Handling Handles: Non-Planar AdS/CFT Integrability

Till Bargheer

Leibniz University Hannover

1711.05326: TB, J. Caetano, T. Fleury, S. Komatsu, P. Vieira
18xx.xxxxx: TB, J. Caetano, T. Fleury, S. Komatsu, P. Vieira
18xx.xxxxx: TB, F. Coronado, P. Vieira
+ further work in progress

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General Idea

In AdS$_5$, string amplitudes can be cut into basic patches (rectangles, pentagons, or hexagons), which can be \textit{bootstrapped} using \textit{integrability} at \textit{any value of the 't Hooft coupling}.

- Amplitudes are given as infinite sums and integrals over intermediate states from \textit{gluing together} these integrable patches.
- Sometimes, these sums and integrals can be re-summed, giving hints of \textit{yet-to-be uncovered structures}.
- This holds at the planar level as well as for \textit{non-planar processes} suppressed by $1/N_c$. 
**$\mathcal{N} = 4$ SYM & The Planar Limit**

$\mathcal{N} = 4$ super Yang–Mills: Gauge field $A_\mu$, scalars $\Phi_I$, fermions $\psi_{\alpha A}$.

Gauge group: $U(N_c) / SU(N_c)$.

Adjoint representation: All fields are $N_c \times N_c$ matrices.

**Double-line notation:**

- **Propagators:**
  \[
  \langle \Phi_I^i \Phi_J^j \rangle \sim g_{YM}^2 \delta^i_l \delta^j_k = \begin{array}{c}
  i \\
  j \\
  k \end{array} \begin{array}{c}
  l \end{array}
  \]

- **Vertices:**
  \[
  \text{Tr}(\Phi \Phi \Phi \Phi) \sim \frac{1}{g_{YM}^2}
  \]

- Diagrams consist of color index loops $\sim$ oriented disks $\sim \delta^i_i = N_c$
- Disks are glued along propagators $\rightarrow$ oriented compact surfaces

**Local operators:**

\[
O_i = \text{Tr}(\Phi \ldots) \sim O_i
\]

- One fewer color loop $\rightarrow$ factor $1/N_c$
- Surface: Hole $\sim$ boundary component
Every diagram is associated to an oriented compact surface.

**Genus Expansion:**

Absorb one factor of $N_c$ in the 't Hooft coupling $\lambda = g_{YM}^2 N_c$

Use Euler formula $V - E + F = 2 - 2g$

$\Rightarrow$ **Correlators** of single trace operators $\mathcal{O}_i = \text{Tr}(\Phi_1 \Phi_2 \ldots )$:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{1}{N_{c}^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_{c}^{2g}} G_g(\lambda)$$

$$\sim \frac{1}{N_{c}^{2}} + \frac{1}{N_{c}^{4}} + \frac{1}{N_{c}^{6}} + \ldots$$
Spectrum: Planar Limit

Goal: Correlation functions in $\mathcal{N} = 4$ SYM

Step 1: Planar spectrum of single-trace local operators $\text{Tr}(\Phi \ldots)$
  ▶ Spectrum of (anomalous) scaling dimensions $\Delta$
  ▶ Scale transformations represented by dilatation operator $\Gamma$
  ▶ $\Gamma$ mixes single-trace (& multi-trace) operators
  ▶ Resolve mixing $\rightarrow$ Eigenstates & eigenvalues (dimensions)

Planar limit:
  ▶ Multi-trace operators suppressed by $1/N_c$
  ▶ Dilatation operator acts locally in color space (neighboring fields)

Organize space of single-trace operators around protected states

$$\text{Tr} \ Z^L, \quad Z = \alpha^I \Phi_I, \quad \alpha^I \alpha_I = 0 \quad \text{(half-BPS, “vacuum”).}$$

Other single-trace operators: Insert impurities $\{\Phi_I, \psi_{\alpha A}, D_\mu\}$ into $\text{Tr} \ Z^L$. 
Initial observation: One-loop dilatation operator for scalar single-trace operators is integrable. Diagonalization by Bethe Ansatz.

