Handling Handles: Non-Planar AdS/CFT Integrability

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Workshop on Holography, Gauge Theories, and Black Holes
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In AdS$_5$, string amplitudes can be cut into basic patches (rectangles, pentagons, or hexagons), which can be \textit{bootstrapped} using \textit{integrability} at \textit{any value of the ’t Hooft coupling}.

- Amplitudes are given as infinite sums and integrals over intermediate states from \textit{gluing together} these integrable patches.
- Sometimes, these sums and integrals can be re-summed, giving hints of \textit{yet-to-be uncovered structures}.
- This holds at the planar level as well as for non-planar processes suppressed by $1/N_c$. 
\[ \mathcal{N} = 4 \text{ SYM & The Planar Limit} \]

\[ \mathcal{N} = 4 \text{ super Yang–Mills:} \text{ Gauge field } A_\mu, \text{ scalars } \Phi_I, \text{ fermions } \psi_{\alpha A}. \]

Gauge group: \( U(N_c) / SU(N_c) \).

Adjoint representation: All fields are \( N_c \times N_c \) matrices.

**Double-line notation:**

Propagators:
\[
\langle \Phi^i_I \Phi^k_J \rangle \sim g_{YM}^2 \delta^i_i \delta^k_k = \left(\begin{array}{c} i \\ j \end{array}\right) \left(\begin{array}{c} l \\ k \end{array}\right)
\]

Vertices:
\[
\text{Tr}(\Phi \Phi \Phi \Phi) \sim \frac{1}{g_{YM}^2}
\]

- Diagrams consist of color index loops \( \sim \) oriented disks \( \sim \delta^i_i = N_c \)
- Disks are glued along propagators \( \rightarrow \) oriented compact surfaces

**Local operators:**

\[
\mathcal{O}_i = \text{Tr}(\Phi \ldots) \sim \mathcal{O}_i
\]

- One fewer color loop \( \rightarrow \) factor \( 1/N_c \)
- Surface: Hole \( \sim \) boundary component
Every diagram is associated to an oriented compact surface.

**Genus Expansion:**
Absorb one factor of $N_c$ in the 't Hooft coupling $\lambda = g_{YM}^2 N_c$
Use Euler formula $V - E + F = 2 - 2g$

$\Rightarrow$ **Correlators** of single trace operators $\mathcal{O}_i = \text{Tr}(\Phi_1 \Phi_2 \ldots)$:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{1}{N_c^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_c^{2g}} G_g(\lambda)$$

$$\sim \frac{1}{N_c^2} + \frac{1}{N_c^4} + \frac{1}{N_c^6} + \ldots$$
**Spectrum: Planar Limit**

**Goal:** Correlation functions in $\mathcal{N} = 4$ SYM

**Step 1:** Planar spectrum of single-trace local operators $\text{Tr}(\Phi \ldots)$
- Spectrum of (anomalous) scaling dimensions $\Delta$
- Scale transformations represented by dilatation operator $\Gamma$
- $\Gamma$ mixes single-trace (& multi-trace) operators
- Resolve mixing $\rightarrow$ Eigenstates & eigenvalues (dimensions)

**Planar limit:**
- Multi-trace operators suppressed by $1/N_c$
- Dilatation operator acts *locally* in color space (neighboring fields)

Organize space of single-trace operators around protected states

$$\text{Tr} \ Z^L, \quad Z = \alpha^I \Phi_I, \quad \alpha^I \alpha_I = 0 \quad \text{(half-BPS, “vacuum”).}$$

Other single-trace operators: Insert *impurities* $\{\Phi_I, \psi_{\alpha A}, D_\mu\}$ into $\text{Tr} \ Z^L$. 
Initial observation: One-loop dilatation operator for scalar single-trace operators is integrable. Diagonalization by Bethe Ansatz.

