Handling Handles: Non-Planar AdS/CFT Integrability

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+ further work in progress

DESY STRING THEORY SEMINAR
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String amplitudes in AdS$_5$ can be cut into basic patches (rectangles, pentagons, or hexagons), which can be bootstrapped using integrability at any value of the 't Hooft coupling.

- Amplitudes are given as infinite sums and integrals over intermediate states from gluing together these integrable patches.
- This holds at the planar level as well as for non-planar processes suppressed by $1/N_c$. 
\( \mathcal{N} = 4 \) SYM & The Planar Limit

\( \mathcal{N} = 4 \) super Yang–Mills: Gauge field \( A_\mu \), scalars \( \Phi_I \), fermions \( \psi_{\alpha A} \).

Gauge group: \( \text{U}(N_c) / \text{SU}(N_c) \).

Adjoint representation: All fields are \( N_c \times N_c \) matrices.

**Double-line notation:**

Propagators:
\[
\langle \Phi^i_I \Phi^k_J \rangle \sim g_{\text{YM}}^2 \delta^i_I \delta^k_J = \begin{array}{c}
  \begin{array}{c}
    i
  \end{array}
  \begin{array}{c}
    \rightarrow
  \end{array}
  \begin{array}{c}
    l
  \end{array}
  \begin{array}{c}
    j
  \end{array}
  \begin{array}{c}
    \rightarrow
  \end{array}
  \begin{array}{c}
    k
  \end{array}
\end{array}
\]

Vertices:
\[
\text{Tr}(\Phi \Phi \Phi \Phi) \sim \frac{1}{g_{\text{YM}}^2}
\]

- Diagrams consist of color index loops \( \sim \) oriented disks \( \sim \delta^i_i = N_c \)
- Disks are glued along propagators \( \rightarrow \) oriented compact surfaces

**Local operators:**

\[
\mathcal{O}_i = \text{Tr}(\Phi \ldots) \sim \begin{array}{c}
  \begin{array}{c}
    \bullet
  \end{array}
  \begin{array}{c}
    \bullet
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  \begin{array}{c}
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  \begin{array}{c}
    \bullet
  \end{array}
\end{array}
\]

- One fewer color loop \( \rightarrow \) factor \( 1/N_c \)
- Surface: Hole \( \sim \) boundary component
Planar Limit & Genus Expansion

Every diagram is associated to an oriented compact surface.

Genus Expansion:
Absorb one factor of $N_c$ in the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N_c$
Use Euler formula $V - E + F = 2 - 2g$

$\Rightarrow$ **Correlators** of single trace operators $\mathcal{O}_i = \text{Tr}(\Phi_1 \Phi_2 \ldots)$:
't Hooft genus expansion

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{1}{N_c^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_c^{2g}} G_g(\lambda)$$

$$\sim \frac{1}{N_c^2} \begin{array}{c} \times \times \times \\ \times \times \times \end{array} + \frac{1}{N_c^4} \begin{array}{c} \times \times \times \\ \times \times \times \\ \times \times \times \end{array} + \frac{1}{N_c^6} \begin{array}{c} \times \times \times \\ \times \times \times \\ \times \times \times \\ \times \times \times \end{array} + \ldots$$
Planar Spectrum: Features of Integrability

Simplest observables: Planar two-point functions $\sim$ scaling dimensions

$O_1 \leftrightarrow O_2 \quad O_i = \text{Tr}(\Phi \ldots)$

single-trace

Perturbatively: Degeneracies in the spectrum $\rightarrow$ Higher charges

Spin chain picture: Organize operators around vacuum operators

$$\text{Tr} \ Z^L, \quad Z = \alpha^I \Phi_I, \quad \alpha^I \alpha_I = 0 \quad \text{(half-BPS, protected)}.$$  

Other operators: Insert impurities $\{\Phi_I, \psi_{\alpha A}, D_\mu\}$ into $\text{Tr} \ Z^L$.

Dilatation operator acts locally in color space (neighboring fields)  
$\rightarrow$ Impurities are magnons, with rapidity (momentum) $u$ and $\mathfrak{su}(2|2)^2 \subset \mathfrak{psu}(2, 2|4)$ flavor index.
Planar Spectrum: Features of Integrability

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**Spin chain picture:** Organize operators around vacuum operators

$$\text{Tr } Z^L, \quad Z = \alpha^I \Phi_I, \quad \alpha^I \alpha_I = 0$$  \hspace{1cm} (half-BPS, protected).