- Impurities are magnons in color space, characterized by rapidity (momentum) $u$ and $\mathfrak{su}(2|2)^2$ flavor index. $[\mathfrak{su}(2|2)^2 \subset \mathfrak{psu}(2, 2|4)]$ preserves the vacuum $\text{Tr} \ Z^L$.

- Dynamics of magnons: integrability:
  - No particle production
  - Individual momenta preserved
  - Factorized scattering

- Two-body ($\to n$-body) S-matrix completely fixed to all loops

$\Rightarrow$ Asymptotic spectrum (for $L \to \infty$) solved to all loops / exactly.
**Finite-Size Effects**

**Asymptotic spectrum** solved by Bethe Ansatz.
Resums $\infty$ Feynman diagrams that govern dynamics of $\infty$ strip:

\[
\begin{array}{c}
\text{L} \\
\text{\rightarrow} \\
L \rightarrow \infty
\end{array}
\]

**Re-compactify:** Finite-size effects.
Leading effect: Momentum quantization constraint $\equiv$ Bethe equations

\[
1 = e^{ip_j L} \prod_{j \neq k} S(p_k, p_j)
\]

Moreover: **Wrapping interactions.**

- No notion of locality for dilatation operator
- Previous techniques (Bethe ansatz) no longer apply
Mirror Theory

Key to all-loop finite-size spectrum: **Mirror map**

Double Wick rotation: \((\sigma, \tau) \rightarrow (i\tilde{\tau}, i\tilde{\sigma})\) — exchanges space and time

Magnon states: Energy and momentum interchange: \(\tilde{E} = ip, \tilde{p} = iE\)

Finite size \(L\) becomes finite, periodic (discrete) time.

Energy \(\sim\) Partition function at finite temperature \(1/L\), with \(R \rightarrow \infty\).

\(\rightarrow\) **Thermodynamic Bethe ansatz**.

Simplifications and refinements:

- **Y-system (T-system, Q-system)**
- **Quantum Spectral Curve**

\(\Rightarrow\) Scaling dimensions computable at finite coupling.
Three-Point Functions: Hexagons

Differences: Topology: Pair of pants instead of cylinder
Non-vanishing for three generic operators (two-point: diagonal)
⇒ Previous techniques not directly applicable

Observation:

The green parts are similar to two-point functions:
Two segments of physical operators joined by parallel propagators ("bridges", $\ell_{ij} = (L_i + L_j - L_k)/2$).

The red part is new: "Worldsheet splitting", "three-point vertex" (open strings)

Take this serious → cut worldsheet along "bridges":
Glue hexagons along three mirror channels:

- Sum over complete state basis (magnons) in the mirror theory
- Mirror magnons: Boltzmann weight \( \exp(-\tilde{E}_{ij}\ell_{ij}) \), \( \tilde{E}_{ij} = \mathcal{O}(g^2) \)
  \( \rightarrow \) mirror excitations are strongly suppressed.

Hexagonal worldsheets patches (form factors):

- Function of rapidities \( u \) and \( \mathfrak{su}(2|2)^2 \) labels \( (A, \dot{A}) \) of all magnons.
- Conjectured exact expression, based on diagonal \( \mathfrak{su}(2|2) \) symmetry as well as form factor axioms.

Finite-coupling hexagon proposal: Supported by very non-trivial matches.
Move on to planar four-point functions:
One way to cut (now that three-point is understood): **OPE cut**

**Problem:** Sum over physical states!
- No loop suppression, all states contrib.
- Double-trace operators.

**Instead:** Cut along propagator bridges

**Benefits:**
- Mirror states highly suppressed in $g$.
- No double traces.

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[Eden '16] [Sfondrini]
Hexagonalization: Formula

\[
\langle O_1 O_2 O_3 \rangle = \prod_{c \in \{1,2,3\}} d_c^\ell \sum_{\psi_c} \mu(\psi_c) \mathcal{H}_1(\psi_1, \psi_2, \psi_3) \mathcal{H}_2(\psi_1, \psi_2, \psi_3)
\]

\[
\langle O_1 O_2 O_3 O_4 \rangle = \sum_{\text{planar prop. graphs}} \prod_{c \in \{1, \ldots, 6\}} d_c^\ell \sum_{\psi_c} \mu(\psi_c) \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4
\]

**New Features:**

- Bridge lengths vary, may go to zero \(\Rightarrow\) Mirror corrections at one loop
- Hexagons are in different “frames” \(\Rightarrow\) Weight factors

Fleury'16, Komatsu
Hexagonalization: Frames

Hexagon depends on positions $x_i$ and polarizations $\alpha_i$ of the three half-BPS “vacuum” operators $\mathcal{O}_i = \text{Tr}[(\alpha_i \cdot \Phi(x_i))^L]$.