- Impurities are magnons in color space, characterized by rapidity (momentum) \( u \) and \( su(2|2)^2 \) flavor index.
  
  \[ su(2|2)^2 \subset psu(2,2|4) \] preserves the vacuum \( \text{Tr} \ Z^L \)

- Dynamics of magnons: integrability:
  - No particle production
  - Individual momenta preserved
  - Factorized scattering

- Two-body (\( \rightarrow n \)-body) S-matrix completely fixed to all loops

\[ \Rightarrow \text{Asymptotic spectrum (for } L \rightarrow \infty \text{) solved to all loops / exactly.} \]
Finite-Size Effects

**Asymptotic spectrum** solved by Bethe Ansatz.
Resums $\infty$ Feynman diagrams that govern dynamics of $\infty$ strip:

Re-compactify: Finite-size effects.
Leading effect: Momentum quantization constraint $\equiv$ Bethe equations

\[
1 = e^{i p_j L} \prod_{j \neq k} S(p_k, p_j)
\]

Moreover: Wrapping interactions.
- No notion of locality for dilatation operator
- Previous techniques (Bethe ansatz) no longer apply
Mirror Theory

Key to all-loop finite-size spectrum: **Mirror map**

Double Wick rotation: \((\sigma, \tau) \rightarrow (i\tilde{\tau}, i\tilde{\sigma})\) — exchanges space and time

Magnon states: Energy and momentum interchange: \(\tilde{E} = ip, \tilde{p} = iE\)

Finite size \(L\) becomes finite, periodic (discrete) time.  
Energy \(\sim\) Partition function at finite temperature \(1/L\), with \(R \rightarrow \infty\).  
\(\rightarrow\) **Thermodynamic Bethe ansatz**.

Simplifications and refinements:

- Y-system (T-system, Q-system)
- Quantum Spectral Curve

\(\Rightarrow\) Scaling dimensions computable at finite coupling.
Three-Point Functions: Hexagons

Differences: Topology: Pair of pants instead of cylinder
Non-vanishing for three generic operators (two-point: diagonal)
⇒ Previous techniques not directly applicable

Observation:

The green parts are similar to two-point functions:
Two segments of physical operators joined by parallel propagators ("bridges", $\ell_{ij} = (L_i + L_j - L_k)/2$).

The red part is new: "Worldsheet splitting", "three-point vertex" (open strings)

Take this serious $\rightarrow$ cut worldsheet along "bridges":

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[Basso,Komatsu Vieira '15]
**Hexagons & Gluing**

- **Glue** hexagons along three *mirror channels*:  
  - Sum over complete state basis (magnons) in the mirror theory  
  - Mirror magnons: Boltzmann weight \( \exp(-\tilde{E}_{ij}\ell_{ij}) \), \( \tilde{E}_{ij} = \mathcal{O}(g^2) \)  
  \( \rightarrow \) mirror excitations are strongly suppressed.

**Hexagonal worldsheet patches (form factors):**

- Function of rapidities \( u \) and \( \mathfrak{su}(2|2)^2 \) labels \((A, \dot{A})\) of all magnons.
- Conjectured exact expression, based on diagonal \( \mathfrak{su}(2|2) \) symmetry as well as form factor axioms.

Finite-coupling hexagon proposal: Supported by very non-trivial matches.
Planar Four-Point Functions: Hexagonalization

Move on to planar four-point functions:
One way to cut (now that three-point is understood): **OPE cut**

![Diagram of OPE cut](image)

**Problem:** Sum over physical states!
- No loop suppression, all states contrib.
- Double-trace operators.

**Instead:** Cut along propagator bridges

![Diagram of propagator cut](image)

**Benefits:**
- Mirror states highly suppressed in $g$.
- No double traces.

**References:**
- Fleury '16
- Eden '16
- Komatsu
- Sfondrini
Hexagonalization: Formula

\[ \langle O_1 O_2 O_3 \rangle = \prod_{c \in \{1,2,3\}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \mathcal{H}_1(\psi_1, \psi_2, \psi_3) \mathcal{H}_2(\psi_1, \psi_2, \psi_3) \]

\[ \langle O_1 O_2 O_3 O_4 \rangle = \sum_{\text{planar prop. graphs}} \prod_{c \in \{1,\ldots,6\}} d_c^{\ell_c} \sum_{\psi_c} \mu(\psi_c) \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \mathcal{H}_4 \]

