

Handling Handles: Non-Planar AdS/CFT Integrability

Part 2 (Part 1 by J. Caetano)

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+ further work in progress

WORKSHOP ON CORRELATION FUNCTIONS IN SOLVABLE MODELS

NORDITA, STOCKHOLM, MAY 2018

Non-Planar Processes: Idea

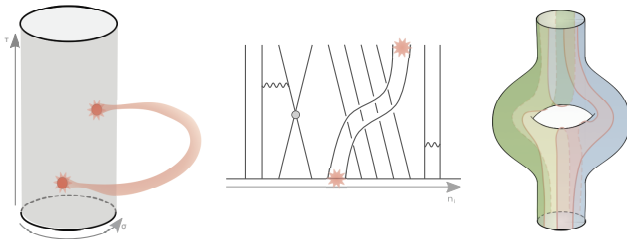
Hexagonalization: Works for planar (4,5)-point functions [Fleury '16] [Komatsu] [Fleury '17] [Komatsu]

Extend to non-planar processes?

- ▶ Fix worldsheet topology
- ▶ Dissect into planar hexagons
- ▶ Glue hexagons (mirror states)

Simple Proposal:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle^{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\substack{\text{graphs} \\ (\text{genus } g)}} \prod_c d_c^{\ell_c} \sum_{\substack{\text{mirror} \\ \text{states}}} \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \dots \mathcal{H}_F$$



Sum over Graphs: Cutting the Torus

Sum over propagator graphs: Split into

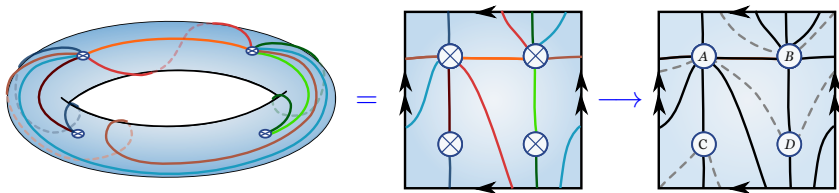
- ▶ Sum over graphs with non-parallel edges ($\equiv$ “bridges”)
- ▶ Sum over distributions of parallel propagators on bridges

Torus with four punctures: *How many hexagons/bridges?*

Euler: $F + V - E = 2 - 2g$.

Our case: $g = 1$, $V = 4$, $E = \frac{3}{2}F \Rightarrow F = 8$, $E = 12$.

→ **Construct** all genus-one graphs with 4 punctures and **up to** 12 edges.



Propagators may populate < 12 bridges and still form a genus-one graph. Such graphs will contain **higher polygons** besides hexagons.

→ **Subdivide** into hexagons by inserting **zero-length bridges (ZLBs)**

Maximal Graphs

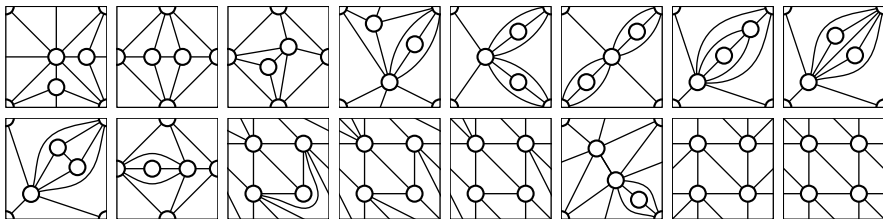
Focus on Maximal Graphs: Graphs with a maximal number of edges.

- ▶ Adding any further edge would increase the genus
- ▶ Maximal graphs $\Leftrightarrow$ triangulations of the torus.

Construction:

- ▶ **Manually:** Add one operator at a time, in all possible ways.
- ▶ **Computer algorithm:** Start with the empty graph, add one bridge in all possible ways, iterate. $\rightarrow$ **Systematic.**

Complete list of maximal graphs:



Submaximal Graphs

Submaximal graphs: Graphs with a **non-maximal** number of edges.