Any three $x_i$ and $\alpha_i$ preserve a diagonal $su(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

**Three-point function:** Both hexagons connect to the same three operators, so their frames ($su(2|2)$ and state basis) are identical.

**Higher-point function:** Two neighboring hexagons always share two operators, but the third/fourth operator may not be identical. ⇒ The two hexagon frames are misaligned.

In order to consistently sum over mirror states, need to align the two frames by a $\text{PSU}(2, 2|4)$ transformation that maps $\mathcal{O}_3$ onto $\mathcal{O}_2$. 
Hexagonalization: Weight Factors

By conformal and R-symmetry transformation, bring $O_1$, $O_2$, and $O_4$ to canonical configuration:

$$e^{-D \log |z|}$$

Transformation that maps $O_3$ to $O_2$: $g = e^{-D \log |z|} e^{i\phi L} e^{J \log |\alpha|} e^{i\theta R}$, where $e^{2i\phi} = z/\bar{z}$, $e^{2i\theta} = \alpha/\bar{\alpha}$, and $(\alpha, \bar{\alpha})$ is the R-coordinate of $O_3$.

Hexagon $\mathcal{H}_1 = \hat{\mathcal{H}}$ is canonical, and $\mathcal{H}_2 = g^{-1}\hat{\mathcal{H}}g$.

Sum over states in mirror channel:

$$\sum_{\psi} \mu(\psi) \langle \mathcal{H}_2 | \psi \rangle \langle \psi | \mathcal{H}_1 \rangle = \sum_{\psi} \mu(\psi) \langle g^{-1}\hat{\mathcal{H}} | \psi \rangle \langle \psi | g \psi \rangle \langle \psi | \hat{\mathcal{H}} \rangle$$

Weight factor: $\langle \psi | g | \psi \rangle = e^{-2i\tilde{p}_\psi \log |z|} e^{J_{\psi} \varphi} e^{i\phi L_{\psi}} e^{i\theta R_{\psi}}$, $i\tilde{p} = (D - J)/2$.

→ Contains all non-trivial dependence on cross ratios $z, \bar{z}$ and $\alpha, \bar{\alpha}$. 
Non-Planar Processes: Idea

Hexagonalization: Works for planar (4,5)-point functions

Extend to non-planar processes?
- Fix worldsheet topology
- Dissect into planar hexagons
- Glue hexagons (mirror states)

Simple Proposal:

\[ \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\text{genus } g} \prod_c d_{c}^{\ell_c} \sum_{\text{mirror states}} \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \cdots \mathcal{H}_F \]
The Observable

Object of Study:
Four half-BPS operators on the torus
- Half-BPS: No physical magnons
- Non-trivial spacetime dependence
- Non-trivial coupling dependence
- Probes a lot of CFT data
- Non-planar data available!

To Do:
- Understand sum over propagator graphs
- Understand all mirror contributions

See also: Non-planar two-point analysis of
- Non-planar two point function for non-BPS states (at tree level)
- Divergence due to hexagons with two edges on the same operator
Sum over Graphs: Cutting the Torus

**Sum over propagator graphs:** Split into
- Sum over graphs with non-parallel edges (≡ “bridges”)
- Sum over distributions of parallel propagators on bridges

**Torus with four punctures:** How many hexagons/bridges?

Euler: \( F + V - E = 2 - 2g \).

Our case: \( g = 1 \), \( V = 4 \), \( E = \frac{3}{2}F \) \( \Rightarrow \) \( F = 8 \), \( E = 12 \).

→ **Construct** all genus-one graphs with 4 punctures and up to 12 edges.

Propagators may populate < 12 bridges and still form a genus-one graph. Such graphs will contain **higher polygons** besides hexagons.