**New Features:**
- Bridge lengths vary, may go to zero \(\Rightarrow\) Mirror corrections at one loop
- Hexagons are in different “frames” \(\Rightarrow\) Weight factors

[Komatsu] [Fleury '16]
Hexagonalization: Frames

Hexagon depends on positions $x_i$ and polarizations $\alpha_i$ of the three half-BPS “vacuum” operators $\mathcal{O}_i = \text{Tr}[(\alpha_i \cdot \Phi(x_i))^L]$.

Any three $x_i$ and $\alpha_i$ preserve a diagonal $\mathfrak{su}(2|2)$ that defines the state basis and S-matrix of excitations on the hexagon.

**Three-point function:** Both hexagons connect to the same three operators, so their frames ($\mathfrak{su}(2|2)$ and state basis) are identical.

**Higher-point function:** Two neighboring hexagons always share two operators, but the third/fourth operator may not be identical. $
\Rightarrow$ The two hexagon frames are misaligned.

In order to consistently sum over mirror states, need to align the two frames by a $\text{PSU}(2,2|4)$ transformation that maps $\mathcal{O}_3$ onto $\mathcal{O}_2$. 

\[
\begin{array}{cccc}
1 & 2 & H_2 \\
\uparrow & \uparrow & \downarrow \\
3 & 4 & H_1 \\
\end{array}
\]
Hexagonalization: Weight Factors

By conformal and R-symmetry transformation, bring $O_1$, $O_2$, and $O_4$ to canonical configuration:

Transformation that maps $O_3$ to $O_2$: $g = e^{-D \log |z|} e^{i \phi L} e^{J \log |\alpha|} e^{i \theta R}$, where $e^{2i \phi} = z/\bar{z}$, $e^{2i \theta} = \alpha/\bar{\alpha}$, and $(\alpha, \bar{\alpha})$ is the R-coordinate of $O_3$.

Hexagon $H_1 = \hat{H}$ is canonical, and $H_2 = g^{-1} \hat{H} g$.

Sum over states in mirror channel:

$$\sum_{\psi} \mu(\psi) \langle H_2 | \psi \rangle \langle \psi | H_1 \rangle = \sum_{\psi} \mu(\psi) \langle g^{-1} \hat{H} | \psi \rangle \langle \psi | g | \psi \rangle \langle \psi | \hat{H} \rangle$$

Weight factor: $\langle \psi | g | \psi \rangle = e^{-2i \tilde{p} \psi \log |z|} e^{J \psi \varphi} e^{i \phi L \psi} e^{i \theta R \psi}$, $i \tilde{p} = (D - J)/2$.

→ Contains all non-trivial dependence on cross ratios $z$, $\bar{z}$ and $\alpha$, $\bar{\alpha}$. 
Non-Planar Processes: Idea

Hexagonalization: Works for planar (4,5)-point functions

Extend to non-planar processes?
- Fix worldsheet topology
- Dissect into planar hexagons
- Glue hexagons (mirror states)

Simple Proposal:

\[ \langle O_1 \ldots O_n \rangle_{\text{full}} = \frac{1}{N_c^{n-2}} \sum_g \frac{1}{N_c^{2g}} \sum_{\text{graphs}} \prod_c d_c^\ell \sum_{\text{mirror states}} H_1 H_2 H_3 \ldots H_F \]
The Observable

Object of Study:
Four half-BPS operators on the torus
- No physical magnons
- Non-trivial spacetime dependence
- Non-trivial coupling dependence
- Probes a lot of CFT data
- Non-planar data available!

To Do:
- Understand sum over propagator graphs
- Understand all mirror contributions

See also: Non-planar two-point analysis of Eden, Jiang ’17, le Plat, Sfondrini
**Sum over Graphs: Cutting the Torus**

**Sum over propagator graphs:** Split into

- Sum over graphs with non-parallel edges (≡ “bridges”)
- Sum over distributions of parallel propagators on bridges

**Torus with four punctures:** *How many hexagons/bridges?*

Euler: \(F + V - E = 2 - 2g\).