- ▶ Obtained from maximal graphs by deleting bridges.
- ▶ Number of genus-one graphs by number of bridges:

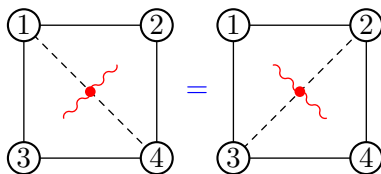
#bridges:	12	11	10	9	8	7	6	5	≤ 4
#graphs:	7	28	117	254	323	222	79	11	0

Hexagonalization:

Submaximal graphs contain **higher polygons** (octagons, decagons, ...).

- ▶ Must be subdivided into hexagons by zero-length bridges.
- ▶ Subdivision is not physical: Can pick any (flip invariance):

[Fleury '16
Komatsu]



The Data: Kinematics

Half-BPS operators:

$$\mathcal{Q}_i^k \equiv \text{Tr}[(\alpha_i \cdot \Phi(x_i))^k], \quad \Phi = (\phi_1, \dots, \phi_6), \quad \alpha_i^2 = 0.$$

For **equal weights** (k, k, k, k) : Expand in X, Y, Z :

$$X \equiv \frac{\alpha_1 \cdot \alpha_2 \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} = \begin{array}{c} \textcircled{1} \text{---} \textcircled{2} \\ \textcircled{3} \text{---} \textcircled{4} \end{array}, \quad Y \equiv \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{3} \end{array} \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{4} \end{array}, \quad Z \equiv \begin{array}{c} \textcircled{1} \text{---} \textcircled{2} \\ \diagdown \quad \diagup \\ \textcircled{3} \text{---} \textcircled{4} \end{array}.$$

Focus on $Z = 0$ (polarizations):

[Arutyunov, Sokatchev '03] [Arutyunov, Penati '03, Santambrogio, Sokatchev]

$$G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle_{\text{loops}} = R \sum_{m=0}^{k-2} \mathcal{F}_{k,m} X^m Y^{k-2-m}$$

Supersymmetry factor: $R = z\bar{z}X^2 - (z + \bar{z})XY + Y^2$

Main data: Coefficients $\mathcal{F}_{k,m} = \mathcal{F}_{k,m}(g; z, \bar{z})$

$$\text{Cross ratios: } z\bar{z} = s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1-z)(1-\bar{z}) = t = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}.$$

The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

$$\mathcal{F}_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} \mathcal{F}_{k,m}^{(\ell)}(z, \bar{z}), \quad \text{'t Hooft coupling: } g^2 = \frac{g_{\text{YM}}^2 N_c}{16\pi^2}.$$

One and two loops: Two ingredients: **Box integrals**

$$F^{(1)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = \text{Box diagram},$$

$$\frac{F^{(2)}(z, \bar{z})}{x_{14}^2} = \frac{x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} = \text{Two-loop box diagram},$$

& Color factors:

$$C_{k,m}^i \in \left\{ \begin{array}{c} \text{Four diagrams showing color factor structures for the box and two-loop box integrals, with vertices labeled 1, 2, 3, 4 and internal lines with dots representing color indices.} \end{array} \right\}$$

$$\textcircled{1} = \text{Tr}(T^{a_1} \dots T^{a_k}), \quad \text{dot on line} = f_{ab}^c$$

The Data: Color Factors

To obtain non-planar corrections: Need to **expand color factors**.

$$C_{k,m}^i = N_c^{2k} k^4 \left(\bullet C_{k,m}^i + {}^\circ C_{k,m}^i N_c^{-2} + \mathcal{O}(N_c^{-4}) \right), \quad i \in \{a, b, c, d\},$$

Compute by **brute force**:

