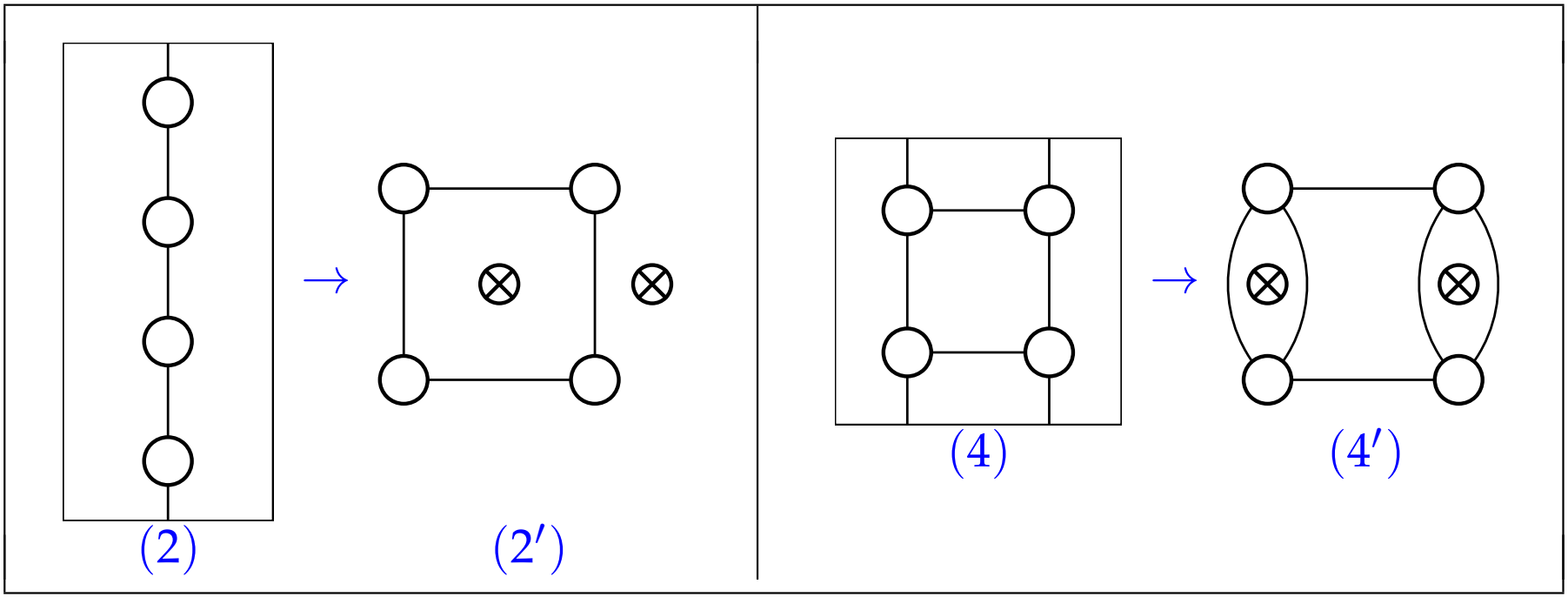
New Problem: This adds the missing contributions (mirror states traversing a handle that contains no propagators). But it also adds many genuinely planar (and therefore unwanted) contributions.

Get rid of these unwanted contributions by subtracting the same graphs, but now drawn on a degenerate torus where the empty handle has been pinched, such that the torus becomes a sphere with two marked points.

This goes under the name of stratification ($\mathcal{S}$ in our formula).