Our case: \(g = 1, \ V = 4, \ E = \frac{3}{2}F\)  \(\Rightarrow\) \(F = 8, \ E = 12\).

→ **Construct** all genus-one graphs with 4 punctures and **up to** 12 edges.

Propagators may populate < 12 bridges and still form a genus-one graph. Such graphs will contain **higher polygons** besides hexagons.

→ **Subdivide** into hexagons by inserting zero-length bridges (ZLBs)
Maximal Graphs

Focus on **Maximal Graphs**: Graphs with a maximal number of edges.

- Maximal graphs $\Leftrightarrow$ triangulations of the torus.

**Construction:**

- **Manually**: Add one operator at a time, in all possible ways.
- **Computer algorithm**: Start with the empty graph, add one bridge in all possible ways, iterate.

**Complete list of maximal graphs:**

![Maximal Graphs Diagram](image-url)
Submaximal graphs: Graphs with a non-maximal number of edges.

- Obtained from maximal graphs by deleting bridges.
- Number of genus-one graphs by number of bridges:

<table>
<thead>
<tr>
<th>#bridges</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>≤4</th>
</tr>
</thead>
<tbody>
<tr>
<td>#graphs</td>
<td>7</td>
<td>28</td>
<td>117</td>
<td>254</td>
<td>323</td>
<td>222</td>
<td>79</td>
<td>11</td>
<td>0</td>
</tr>
</tbody>
</table>

Hexagonalization:
Submaximal graphs contain higher polygons (octagons, decagons, ...).

- Must be subdivided into hexagons by zero-length bridges.
- Subdivision is not physical: Can pick any (flip invariance):
The Data: Kinematics

Half-BPS operators:

\[ Q^k_i \equiv \text{Tr}[(\alpha_i \cdot \Phi(x_i))^k], \quad \Phi = (\phi_1, \ldots, \phi_6), \quad \alpha^2_i = 0. \]

For equal weights \((k, k, k, k)\): Expand in \(X, Y, Z\):

\[ X \equiv \frac{\alpha_1 \cdot \alpha_2 \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} = \frac{1}{3} \frac{2}{4}, \quad Y \equiv \frac{1}{3} \frac{2}{4}, \quad Z \equiv \frac{1}{3} \frac{2}{4}. \]

Focus on \(Z = 0\) (polarizations):

\[ G_k \equiv \langle Q^k_1 Q^k_2 Q^k_3 Q^k_4 \rangle_{\text{loops}} = R \sum_{m=0}^{k-2} F_{k,m} X^m Y^{k-2-m} \]

Supersymmetry factor: \(R = z\bar{z}X^2 - (z + \bar{z})XY + Y^2\)

Main data: Coefficients \(F_{k,m} = F_{k,m}(g; z, \bar{z})\)

Cross ratios: \(z\bar{z} = s = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad (1 - z)(1 - \bar{z}) = t = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \)
The Data: Quantum Coefficients

Data Functions: Correlator coefficients:

\[ F_{k,m} = \sum_{\ell=1}^{\infty} g^{2\ell} F^{(\ell)}_{k,m}(z, \bar{z}), \quad \text{'t Hooft coupling: } g^2 = \frac{g_{YM}^2 N_c}{16\pi^2}. \]

One and two loops: Two ingredients: Box integrals

\[ F^{(1)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = , \]

\[ F^{(2)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{d^4x_5 d^4x_6}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} = , \]

& Color factors: \[ C^i_{k,m} \in \left\{ \begin{array}{c}
1 & 2 & 1 & 2 \\
3 & 4 & 3 & 4 \\
\end{array} \right\} \]

\[ 1 = \text{Tr}(T^{(a_1 \ldots T^{a_k})}), \quad f_{ab}^c = \]

Till Bargheer — Handling Handles — Southampton — 26 March 2018
The Data: Color Factors