→ **Subdivide** into hexagons by inserting zero-length bridges (ZLBs)
Focus on **Maximal Graphs**: Graphs with a maximal number of edges.

- Maximal graphs ⇔ triangulations of the torus.

**Construction:**

- **Manually**: Add one operator at a time, in all possible ways.
- **Computer algorithm**: Start with the empty graph, add one bridge in all possible ways, iterate.

**Complete list of maximal graphs:**

![Diagrams of maximal graphs](image-url)
Submaximal Graphs

Submaximal graphs: Graphs with a non-maximal number of edges.
- Obtained from maximal graphs by deleting bridges.
- Number of genus-one graphs by number of bridges:

<table>
<thead>
<tr>
<th>Bridges</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>≤4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphs</td>
<td>7</td>
<td>28</td>
<td>117</td>
<td>254</td>
<td>323</td>
<td>222</td>
<td>79</td>
<td>11</td>
<td>0</td>
</tr>
</tbody>
</table>

Hexagonalization:
Submaximal graphs contain higher polygons (octagons, decagons, ...).
- Must be subdivided into hexagons by zero-length bridges.
- Subdivision is not physical: Can pick any (flip invariance):
The Data: Kinematics

Half-BPS operators:

\[ Q_i^k \equiv \text{Tr}\left[ (\alpha_i \cdot \Phi(x_i))^k \right], \quad \Phi = (\phi_1, \ldots, \phi_6), \quad \alpha_i^2 = 0. \]

For equal weights \((k, k, k, k)\): Expand in \(X, Y, Z\):

\[ X \equiv \frac{\alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2}, \quad Y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}. \]

Focus on \(Z = 0\) (polarizations):

\[ G_k \equiv \langle Q_1^k Q_2^k Q_3^k Q_4^k \rangle_{\text{loops}} = R \sum_{m=0}^{k-2} F_{k,m} X^m Y^{k-2-m} \]

Supersymmetry factor: \(R = z\bar{z}X^2 - (z + \bar{z})XY + Y^2\)

**Main data:** Coefficients \(F_{k,m} = F_{k,m}(g; z, \bar{z})\)

Cross ratios: \(z\bar{z} = s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1 - z)(1 - \bar{z}) = t = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \)
The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

\[ F_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} F_{k,m}^{(\ell)}(z, \bar{z}), \quad \text{'}t\text{ Hooft coupling: } g^2 = \frac{g_{YM}^2 N_c}{16\pi^2}. \]

One and two loops: Two ingredients: Box integrals

\[ F^{(1)}(z, \bar{z}) = \frac{x_1^2 x_2^2}{\pi^2} \int \frac{d^4 x_5}{x_1^2 x_2^2 x_5^2 x_{35}^2 x_{45}^2} = \]

\[ F^{(2)}(z, \bar{z}) = \frac{x_1^2 x_2^2}{(\pi^2)^2} \int \frac{d^4 x_5 d^4 x_6}{x_1^2 x_2^2 x_5^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} = \]

& Color factors: \( C_{k,m}^i \in \{ \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \} \)

\( \mathbf{1} = \text{Tr}(T^{a_1} \ldots T^{a_k}) \), \( \mathbf{1} = f_{ab}^c \)
The Data: Color Factors

To obtain non-planar corrections: Need to expand color factors.

\[ C_{k,m}^i = N_c^{2k} k^4 \left( \cdot C_{k,m}^i + \circ C_{k,m}^i N_c^{-2} + \mathcal{O}(N_c^{-4}) \right) , \quad i \in \{a, b, c, d\} , \]

Compute by brute force:

\[
\begin{array}{cccccccccccc}
 k & m & \frac{1}{2} \circ C_{k,m}^{1,U} & \frac{1}{2} \circ C_{k,m}^{1,SU} & \circ C_{k,m}^{a,U} & \circ C_{k,m}^{b,U} & 2 \circ C_{k,m}^{c,U} & \circ C_{k,m}^{d,U} & \circ C_{k,m}^{a,SU} & \circ C_{k,m}^{b,SU} & 2 \circ C_{k,m}^{c,SU} & \circ C_{k,m}^{d,SU} \\
 2 & 0 & 1 & 1 & 0 & -2 & -1 & -1 & 0 & -2 & -1 & -1 \\
 3 & 0 & 1 & 9 & -5 & -2 & -1 & -1 & -9 & -18 & -9 & -9 \\
 3 & 1 & 1 & 9 & 0 & 3 & -1 & -1 & 0 & -5 & -9 & -9 \\
 4 & 0 & -5 & 13 & -7 & 10 & 5 & 5 & -25 & -26 & -13 & -13 \\
 4 & 1 & -12 & 24 & 4 & 15 & 13 & 14 & -23 & -21 & -23 & -22 \\
 4 & 2 & -5 & 13 & 0 & 21 & 5 & 5 & 0 & 3 & -13 & -13 \\
 5 & 0 & -23 & 9 & -1 & 46 & 23 & 23 & -33 & -18 & -9 & -9 \\
 5 & 1 & -51 & 13 & 31 & 47 & 55 & 59 & -33 & -17 & -9 & -5 \\
 5 & 2 & -51 & 13 & 39 & 76 & 55 & 59 & -9 & 12 & -9 & -5 \\
 5 & 3 & -23 & 9 & 0 & 63 & 23 & 23 & 0 & 31 & -9 & -9 \\
 6 & 0 & -61 & -11 & 20 & 122 & 61 & 61 & -30 & 22 & 11 & 11 \\
 6 & 1 & -126 & -26 & 92 & 107 & 135 & 144 & -8 & 7 & 35 & 44 \\
 6 & 2 & -159 & -59 & 139 & 187 & 175 & 191 & 39 & 87 & 75 & 91 \\
 6 & 3 & -126 & -26 & 110 & 201 & 135 & 144 & 35 & 101 & 35 & 44 \\
 6 & 4 & -61 & -11 & 0 & 139 & 61 & 61 & 0 & 89 & 11 & 11 \\
\end{array}
\]

also: \( k = 7, 8, 9 \). All color factors are quartic polynomials in \( m \) and \( k \).
\[ \mathcal{F}_{k,m}^{(1)}(z, \bar{z}) = 2k^2 \left\{ 1 + \frac{1}{N_c^2} \left[ \left( \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right) k^4 + \left( \frac{9}{2} r^2 - \frac{13}{8} \right) k^3 + \left( \frac{1}{6} r^2 + \frac{15}{8} \right) k^2 - \frac{1}{2} k \right] \right\} F^{(1)}, \]

\[ \mathcal{F}_{k,m}^{(2)}(z, \bar{z}) = \]

\[ \frac{4k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left( \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right) k^4 + \left( \frac{9}{2} r^2 - \frac{13}{8} \right) k^3 + \left( \frac{1}{6} r^2 + \frac{15}{8} \right) k^2 - \frac{1}{2} k \right] \right\} F^{(2)} \]

\[ + \left\{ \frac{t}{4} + \frac{1}{N_c^2} \left[ \left( \frac{7}{4} r^2 - \frac{1}{8} \right) k^2 + \frac{5}{8} k - \frac{1}{4} \right] s_+ - r \left( \frac{17}{6} r^2 - \frac{7}{8} \right) k^3 + 3k^2 - \frac{13}{12} k \right\} s_- \]

\[ + \left( \left[ \frac{29}{24} r^4 - \frac{11}{16} r^2 + \frac{15}{128} \right] k^4 + \left[ \frac{17}{8} r^2 - \frac{21}{32} \right] k^3 - \left[ \frac{23}{24} r^2 - \frac{39}{32} \right] k^2 - \frac{9}{8} k + \frac{1}{2} \right) t \right\} \left( F^{(1)} \right)^2 \]

\[ - \frac{1}{N_c^2} \left[ r \left\{ \left( \frac{7}{6} r^2 - \frac{1}{8} \right) k^3 + \frac{3}{2} k^2 + \frac{10}{3} k \right\} F_{C,-}^{(2)} \right. \]

\[ + \left\{ \left[ \frac{5}{4} r^2 - \frac{19}{48} \right] k^3 + \left[ \frac{3}{2} r^2 + \frac{7}{8} \right] k^2 + \frac{1}{3} k \right\} F_{C,+}^{(2)} \right) \]