Other operators: Insert impurities $\{\Phi_I, \psi_{\alpha A}, D_{\mu}\}$ into $\text{Tr } Z^L$.

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$\rightarrow$ Impurities are *magnons*, with *rapidity* (momentum) $u$
and $\mathfrak{su}(2|2)^2 \subset \mathfrak{psu}(2,2|4)$ *flavor index*.

**Dynamics of magnons:** Integrability:
$\rightarrow$ No particle production
$\rightarrow$ Individual momenta preserved
$\rightarrow$ Factorized scattering

Two-body **S-matrix** completely fixed *to all loops* [Beisert 2005, Janik 2006, Beisert, Hernandez Lopez 2006]

Planar spectrum (asymptotic) solved exactly by *Bethe ansatz*.
**Non-Planar Corrections: (Past) Status**

**Degeneracies** are lifted at subleading orders in $1/N_c$.

**Interactions** are long-ranged, non-local from the start:

→ Hilbert space much bigger
→ Spin chain picture?
→ Fate of local S-matrix? Definition?
→ No integrable spin chain!

**No dual superconformal symmetry**

Classical integrability of $\sigma$ model (strong coupling) not clear

⇒ “Integrability is lost”.
**Three-Point Functions:** Hexagons

**Differences:** Topology: Pair of pants instead of cylinder  
Non-vanishing for three generic operators (two-point: diagonal)  
⇒ Previous techniques not directly applicable

**Observation:**

The green parts are similar to two-point functions:  
Two segments of physical operators joined by parallel propagators (“bridges”, \( \ell_{ij} = (L_i + L_j - L_k)/2 \)).

The red part is new: “Worldsheet splitting”,  
“three-point vertex” (open strings)

Take this serious → cut worldsheet along “bridges”:

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[Basso, Komatsu Vieira ’15]
Hexagons & Gluing

Glue hexagons along three mirror channels:

- Sum over complete state basis (magnons) in the mirror theory
- Mirror magnons: Boltzmann weight $\exp(-\tilde{E}_{ij} \ell_{ij})$, $\tilde{E}_{ij} = \mathcal{O}(g^2)$ → mirror excitations are strongly suppressed.

Hexagonal worldsheet patches (form factors):

- Function of rapidities $u$ and $\mathfrak{su}(2|2)$ labels $(A, \dot{A})$ of all magnons.
- Conjectured exact expression, based on diagonal $\mathfrak{su}(2|2)$ symmetry, form factor axioms, and integrability assumptions.

Hexagon proposal supported by very non-trivial matches.
The Hexagon Form Factor

All excitations on the same physical edge (canonical frame):

\[ \mathcal{H}(\chi^{A_1} \chi^{\dot{A}_1} \chi^{A_2} \chi^{\dot{A}_2} \ldots \chi^{A_n} \chi^{\dot{A}_n}) = (-1)^{\delta} \left( \prod_{i<j} h_{ij} \right) \langle \chi^{A_1} \chi^{A_2} \ldots \chi^{A_n} | S | \chi^{\dot{A}_n} \ldots \chi^{\dot{A}_2} \chi^{\dot{A}_1} \rangle \]

- \( \chi^{A} = \phi^a | \psi^{\alpha} \): Left \( su(2|2) \) fundamental magnon
- \( \chi^{\dot{A}} = \phi^{\dot{a}} | \psi^{\dot{\alpha}} \): Right \( su(2|2) \) fundamental magnon
- \( \delta \): Fermion number operator
- \( S \): Beisert S-matrix

\[ h_{ij} = \frac{x_i^- - x_j^- x_j^+ - 1/x_i^-}{x_i^- - x_j^+ x_j^+ - 1/x_j^+} \frac{1}{\sigma_{ij}} \]

\( x^\pm(u) = x(u \pm \frac{i}{2}) \), \( \frac{u}{g} = x + \frac{1}{x} \)

Example:

Two magnons (○■, ○□)
Mirror Map

Double Wick rotation: \((\sigma, \tau) \rightarrow (i\tilde{\tau}, i\tilde{\sigma})\) — exchanges space and time