Stratification: Examples



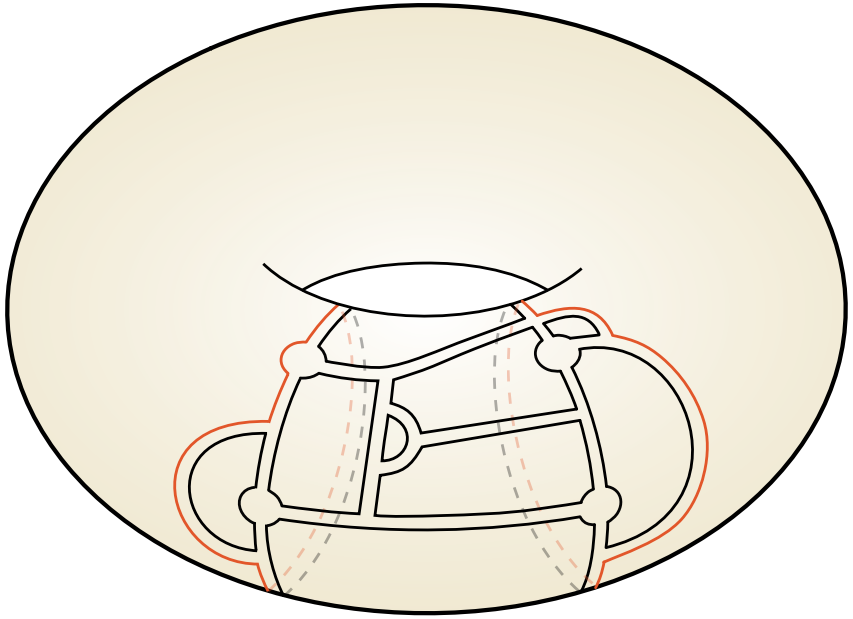
Stratification & Moduli Space

Stratification is also natural from the string theory point of view:
The sum over graphs discretizes the integration over the moduli space of worldsheet Riemann surfaces.

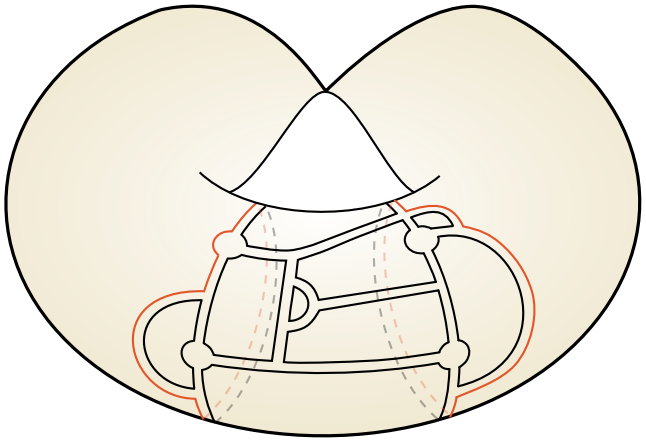
The moduli space includes boundaries. In continuous integrations, these boundaries are measure-zero sets and hence do not contribute. But in a discretized sum, it matters which terms are included or dropped.

Moduli space discretizations have been considered before in the context of matrix models, and the right treatment of boundary contributions in the known cases is in line with the above prescription (stratification). [Chekhov 1995]

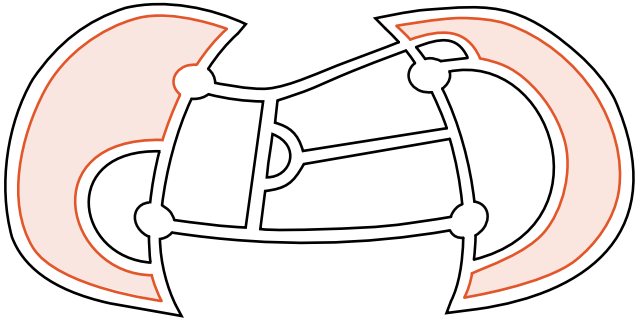
Stratification: Degeneration Type I



(a)