\[ + \frac{1}{4} \left\{ 1 + \frac{(k-1)(k^3 + 3k^2 - 46k + 36)}{12N_c^2} \right\} \left( s_{m,0} + \delta_{m,k-2} \right) \left( F^{(1)} \right)^2 \]

\[ + \left\{ 1 + \frac{(k-2)4}{12N_c^2} \right\} \left( \delta_{m,0} F_{z-1}^{(2)} + \delta_{m,k-2} F_{1-z}^{(2)} \right), \]

where \( r = (m + 1)/k - 1/2. \)

\[ \mathcal{F}_{k,m}: \text{Coefficient of } X^m Y^{k-2-m}. \]
Focus on leading order in large $k \to$ several simplifications:

Data:
\[
\mathcal{F}_{k,m}^{(1),U}(z,\bar{z}) = -\frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(1)},
\]
\[
\mathcal{F}_{k,m}^{(2),U}(z,\bar{z}) = \frac{4k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(2)}
\]
\[
+ \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} \frac{t}{4} \left( F^{(1)} \right)^2.
\]

Combinatorics of distributing propagators on bridges:
Sum over distributions of $m$ propagators on $j + 1$ bridges $\to m^j/j!$

- $\Rightarrow$ Only graphs with maximum bridge number contribute.
- $\Rightarrow$ All bridges carry a large number of propagators.
## First Test: Large $k$: Graphs and Labelings

### Graphs:

<table>
<thead>
<tr>
<th>Graph</th>
<th>Labelings</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)</td>
</tr>
<tr>
<td>B</td>
<td>(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)</td>
</tr>
<tr>
<td>G</td>
<td>(1, 2, 4, 3), (3, 4, 2, 1)</td>
</tr>
<tr>
<td>G</td>
<td>(1, 3, 4, 2), (2, 4, 3, 1)</td>
</tr>
<tr>
<td>L</td>
<td>(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)</td>
</tr>
<tr>
<td>M</td>
<td>(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)</td>
</tr>
<tr>
<td>P</td>
<td>(1, 2, 4, 3)</td>
</tr>
<tr>
<td>Q</td>
<td>(1, 2, 4, 3)</td>
</tr>
</tbody>
</table>

### Sum over labelings:

<table>
<thead>
<tr>
<th>Case</th>
<th>Inequivalent Labelings (clockwise)</th>
<th>Combinatorial Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)$</td>
<td>$m^3(k - m)/6$</td>
</tr>
<tr>
<td>B</td>
<td>$(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)$</td>
<td>$m(k - m)^3/6$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1)$</td>
<td>$m^4/24$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 3, 4, 2), (2, 4, 3, 1)$</td>
<td>$(k - m)^4/24$</td>
</tr>
<tr>
<td>L</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)$</td>
<td>$m^2/2 \cdot (k - m)^2/2$</td>
</tr>
<tr>
<td>M</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>P</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>Q</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2$</td>
</tr>
</tbody>
</table>
All graphs consist of only octagons!
Split each octagon into two hexagons with a zero-length bridge.

Example:
Loop Counting:
Expand mirror measure $\mu(u) \sim e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling $g$ → $n$ particles on bridge of size $\ell$: $\mathcal{O}(g^2(n\ell + n^2))$
All graphs consist of octagons framed by parametrically large bridges → Only excitations on zero-length bridges inside octagons survive

Excited Octagons:
$n$ particles on a zero-length bridge → $\mathcal{O}(g^{2n^2})$ → Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops

Octagon 1–2–4–3 with 1 particle:
\[
\mathcal{M}(z, \alpha) = \left[ z + \bar{z} - (\alpha + \bar{\alpha}) \frac{\alpha \bar{\alpha} + z \bar{z}}{2\alpha \bar{\alpha}} \right] \\
\cdot \left( g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \ldots \right)
\]
For $Z = 0$: R-charge cross ratios
\[
\alpha = z\bar{z} \frac{X}{Y} \text{ and } \bar{\alpha} = 1.
\]
First Test: Large $k$: Match and Prediction