![Diagram](image)

**Magnon states:** Energy and momentum interchange:

\[
p\sigma \rightarrow p^\gamma i\tilde{\tau} \equiv \tilde{E}\tilde{\tau}, \quad E\tau \rightarrow E^\gamma i\tilde{\sigma} \equiv \tilde{\rho}\tilde{\sigma} \quad \Rightarrow \quad (\tilde{E}, \tilde{\rho}) = (ip^\gamma, iE^\gamma).
\]

**Continuations:** \(u \rightarrow u^\gamma\). All quantities given in terms of \(x^\pm(u)\). 

![Diagram](image)
Move on to planar four-point functions:
One way to cut (now that three-point is understood): **OPE cut**

**Problem:** Sum over physical states!
- No loop suppression, all states contrib.
- Double-trace operators.

**Instead:** Cut along propagator bridges

**Benefits:**
- Mirror states highly suppressed in $g$.
- No double traces.
Hexagonalization: Formula

\[ \langle O_1 O_2 O_3 \rangle = \left[ \prod_{c \in \{1,2,3\}} d_c^\ell \sum_{\psi_c} \mu(\psi_c) \right] H_1(\psi_1, \psi_2, \psi_3) H_2(\psi_1, \psi_2, \psi_3) \]

\[ \langle O_1 O_2 O_3 O_4 \rangle = \sum_{\text{planar prop. graphs}} \left[ \prod_{c \in \{1,\ldots,6\}} d_c^\ell \sum_{\psi_c} \mu(\psi_c) \right] H_1 H_2 H_3 H_4 \]

**New Features:**

- Bridge lengths vary, may go to zero \( \Rightarrow \) Mirror corrections at one loop
- Hexagons are in different “frames” \( \Rightarrow \) Weight factors
Hexagon depends on positions $x_i$ and polarizations $\alpha_i$ of the three half-BPS “vacuum” operators $O_i = \text{Tr}[(\alpha_i \cdot \Phi(x_i))^k]$.

Any three $x_i$ and $\alpha_i$ preserve a diagonal $su(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

**Three-point function:** Both hexagons connect to the same three operators, so their frames ($su(2|2)$ and state basis) are identical.

**Higher-point function:** Two neighboring hexagons always share two operators, but the third/fourth operator may not be identical. $\Rightarrow$ The two hexagon frames are misaligned.

In order to consistently sum over mirror states, need to align the two frames by a $PSU(2,2|4)$ transformation that maps $O_3$ onto $O_2$. 
Hexagonalization: Weight Factors

By conformal and R-symmetry transformation, bring $O_1$, $O_2$, and $O_4$ to canonical configuration:

\[
\begin{align*}
&\text{Transformation that maps } O_3 \text{ to } O_2: \quad g = e^{-D \log |z|} e^{i\phi L} e^{J \log |\alpha|} e^{i\theta R}, \\
&\text{where } e^{2i\phi} = z/\bar{z}, \ e^{2i\theta} = \alpha/\bar{\alpha}, \text{ and } (\alpha, \bar{\alpha}) \text{ is the R-coordinate of } O_3.
\end{align*}
\]

Hexagon $\mathcal{H}_1 = \hat{\mathcal{H}}$ is canonical, and $\mathcal{H}_2 = g^{-1} \hat{\mathcal{H}} g$.

Sum over states in mirror channel:

\[
\sum_{\psi} \mu(\psi) \langle \mathcal{H}_2 | \psi \rangle \langle \psi | \mathcal{H}_1 \rangle = \sum_{\psi} \mu(\psi) \langle g^{-1} \hat{\mathcal{H}} | \psi \rangle \langle \psi | g | \psi \rangle \langle \psi | \hat{\mathcal{H}} \rangle
\]

Weight factor:

\[
\langle \psi | g | \psi \rangle = e^{-2i\tilde{\rho}_\psi \log |z|} e^{J \psi \varphi} e^{i\phi L \psi} e^{i\theta R \psi}, \quad i\tilde{\rho} = (D - J)/2.
\]

→ Contains all non-trivial dependence on cross ratios $z, \bar{z}$ and $\alpha, \bar{\alpha}$. 