k	m	$\frac{1}{2} {}^\circ C_{k,m}^{1,U}$	$\frac{1}{2} {}^\circ C_{k,m}^{1,SU}$	${}^\circ C_{k,m}^{a,U}$	$2 {}^\circ C_{k,m}^{b,U}$	$\frac{1}{2} {}^\circ C_{k,m}^{c,U}$	${}^\circ C_{k,m}^{d,U}$	${}^\circ C_{k,m}^{a,SU}$	$2 {}^\circ C_{k,m}^{b,SU}$	$\frac{1}{2} {}^\circ C_{k,m}^{c,SU}$	${}^\circ C_{k,m}^{d,SU}$
2	0	1	1	0	-2	-1	-1	0	-2	-1	-1
3	0	1	9	-5	-2	-1	-1	-9	-18	-9	-9
3	1	1	9	0	3	-1	-1	0	-5	-9	-9
4	0	-5	13	-7	10	5	5	-25	-26	-13	-13
4	1	-12	24	4	15	13	14	-23	-21	-23	-22
4	2	-5	13	0	21	5	5	0	3	-13	-13
5	0	-23	9	-1	46	23	23	-33	-18	-9	-9
5	1	-51	13	31	47	55	59	-33	-17	-9	-5
5	2	-51	13	39	76	55	59	-9	12	-9	-5
5	3	-23	9	0	63	23	23	0	31	-9	-9
6	0	-61	-11	20	122	61	61	-30	22	11	11
6	1	-126	-26	92	107	135	144	-8	7	35	44
6	2	-159	-59	139	187	175	191	39	87	75	91
6	3	-126	-26	110	201	135	144	35	101	35	44
6	4	-61	-11	0	139	61	61	0	89	11	11

also: $k = 7, 8, 9$. All color factors are **quartic polynomials** in m and k .

The Data: Result

$$\begin{aligned}
 \mathcal{F}_{k,m}^{(1),U}(z, \bar{z}) = & -\frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[\left[\frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] k^4 + \left[\frac{9}{2}r^2 - \frac{13}{8} \right] k^3 + \left[\frac{1}{6}r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2}k \right] \right\} F^{(1)}, \\
 \mathcal{F}_{k,m}^{(2),U}(z, \bar{z}) = & \frac{4k^2}{N_c^2} \left[\left\{ 1 + \frac{1}{N_c^2} \left[\left[\frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] k^4 + \left[\frac{9}{2}r^2 - \frac{13}{8} \right] k^3 + \left[\frac{1}{6}r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2}k \right] \right\} F^{(2)} \right. \\
 & + \left\{ \frac{t}{4} + \frac{1}{N_c^2} \left[\left(\left[\frac{7}{2}r^2 - \frac{1}{8} \right] k^2 + \frac{5}{8}k - \frac{1}{4} \right) s_+ - r \left(\left[\frac{17}{6}r^2 - \frac{7}{8} \right] k^3 + 3k^2 - \frac{13}{12}k \right) s_- \right. \right. \\
 & \left. \left. + \left(\left[\frac{29}{24}r^4 - \frac{11}{16}r^2 + \frac{15}{128} \right] k^4 + \left[\frac{17}{8}r^2 - \frac{21}{32} \right] k^3 - \left[\frac{23}{24}r^2 - \frac{39}{32} \right] k^2 - \frac{9}{8}k + \frac{1}{2} \right) t \right] \right\} \left(F^{(1)} \right)^2 \\
 & - \frac{1}{N_c^2} \left[r \left\{ \left[\frac{7}{6}r^2 - \frac{1}{8} \right] k^3 + \frac{3}{2}k^2 + \frac{10}{3}k \right\} F_{C,-}^{(2)} \right. \\
 & \left. + \left\{ \left[\frac{5}{4}r^2 - \frac{19}{48} \right] k^3 + \left[\frac{3}{2}r^2 + \frac{7}{8} \right] k^2 + \frac{1}{3}k \right\} F_{C,+}^{(2)} \right] \\
 & + \frac{1}{4} \left\{ 1 + \frac{(k-1)(k^3 + 3k^2 - 46k + 36)}{12N_c^2} \right\} (s\delta_{m,0} + \delta_{m,k-2}) \left(F^{(1)} \right)^2 \\
 & + \left\{ 1 + \frac{(k-2)_4}{12N_c^2} \right\} \left(\delta_{m,0} F_{z-1}^{(2)} + \delta_{m,k-2} F_{1-z}^{(2)} \right) \Big],
 \end{aligned}$$

where $r = (m+1)/k - 1/2$.

$\mathcal{F}_{k,m}$: Coefficient of $X^m Y^{k-2-m}$.