We are Done:
Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
⇒ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

$$\mathcal{F}_{k,m}^U(z, \bar{z})\bigg|_{\text{torus}} = -\frac{2k^6}{N_c^4} \left\{ \right.$$  
\begin{align*}
g^2 & \left[ \frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] F^{(1)} \checkmark \text{ match} \\
- 2g^4 & \left[ \frac{17}{6}r^4 - \frac{7}{4}r^2 + \frac{11}{32} \right] F^{(2)} + \left[ \frac{29}{6}r^4 - \frac{11}{4}r^2 + \frac{15}{32} \right] \frac{t}{4} (F^{(1)})^2 \checkmark \text{ match} \\
+ g^6 & \left[ \ldots \right] F^{(3)} + \left[ \ldots \right] (F^{(2)})(F^{(1)}) + \left[ \ldots \right] (F^{(1)})^3 \right. \text{ prediction!} \\
+ \mathcal{O}(g^8) + \mathcal{O}(1/k) \left. \right\} .
\end{align*}
More Tests: \( k = 2, 3, 4, 5, \ldots \)

**Small and finite \( k \):**
Few propagators \( \rightarrow \) Fewer bridges \( \rightarrow \) Graphs with fewer edges
\( \Rightarrow \) Graphs composed of not only octagons, but bigger polygons

**Example:** Graphs for \( k = 3 \):

![Graphs For k = 3](image)

**Hexagonalization:**
Each \( 2n \)-gon: Split into \( n - 2 \) hexagons by \( n - 3 \) zero-length bridges.

**Loop Expansion:** Much more complicated!
All kinds of excitation patterns already at low loop orders

- Single particles on several adjacent zero-length (or \( \ell = 1 \)) bridges
- Strings of excitations wrapping around operators
Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

⇒ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:

One-strings: understood ✓

Longer strings: need to compute!
The Two-String Excitation

Has been computed for the planar five-point function.

**Very non-trivial computation:**
- 3 hexagons $\rightarrow$ 2 weight factors
- Two integrations over rapidities $u_1, u_2$
- Two infinite sums over bound states $a_1, a_2$
- A complicated matrix part $M_{a_1 a_2}$

\[
M^{(2)} = \int \frac{d u_1}{2 \pi} \frac{d u_2}{2 \pi} \sum_{a_1 = 1}^\infty \sum_{a_2 = 1}^\infty \left[ \prod_{j=1,2} \tilde{\mu}_{a_j}(u_j) e^{-i \tilde{p}_{a_j} \log |z_j|} \right] \frac{M_{a_1 a_2}}{h_{a_2 a_1}(u_2^\gamma, u_1^\gamma)}
\]
Two-String Excitation: Matrix Part

Figure 6 from Fleury/Komatsu

\[ \sum \]

\[ \equiv \mathcal{F}_{ab} \]
Two-String: Result

One-String: Can be written as

\[ \mathcal{M}^{(1)}(z, \alpha) = m(z) + m(z^{-1}), \]

with building block

\[ m(z) = m(z, \alpha) = g^2 \frac{(z + \bar{z}) - (\alpha + \bar{\alpha})}{2} F^{(1)}(z, \bar{z}) \]

Two-string: Despite complicated computation, simplifies to

\[ \mathcal{M}^{(2)}(z_1, z_2, \alpha_1, \alpha_2) = m \left( \frac{z_1 - 1}{z_1 z_2} \right) + m \left( \frac{1 - z_1 + z_1 z_2}{z_2} \right) \]
\[ + m(z_1(1 - z_2)) - m(z_1) - m(z_2^{-1}), \]

with the same building block \( m(z) \)!
**Finite $k$: Larger Strings**

**Larger strings:** Computation will be even more complicated!  
**But:** Can in fact bootstrap all of them by using **flip invariance**!

Apply recursively:
- 3-string ≃ 1-strings & 2-strings
- ...iterate ...
- $n$-string ≃ 1-strings & 2-strings

⇒ Can write all polygons in terms of only 1-strings & 2-strings.