Non-Planar Processes: Idea

Hexagonalization: Works for planar (4,5)-point functions

Extend to non-planar processes?
- Fix worldsheet topology
- Dissect into planar hexagons
- Glue hexagons (mirror states)

Simple Proposal:
\[
\langle O_1 \ldots O_n \rangle_{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\text{graphs}} \prod_c d_c^\ell_c \sum_{\text{mirror states}} H_1 H_2 H_3 \ldots H_F
\]
The Data: Kinematics

Half-BPS operators:

\[ Q^k_i \equiv \text{Tr}\left[ (\alpha_i \cdot \Phi(x_i))^k \right], \quad \Phi = (\phi_1, \ldots, \phi_6), \quad \alpha^2_i = 0. \]

For equal weights \((k, k, k, k)\): Expand in \(X, Y, Z\):

\[
X \equiv \frac{\alpha_1 \cdot \alpha_2 \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2}, \quad Y \equiv \begin{array}{c} 1 \hline 2 \end{array}, \quad Z \equiv \begin{array}{c} 1 \hline 2 \\ \hline 3 \hline 4 \end{array}.
\]

Focus on \(Z = 0\) (polarizations):

\[ G_k \equiv \langle Q^k_1 Q^k_2 Q^k_3 Q^k_4 \rangle^{\text{loops}} = R \sum_{m=0}^{k-2} F_{k,m} X^m Y^{k-2-m} \]

Supersymmetry factor: \(R = z\bar{z}X^2 - (z + \bar{z})XY + Y^2\)

Main data: Coefficients \(F_{k,m} = F_{k,m}(g; z, \bar{z})\)

Cross ratios: \(z\bar{z} = s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1 - z)(1 - \bar{z}) = t = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \)
The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

\[ F_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} F^{(\ell)}_{k,m}(z, \bar{z}) , \quad \text{'t Hooft coupling: } g^2 = \frac{g_{YM}^2 N_c}{16\pi^2} . \]

One and two loops: Two ingredients: Box integrals

\[ F^{(1)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \]

\[ F^{(2)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} = \]

& Color factors: \[ C_{k,m}^i \in \{ \]

\[ \mathcal{1} = \text{Tr}(T^{a_1} \ldots T^{a_k}) , \quad \bullet = f_{ab}^c \]
To obtain non-planar corrections: Need to expand color factors.

\[ C_{k,m}^i = N_c^{2k} k^4 \left( \cdot C_{k,m}^i + \circ C_{k,m}^i N_c^{-2} + \mathcal{O}(N_c^{-4}) \right), \quad i \in \{a, b, c, d\}, \]

Compute by brute force:

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<th>( k )</th>
<th>( m )</th>
<th>( \frac{1}{2} C_{k,m}^{1,U} )</th>
<th>( \frac{1}{2} C_{k,m}^{1,SU} )</th>
<th>( C_{k,m}^{a,U} )</th>
<th>( C_{k,m}^{b,U} )</th>
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<th>( \frac{1}{2} C_{k,m}^{a,SU} )</th>
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also: \( k = 7, 8, 9 \). All color factors are quartic polynomials in \( m \) and \( k \).
\[ F_{k,m}(z, \bar{z}) = \]
\[ -\frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \frac{1}{6} r^2 + \frac{15}{8} k^2 - \frac{1}{2} k \right\} F^{(1)}, \]
\[ F_{k,m}^{(2)}(z, \bar{z}) = \]
\[ \frac{4k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \frac{1}{6} r^2 + \frac{15}{8} k^2 - \frac{1}{2} k \right\} F^{(2)} \]
\[ + \left\{ t + \frac{1}{N_c^2} \left[ \left( \frac{7}{2} r^2 - \frac{1}{8} \right] k^2 + \frac{5}{8} k - \frac{1}{4} \right] s_+ - r \left( \frac{17}{6} r^2 - \frac{7}{8} \right] k^3 + 3k^2 - \frac{13}{12} k \right) s_- \]
\[ + \left( \left[ \frac{29}{24} r^4 - \frac{11}{16} r^2 + \frac{15}{128} \right] k^4 + \frac{17}{8} r^2 - \frac{21}{32} \right] k^3 - \left[ \frac{23}{24} r^2 + \frac{39}{32} \right] k^2 - \frac{9}{8} k + \frac{1}{2} \right] t \right\} \left( F^{(1)} \right)^2 \]
\[ - \frac{1}{N_c^2} \left[ r \left\{ \frac{1}{8} \right] k^3 + \frac{3}{2} k^2 + \frac{10}{3} k \right\} F^{(2)}_{C,-} \]
\[ + \left\{ \left[ \frac{5}{4} r^2 - \frac{19}{48} \right] k^3 + \left[ \frac{3}{2} r^2 + \frac{7}{8} \right] k^2 + \frac{1}{3} k \right\} F^{(2)}_{C,+} \]
\[ + \frac{1}{4} \left\{ 1 + \frac{(k-1)(k^3 + 3k^2 - 46k + 36)}{12N_c^2} \right\} \left( s_{m,0} + \delta_{m,k-2} \right) \left( \frac{F^{(1)}}{F^{(1)}} \right)^2 \]
\[ + \left\{ \frac{1}{2} \left( \frac{(k-2)}{12N_c^2} \right) \right\} \left( \delta_{m,0} F^{(2)}_{z=1} + \delta_{m,k-2} F^{(2)}_{1-z} \right), \]