First Test: Large k : Data and Graphs

Focus on leading order in large $k \rightarrow$ **several simplifications:**

Data: $\mathcal{F}_{k,m}^{(1),U}(z, \bar{z}) = -\frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[\left[\frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(1)},$

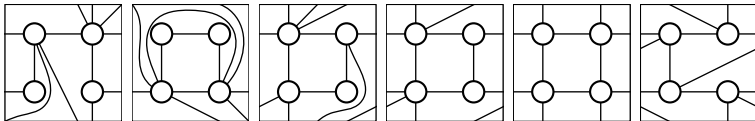
$$\mathcal{F}_{k,m}^{(2),U}(z, \bar{z}) = \frac{4k^2}{N_c^2} \left[\left\{ 1 + \frac{1}{N_c^2} \left[\left[\frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(2)} + \left\{ 1 + \frac{1}{N_c^2} \left[\left[\frac{29}{6}r^4 - \frac{11}{4}r^2 + \frac{15}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} \frac{t}{4} (F^{(1)})^2 \right].$$

Combinatorics of distributing propagators on bridges:

Sum over distributions of m propagators on $j + 1$ bridges $\rightarrow m^j / j!$

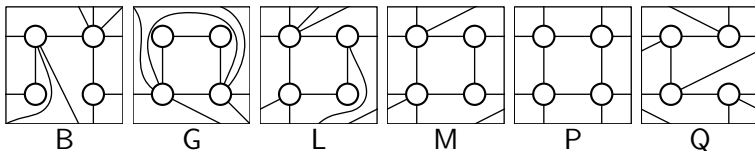
- ▶ $\Rightarrow$ Only graphs with maximum bridge number contribute.
- ▶ $\Rightarrow$ All bridges carry a large number of propagators.

Graphs:
($Z = 0$)



First Test: Large k : Graphs and Labelings

Graphs:

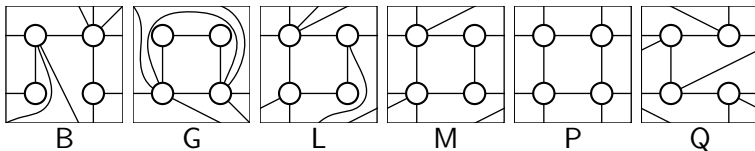


Sum over labelings:

Case	Inequivalent Labelings (clockwise)	Combinatorial Factor
B	$(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)$	$m^3(k - m)/6$
B	$(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)$	$m(k - m)^3/6$
G	$(1, 2, 4, 3), (3, 4, 2, 1)$	$m^4/24$
G	$(1, 3, 4, 2), (2, 4, 3, 1)$	$(k - m)^4/24$
L	$(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)$	$m^2/2 \cdot (k - m)^2/2$
M	$(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)$	$m^2(k - m)^2/2$
P	$(1, 2, 4, 3)$	$m^2(k - m)^2/2$
Q	$(1, 2, 4, 3)$	$m^2(k - m)^2$

First Test: Large k : Octagons

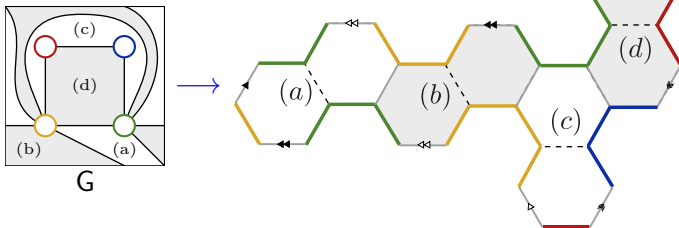
Graphs:



All graphs consist of only **octagons**!

Split each octagon into two **hexagons** with a zero-length bridge.