⇒ All $n$-strings can be written as linear combinations of one-string building blocks $m(z)$. 
**Finite $k$: General Polygons at One Loop**

Polygon with $2n$ edges:
Sum over all strings inside the polygon greatly simplifies to:

$$\mathcal{P}_{2n}^{(1)} = \sum_{\{j,k\} \text{ non-consecutive}} m \left( z_{jk} \equiv \frac{x_{j,k}^2 x_{j+1,k}^2}{x_{jk}^2 x_{j+1,k+1}^2} \right)$$

→ Sum over $m(z)$ evaluated in each subsquare:

Recall the one-loop building block:

$$m(z) = g^2 \frac{(z + \bar{z}) - (\alpha + \bar{\alpha})}{2} F^{(1)}(z, \bar{z})$$
Done! Sum over all graphs, expand all polygons to their one-loop values

<table>
<thead>
<tr>
<th>k</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g = 0:</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>24</td>
</tr>
<tr>
<td>g = 1:</td>
<td>0</td>
<td>32</td>
<td>441</td>
<td>2760</td>
</tr>
</tbody>
</table>

**Result:** For $k = 2, 3, 4, 5, \ldots$:
Matches the $U(N_c)$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m}: \quad \text{Result} = (\text{torus data}) - \frac{1}{N_c^2} (\text{planar data})$$

What does this mean?? ⇒ Puzzle.

Difference between $U(N_c)$ and $SU(N_c)$? → No
Operator normalizations? → No
Need to include planar graphs on the torus? If yes, how?
Finite \( k \): Stratification

We are computing a worldsheet process. The string amplitude involves integration over moduli space \( \mathcal{M}_{g,n} \).

**Sum over graphs:** Reminiscent of moduli space integration. This can be made more precise:

Moduli space ⇔ space of *metric ribbon graphs* \( \text{RGB}^\text{met}_{g,n} \).

**Metric Ribbon Graphs with labeled Boundary:**

Regular graphs, but edges at each vertex have definite ordering. Double-line notation defines \( n \) oriented boundary components (faces). Faces define compact oriented surface of definite genus \( g \).

Assign length \( \ell_j \in \mathbb{R}_+ \) to each edge.

**Bijection:** Via Strebel theory:

\[
\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \longleftrightarrow \text{RGB}^\text{met}_{g,n} = \bigsqcup_{\Gamma \in \text{RG}_{g,n}} \frac{\mathbb{R}^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)}
\]
Finite $k$: Stratification

The graphs we sum over are metric ribbon graphs. The graphs in the bijection are the duals to our graphs. **Dual graphs**: Swap faces and vertices, genus is preserved.

**Translation:**
Labeled boundary components $\leftrightarrow$ Labeled operators
Edge lengths $\ell_j$ $\leftrightarrow$ bridge sizes
In our case, the bridge sizes are integer (numbers of propagators)

Via the bijection, our sum over graphs amounts to a **discretization** of the integration over the moduli space.

The bijection defines a **cell decomposition** of the moduli space. Highest-dimensional cells: Graphs with maximal number of bridges. Cell boundaries: Some bridge size $\ell_j \to 0$. 
**Finite \( k \): Stratification**

**Discretization:** Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size \( \rightarrow \) handle gets pinched.

This problem has been considered before in the context of matrix models.

**Resolution:** In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification.*

\[
\begin{align*}
\Rightarrow & \quad + \quad \begin{array}{c}
\times \\
\times \\
\times \\
\times \\
\times \\
\times \\
\end{array} \\
\end{align*}
\]

Including these contributions indeed accounts for the \((\text{planar})/N_c^2\) term!

\[
\Rightarrow \text{Now have a complete match for } k = 2, 3, 4, 5.
\]
Summary: Method to compute higher-genus terms in $1/N_c$ expansion.

- Sum over free graphs, decompose into planar hexagons, integrate over mirror states.
- Large $k$: Only octagons, match at two loops, three-loop prediction
- Match for various finite $k \rightarrow$ stratification

Outlook: There are many things to do that we currently explore:

- Study more examples: Higher loops/genus, more general operators
- Extract interesting data: Non-planar cups anomalous dimension?
- Understand details/implications of stratification beyond one loop
- Connect to recent supergravity loop computations at strong coupling?
- Promising: Large $k$ at higher genus: Only octagons. Resum $1/N_c$?

[Aharony,Alday ’16] [Alday,Bissi ’17] [Alday ’17] [Aprile,Drummond,Heslop Paul ’17, ’17, ’17, ’18]
Thank You for listening!

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