where \( r = (m+1)/k - 1/2 \). \( F_{k,m} \): Coefficient of \( X^m Y^{k-2-m} \).
Sum over Graphs: Cutting the Torus

Sum over propagator graphs: Split into
- Sum over “skeleton graphs” with non-parallel edges (≡ “bridges”)
- Sum over distributions of parallel propagators on bridges

Torus with four punctures: How many hexagons/bridges?

Euler: \( F + V - E = 2 - 2g \).

Our case: \( g = 1, \ V = 4, \ E = \frac{3}{2}F \quad \Rightarrow \quad F = 8, \ E = 12. \)

→ Construct all genus-one graphs with 4 punctures and up to 12 edges.

Propagators may populate < 12 bridges and still form a genus-one graph. Such graphs will contain higher polygons besides hexagons.

→ Subdivide into hexagons by inserting zero-length bridges (ZLBs)
Maximal Graphs

Focus on Maximal Graphs: Graphs with a maximal number of edges.
- Adding any further edge would increase the genus
- Maximal graphs ⇔ triangulations of the torus.

Construction:
- Manually: Add one operator at a time, in all possible ways.
- Computer algorithm: Start with the empty graph, add one bridge in all possible ways, iterate. → Systematic.

Complete list of maximal graphs:
Submaximal Graphs

Submaximal graphs: Graphs with a non-maximal number of edges.

- Obtained from maximal graphs by deleting bridges.
- Number of genus-one graphs by number of bridges:

<table>
<thead>
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<th>#bridges:</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>≤4</th>
</tr>
</thead>
<tbody>
<tr>
<td>#graphs:</td>
<td>7</td>
<td>28</td>
<td>117</td>
<td>254</td>
<td>323</td>
<td>222</td>
<td>79</td>
<td>11</td>
<td>0</td>
</tr>
</tbody>
</table>

Hexagonalization:
Submaximal graphs contain higher polygons (octagons, decagons, ...).

- Must be subdivided into hexagons by zero-length bridges.
- Subdivision is not physical: Can pick any (flip invariance):
Focus on leading order in large $k \to$ several simplifications:

Data:

\[
\mathcal{F}_{k,m}^{(1),U}(z, \bar{z}) = -\frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(1)},
\]

\[
\mathcal{F}_{k,m}^{(2),U}(z, \bar{z}) = \frac{4k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(2)}
\] + \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} \frac{t}{4} \left( F^{(1)} \right)^2.
\]

Combinatorics of distributing propagators on bridges:
Sum over distributions of $m$ propagators on $j + 1$ bridges $\to m^j / j!$

- Only graphs with maximum bridge number contribute.
- All bridges carry a large number of propagators.
First Test: Large $k$: Graphs and Labelings