Example:



First Test: Large k : Mirror Particles

Loop Counting:

Expand mirror measure $\mu(u) \sim e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling g
→ n particles on bridge of size ℓ : $\mathcal{O}(g^{2(n\ell+n^2)})$

All graphs consist of octagons framed by parametrically large bridges
→ Only excitations on zero-length bridges inside octagons survive

Excited Octagons:

n particles on a zero-length bridge → $\mathcal{O}(g^{2n^2})$

→ Octagons with 1/2/3/4 particles start at 1/4/9/16 loops

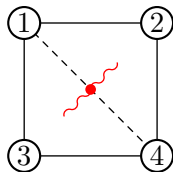
Octagon 1–2–4–3 with 1 particle:

$$\mathcal{M}(z, \alpha) = \left[z + \bar{z} - (\alpha + \bar{\alpha}) \frac{\alpha \bar{\alpha} + z \bar{z}}{2\alpha \bar{\alpha}} \right] \cdot \left(g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \dots \right)$$

For $Z = 0$: R-charge cross ratios

$$\alpha = z \bar{z} X/Y \text{ and } \bar{\alpha} = 1.$$

[Fleury '16] [TB, Caetano, Fleury
Komatsu] [Komatsu, Vieira '18]



First Test: Large k : Match and Prediction

We are Done:

Sum over graph topologies and labelings (with bridge sum factors),

Sum over one-particle excitations of all octagons.

⇒ Result **matches data** and **produces prediction** for higher loops!

Summing all octagons gives:

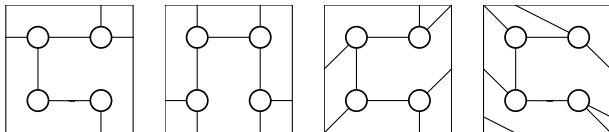
$$\begin{aligned} \mathcal{F}_{k,m}^{\text{U}}(z, \bar{z}) \Big|_{\text{torus}} = & -\frac{2k^6}{N_{\text{c}}^4} \left\{ \right. \\ & g^2 \left[\frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] F^{(1)} \quad \checkmark \text{ match} \\ & - 2g^4 \left[\left[\frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] F^{(2)} + \left[\frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right] \frac{t}{4} (F^{(1)})^2 \right] \quad \checkmark \text{ match} \\ & + g^6 \left[[\dots] F^{(3)} + [\dots] (F^{(2)}) (F^{(1)}) + [\dots] (F^{(1)})^3 \right] \quad \text{prediction!} \\ & + \mathcal{O}(g^8) + \mathcal{O}(1/k) \left. \right\}. \end{aligned}$$

More Tests: $k = 2, 3, 4, 5, \dots$

Small and finite k :

Few propagators $\rightarrow$ Fewer bridges $\rightarrow$ Graphs with fewer edges
 $\Rightarrow$ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:



Hexagonalization:

Each $2n$ -gon: Split into $n - 2$ hexagons by $n - 3$ zero-length bridges.

Loop Expansion: Much more complicated!

All kinds of excitation patterns already at low loop orders

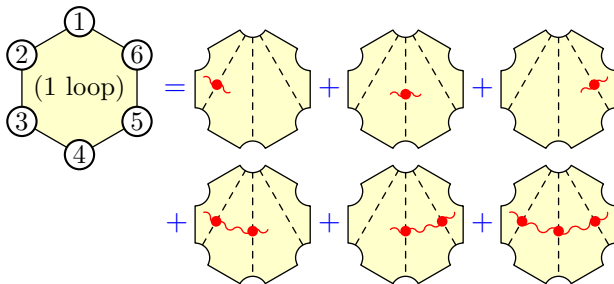
- ▶ Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- ▶ Strings of excitations wrapping around operators

Finite k : One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

⇒ Excitations confined to **single polygons** bounded by propagators.

For each polygon: Sum over all possible one-loop strings:



One-strings: **understood** ✓

Longer strings: **need to compute!**

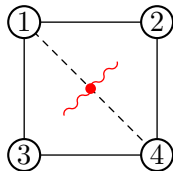
Two-String: Result

One-String: Can be written as

$$\mathcal{M}^{(1)}(z, \alpha) = m(z) + m(z^{-1}),$$

with building block

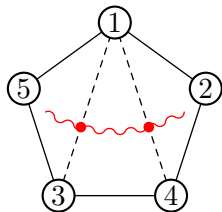
$$m(z) = m(z, \alpha) = g^2 \frac{(z + \bar{z}) - (\alpha + \bar{\alpha})}{2} F^{(1)}(z, \bar{z})$$



Two-string: Despite complicated computation, simplifies to

[Fleury '17
Komatsu]

$$\begin{aligned} \mathcal{M}^{(2)}(z_1, z_2, \alpha_1, \alpha_2) \\ = m\left(\frac{z_1 - 1}{z_1 z_2}\right) + m\left(\frac{1 - z_1 + z_1 z_2}{z_2}\right) \\ + m(z_1(1 - z_2)) - m(z_1) - m(z_2^{-1}), \end{aligned}$$



with the same building block $m(z)$!