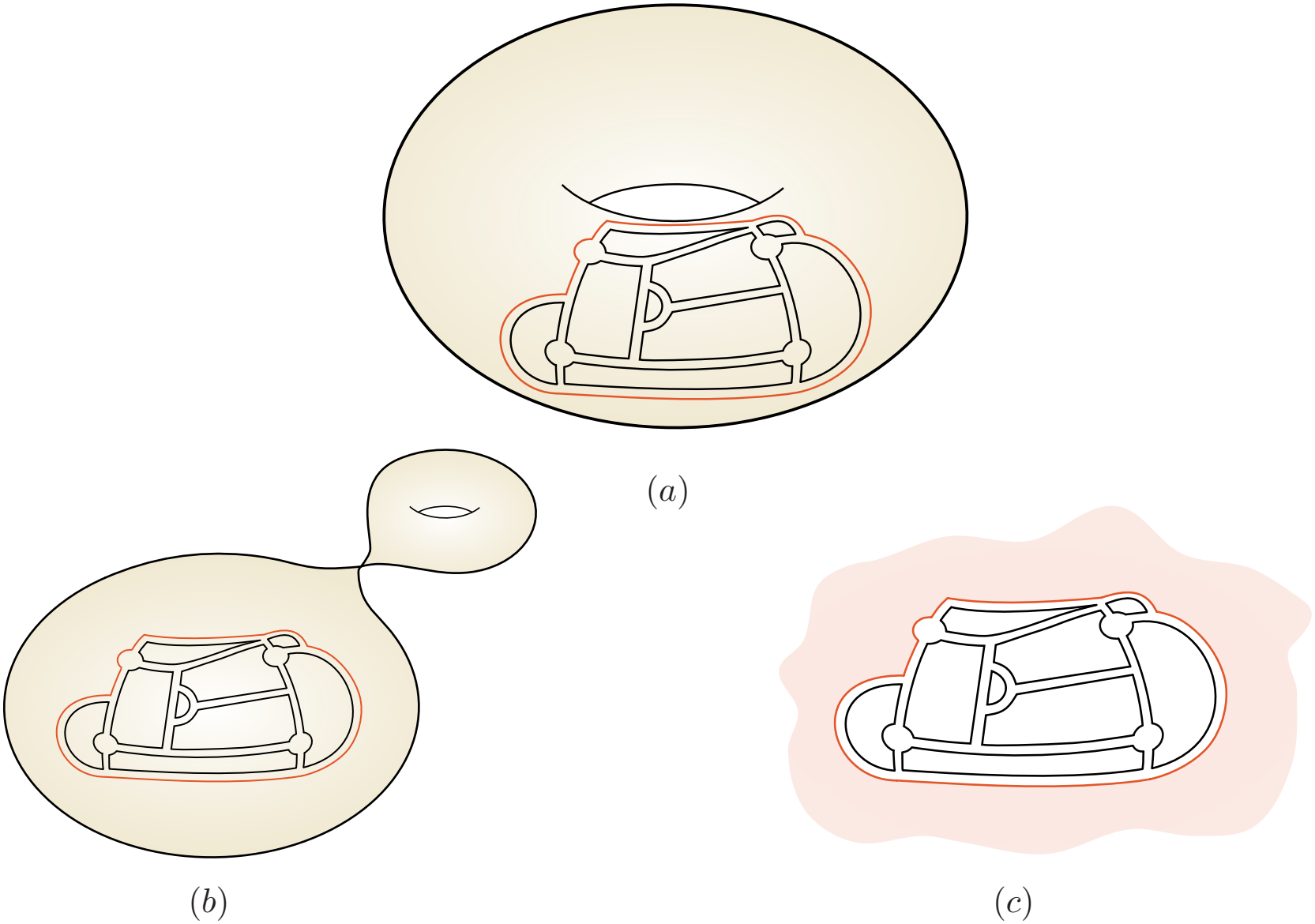


(b)



(c)

Stratification: Degeneration Type II



Stratification: Final Formula

At higher genus, simple degenerations subtract terms multiple times
$\rightarrow$ need to be compensated by *adding* double degenerations etc.
$\rightarrow$ alternating sum.
Also need to account for disconnected degenerations. Final result:

$$\mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \equiv \sum_{g=0}^{\infty} \sum_{c=1}^{2g+n-2} \sum_{\tau \in \boldsymbol{\tau}_{g,c,n}} (-1)^{\sum_i m_i/2} \sum_{\Gamma \in \Sigma_\tau} .$$

c: Number of components of the surface

τ: Genus-g topology with c components and n punctures:

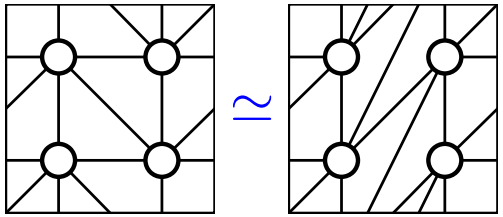
$$\boldsymbol{\tau}_{g,c,n} = \left\{ \{(g_1, n_1, m_1) \dots (g_c, n_c, m_c)\} \,\middle|\, \sum_i n_i = n,\ \sum_i (g_i + \tfrac{m_i}{2}) - c + 1 = g \right\}$$

where (g_i, n_i, m_i) labels the genus, the number of punctures, and the number of marked points on component i.

Σ_τ: Set of all graphs Γ (connected and disconnected) that are compatible with the topology τ and that are embedded in the surface defined by τ in all inequivalent possible ways (Γ may cover all or only some components of the surface).