**Graphs:**

<table>
<thead>
<tr>
<th>Graph</th>
<th>Labelings</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)</td>
</tr>
<tr>
<td>B</td>
<td>(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)</td>
</tr>
<tr>
<td>G</td>
<td>(1, 2, 4, 3), (3, 4, 2, 1)</td>
</tr>
<tr>
<td>G</td>
<td>(1, 3, 4, 2), (2, 4, 3, 1)</td>
</tr>
<tr>
<td>L</td>
<td>(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)</td>
</tr>
<tr>
<td>M</td>
<td>(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)</td>
</tr>
<tr>
<td>P</td>
<td>(1, 2, 4, 3)</td>
</tr>
<tr>
<td>Q</td>
<td>(1, 2, 4, 3)</td>
</tr>
</tbody>
</table>

**Sum over labelings:**

<table>
<thead>
<tr>
<th>Case</th>
<th>Inequivalent Labelings (clockwise)</th>
<th>Combinatorial Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)$</td>
<td>$m^3(k - m)/6$</td>
</tr>
<tr>
<td>B</td>
<td>$(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)$</td>
<td>$m(k - m)^3/6$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1)$</td>
<td>$m^4/24$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 3, 4, 2), (2, 4, 3, 1)$</td>
<td>$(k - m)^4/24$</td>
</tr>
<tr>
<td>L</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)$</td>
<td>$m^2/2 \cdot (k - m)^2/2$</td>
</tr>
<tr>
<td>M</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>P</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>Q</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2$</td>
</tr>
</tbody>
</table>
First Test: Large $k$: Octagons

All graphs consist of only **octagons**!
Split each octagon into two **hexagons** with a zero-length bridge.

**Example:**
**First Test: Large $k$: Mirror Particles**

**Loop Counting:**
Expand mirror propagation $\mu(u) e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling $g$.

$\rightarrow$ $n$ particles on bridge of size $\ell$: $\mathcal{O}(g^2(n\ell+n^2))$

All graphs consist of octagons framed by parametrically large bridges.

$\rightarrow$ Only excitations on zero-length bridges inside octagons survive.

**Excited Octagons:**

$n$ particles on a zero-length bridge $\rightarrow \mathcal{O}(g^{2n^2})$

$\rightarrow$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops.

**Octagon 1–2–4–3 with 1 particle:**

$$
\mathcal{M}(z, \alpha) = \left[ z + \bar{z} - (\alpha + \bar{\alpha}) \frac{\alpha \bar{\alpha} + z \bar{z}}{2\alpha \bar{\alpha}} \right] \\
\cdot \left( g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \ldots \right)
$$

For $Z = 0$: R-charge cross ratios

$$
\alpha = z\bar{z} \frac{X}{Y} \text{ and } \bar{\alpha} = 1.
$$
First Test: Large $k$: Match and Prediction

We are Done:
Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
⇒ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

\[
\mathcal{F}_{k,m}^U(z, \bar{z})\bigg|_{\text{torus}} = -\frac{2k^6}{N_c^4} \left\{ \begin{array}{l}
g^2 \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] F^{(1)} \checkmark \text{ match} \\
- 2g^4 \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] F^{(2)} + \left[ \frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right] \frac{t}{4} (F^{(1)})^2 \right] \checkmark \text{ match} \\
+ g^6 \left[ \ldots \right] F^{(3)} + \left[ \ldots \right] (F^{(2)}) (F^{(1)}) + \left[ \ldots \right] (F^{(1)})^3 \right. \text{ prediction!} \\
\left. + \mathcal{O}(g^8) + \mathcal{O}(1/k) \right\} .
\]
More Tests: \( k = 2, 3, 4, 5, \ldots \)

Small and finite \( k \):
Few propagators \( \rightarrow \) Fewer bridges \( \rightarrow \) Graphs with fewer edges
\( \Rightarrow \) Graphs composed of not only octagons, but bigger polygons

Example: Graphs for \( k = 3 \):

Hexagonalization:
Each \( 2n \)-gon: Split into \( n - 2 \) hexagons by \( n - 3 \) zero-length bridges.