Finite k : Results

Done! Sum over all graphs, expand all polygons to their one-loop values.

Numbers of labeled
graphs with assigned
bridge sizes:

k :	2	3	4	5
$g = 0$:	3	8	15	24
$g = 1$:	0	32	441	2760

Result: For $k = 2, 3, 4, 5, \dots$:

Matches the $U(N_c)$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m} : \text{Result} = (\text{torus data}) + \frac{1}{N_c^2} (\text{planar data})$$

✓✓✓ ???

What does this mean?? $\Rightarrow$ **Puzzle.**

Difference between $U(N_c)$ and $SU(N_c)$? $\rightarrow$ No

Operator normalizations? $\rightarrow$ No

Need to include planar graphs on the torus? If yes, how?

Finite k : Stratification

We are computing a **worksheet process**.

The string amplitude involves integration over moduli space $\mathcal{M}_{g,n}$.

Sum over graphs: Reminiscent of moduli space integration.

This can be made more precise:

Moduli space $\Leftrightarrow$ space of **metric ribbon graphs** $\text{RGB}_{g,n}^{\text{met}}$.

Metric Ribbon Graphs with labeled Boundary:

Regular graphs, but edges at each vertex have definite ordering.

Double-line notation defines n oriented **boundary components** (faces).

Faces define compact oriented surface of definite **genus** g .

Assign **length** $\ell_j \in \mathbb{R}_+$ to each edge.

Bijection: Via Strebel theory:

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \longleftrightarrow \text{RGB}_{g,n}^{\text{met}} = \coprod_{\Gamma \in \text{RG}_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)}$$

Finite k : Stratification

Discretization: Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size $\rightarrow$ handle gets pinched.

This problem has been considered before in the context of matrix models.

[Deligne
Mumford '69]
[Chekhov
1995]

Resolution: In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification*.

$$\Rightarrow + \left(\text{torus with 4 x's} \right) - \left(\text{torus with 4 x's and a bridge} \right) = \left(\text{torus with 4 x's and two pinched bridges} \right)$$

The diagrammatic equation shows a red arrow pointing to a plus sign, followed by a torus with four 'x' marks. This is followed by a minus sign and a large blue bracket containing two diagrams: a torus with four 'x' marks and a bridge, and a torus with four 'x' marks and two pinched bridges (marked with a circled cross). The entire expression is followed by an equals sign and another diagram of a torus with four 'x' marks and two pinched bridges.

Including these contributions indeed accounts for the (planar)/ N_c^2 term!

$\Rightarrow$ Now have a complete match for $k = 2, 3, 4, 5$.

Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_c$ expansion.

- ▶ **Sum** over free graphs, **decompose** into planar hexagons, **integrate** over mirror states.
 - ▶ Large k : Only octagons, match at two loops, three-loop prediction
 - ▶ Match for various finite $k \rightarrow$ stratification
-

Outlook: There are many things to do that we currently explore:

- ▶ Study more examples: Higher loops / genus, more general operators
- ▶ Understand details/implications of stratification beyond one loop
- ▶ Connect to recent supergravity loop computations at strong coupling? [\[Aharony, Alday '16\]](#) [\[Alday, Bissi\]](#) [\[Alday\]](#) [\[Aprile, Drummond, Heslop\]](#)
[\[Bissi, Perlmutter\]](#) [\[Perlmutter '17\]](#) [\[Bissi '17\]](#) [\[Paul '17, '17, '17, '18\]](#)
- ▶ Promising: Large k at higher genus: Only octagons.
Speculate: Resum $1/N_c$?