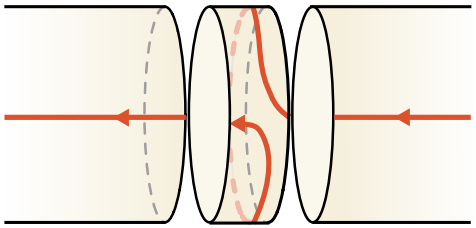
Dehn Twists and Modular Group

We implicitly identified graphs that only differ by "twists" of a handle:



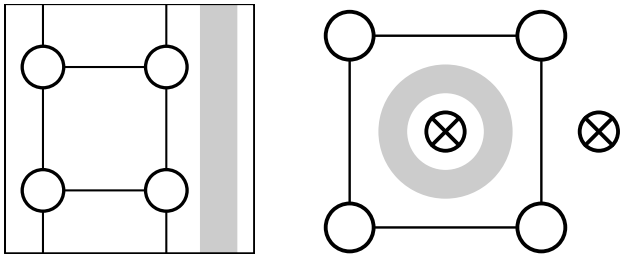
This makes sense at weak coupling: Identity at the level of Feynman graphs.

Also makes sense from string moduli space perspective: *Dehn twists* are modular transformations that leave the complex structure invariant.



Modding out by Dehn twists has non-trivial implications for the summation over mirror states, especially for stratification terms:

Dehn twists along cycles not covered by the propagator graph act trivially in the absence of mirror particles:

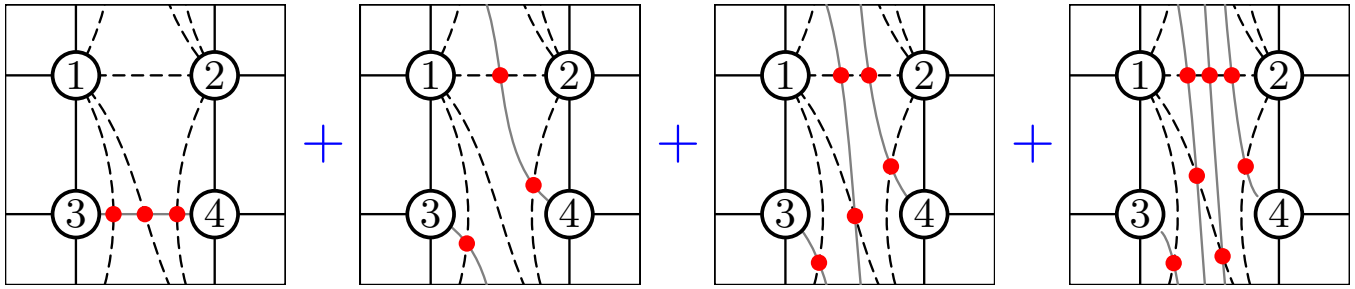


Once the cycle is dressed with zero-length bridges and mirror particles, Dehn twists will non-trivially map sets of mirror magnons onto each other.
$\rightarrow$ Need to mod out by this non-trivial action!

Stratification: Evaluation

Stratification graphs: Cycles that are traversed only by zero-length bridges.
$\rightarrow$ Infinitely many mirror contributions already at one loop!

Example:



Presently: Cannot evaluate strings of magnons that wrap a cycle, or cross any edge more than once.

However, reasonable to assume that almost all such contributions will either *cancel* agains stratification subtractions, or be *projected out* by Dehn twists.

Use (partly heuristic) simple rules: Drop configurations with closed loops; identify one-loop strings that are "superficially" related by Dehn twists.

All remaining contributions can be honestly computed.

Including stratification indeed gives the missing $(\text{planar})/N_c^2$ term!

$\Rightarrow$ Now have a perfect match for $k = 2, 3, 4, 5$!

Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_c$ expansion.

- Sum over free graphs, decompose into planar hexagons.
- Infinite sum over mirror excitations.
- Discretizes string moduli space integration.
- Non-trivial match with various one/two-loop correlators.
- New, bottom-up approach to string perturbation theory?

Outlook: There are many things to do!

- Study more examples: Higher loops / genus, more general operators.
- Understand details/implications of stratification beyond one loop
- Better understand summation/integration of mirror particles!
- Find a limit that can be resummed (λ and/or $1/N_c$).
- Most promising: Large-charge limit. No stratification.
- Other non-planar observables: Anomalous dimensions? Double-traces?
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Thank you!

Graph Statistics

Numbers of maximal graphs for various g and numbers of insertions:

genus :	0	1	2	3	4
$n = 2$:	1	1	4	82	7325
$n = 3$:	1	3	38	661	
$n = 4$:	2	16	760	122307	
$n = 5$:	4	132	18993		
$n = 6$:	14	1571	487293		
$n = 7$:	66	20465			
$n = 8$:	409	278905			
$n = 9$:	3078				
$n = 10$:	26044				