Loop Expansion: Much more complicated!
All kinds of excitation patterns already at low loop orders
- Single particles on several adjacent zero-length (or \( \ell = 1 \)) bridges
- Strings of excitations wrapping around operators
Finite $k$: One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.
⇒ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:

One-strings: understood ✓
Longer strings: need to compute!
**Two-String: Result**

**One-String:** Can be written as

\[ M^{(1)}(z, \alpha) = m(z) + m(z^{-1}), \]

with building block

\[ m(z) = m(z, \alpha) = g^2 \frac{(z + \bar{z}) - (\alpha + \bar{\alpha})}{2} F^{(1)}(z, \bar{z}) \]

**Two-string:** Despite complicated computation, simplifies to

\[ M^{(2)}(z_1, z_2, \alpha_1, \alpha_2) = m \left( \frac{z_1 - 1}{z_1 z_2} \right) + m \left( \frac{1 - z_1 + z_1 z_2}{z_2} \right) + m(z_1(1 - z_2)) - m(z_1) - m(z_2^{-1}), \]

with the same building block \( m(z) \)!
**Finite $k$: Larger Strings**

**Larger strings:** Computation will be even more complicated!

**But:** Can in fact bootstrap all of them by using flip invariance!

Apply recursively:
- 3-string $\sim$
  - 1-strings & 2-strings
- ...iterate ...
- $n$-string $\sim$
  - 1-strings & 2-strings

⇒ Can write all polygons in terms of only 1-strings & 2-strings.

⇒ All $n$-strings can be written as linear combinations of one-string building blocks $m(z)$.
Finite $k$: Results

Done! Sum over all graphs, expand all polygons to their one-loop values.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$g = 0$</th>
<th>$g = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>32</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>441</td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>2760</td>
</tr>
</tbody>
</table>

Numbers of labeled graphs with assigned bridge sizes:

Result: For $k = 2, 3, 4, 5, \ldots$:
Matches the $U(N_c)$ data $F_{k,m}$, up to a copy of the planar term!

$$F_{k,m}: \text{Result} = \text{(torus data)} + \frac{1}{N_c^2} \text{(planar data)}$$

What does this mean?? ⇒ Puzzle.

Difference between $U(N_c)$ and $SU(N_c)$? → No
Operator normalizations? → No
Need to include planar graphs on the torus? If yes, how?
Finite $k$: Stratification

We are computing a worldsheet process.
The string amplitude involves integration over moduli space $\mathcal{M}_{g,n}$.

**Sum over graphs:** Reminiscent of moduli space integration.
This can be made more precise:
Moduli space $\Leftrightarrow$ space of *metric ribbon graphs* $\text{RGB}^\text{met}_{g,n}$.

**Metric Ribbon Graphs with labeled Boundary:**
Regular graphs, but edges at each vertex have definite ordering.
Double-line notation defines $n$ oriented boundary components (faces).
Faces define compact oriented surface of definite genus $g$.
Assign length $\ell_j \in \mathbb{R}_+$ to each edge.

**Bijection:** Via Strebel theory:

$$
\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \leftrightarrow \text{RGB}^\text{met}_{g,n} = \bigsqcup_{\Gamma \in \text{RG}_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}_\partial(\Gamma)}
$$
Finite $k$: Stratification

**Discretization:** Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size $\rightarrow$ handle gets pinched.

This problem has been considered before in the context of matrix models.

**Resolution:** In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification.*

\[
\Rightarrow + \quad \begin{array}{c}
\times \\
\times \\
\times \\
\times 
\end{array} 
- 
\left( 
\begin{array}{c}
\times \\
\times \\
\times \\
\times 
\end{array}
\right) 
= 
\begin{array}{c}
\times \\
\times \\
\times \\
\times 
\end{array}
\]

Including these contributions indeed accounts for the $(\text{planar})/N_c^2$ term!

**⇒** Now have a complete match for $k = 2, 3, 4, 5$. 

Deligne Mumford '69
Chekhov 1995
Summary & Outlook

**Summary:** Method to compute higher-genus terms in $1/N_c$ expansion.

- **Sum** over free graphs, **decompose** into planar hexagons, **integrate** over mirror states.
- Large $k$: Only octagons, match at two loops, three-loop prediction
- Match for various finite $k \rightarrow$ stratification

**Outlook:** There are many things to do that we currently explore:

- Study more examples: Higher loops / genus, more general operators
- Understand details/implications of stratification beyond one loop
- Evaluation of mirror particles at higher loops
- Connect to recent supergravity loop computations at strong coupling?
  - Promising: Large $k$ at higher genus: Only octagons.