To obtain non-planar corrections: Need to expand color factors.

\[ C_{k,m}^i = N_c^{2k} k^4 \left( \cdot C_{k,m}^i + C_{k,m}^i N_c^{-2} + O(N_c^{-4}) \right), \quad i \in \{a, b, c, d\}, \]

Compute by brute force:

<table>
<thead>
<tr>
<th>k</th>
<th>m</th>
<th>( \frac{1}{2} C_{k,m}^{1,U} )</th>
<th>( \frac{1}{2} C_{k,m}^{2,U} )</th>
<th>( \frac{1}{2} C_{k,m}^{1,SU} )</th>
<th>( \frac{1}{2} C_{k,m}^{2,SU} )</th>
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<tr>
<td>2</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
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<td>1</td>
<td>9</td>
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<td>-2</td>
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<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>9</td>
<td>0</td>
<td>3</td>
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<td>-7</td>
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</tr>
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<td>1</td>
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<td>24</td>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
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<td>-5</td>
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<td>0</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>-23</td>
<td>9</td>
<td>-1</td>
<td>46</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-51</td>
<td>13</td>
<td>31</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>-51</td>
<td>13</td>
<td>39</td>
<td>76</td>
</tr>
<tr>
<td>5</td>
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<td>-23</td>
<td>9</td>
<td>0</td>
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<tr>
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<td>0</td>
<td>-61</td>
<td>-11</td>
<td>20</td>
<td>122</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>-126</td>
<td>-26</td>
<td>92</td>
<td>107</td>
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<td>6</td>
<td>2</td>
<td>-159</td>
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<td>139</td>
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<tr>
<td>6</td>
<td>3</td>
<td>-126</td>
<td>-26</td>
<td>110</td>
<td>201</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>-61</td>
<td>-11</td>
<td>0</td>
<td>139</td>
</tr>
</tbody>
</table>

also: \( k = 7, 8, 9 \). All color factors are quartic polynomials in \( m \) and \( k \).
\[
\mathcal{F}_{k,m}^{(1)} (z, \bar{z}) = \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \left[ \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \left[ \frac{1}{6} r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2} k \right] \right\} \mathcal{F}^{(1)} + \mathcal{F}_{k,m}^{(2)} (z, \bar{z}) = \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \left[ \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \left[ \frac{1}{6} r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2} k \right] \right\} \mathcal{F}^{(2)} + \mathcal{F}_{k,m}^{(2)} (z, \bar{z}) = \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \left[ \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \left[ \frac{1}{6} r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2} k \right] \right\} \mathcal{F}^{(2)} \]
\[
- \frac{2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \left[ \frac{9}{2} r^2 - \frac{13}{8} \right] k^3 + \left[ \frac{1}{6} r^2 + \frac{15}{8} \right] k^2 - \frac{1}{2} k \right] \right\} \mathcal{F}^{(1)},
\]
where \( r = (m + 1)/k - 1/2 \).

\( \mathcal{F}_{k,m}^{(1)} \): \text{Coefficient of } X^m Y^{k-2-m} \text{.}
Focus on leading order in large $k \to$ several simplifications:

Data:

$$
\mathcal{F}_{k,m}^{(1),U}(z, \bar{z}) = \frac{-2k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(1)},$$

$$
\mathcal{F}_{k,m}^{(2),U}(z, \bar{z}) = \frac{4k^2}{N_c^2} \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} F^{(2)},$$

$$+ \left\{ 1 + \frac{1}{N_c^2} \left[ \left[ \frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right] k^4 + \mathcal{O}(k^3) \right] \right\} \frac{t}{4} \left( F^{(1)} \right)^2 .$$

Combinatorics of distributing propagators on bridges:

Sum over distributions of $m$ propagators on $j+1$ bridges $\to m^j/j!$

- $\Rightarrow$ Only graphs with maximum bridge number contribute.
- $\Rightarrow$ All bridges carry a large number of propagators.

Graphs: $Z = 0$
### First Test: Large $k$: Graphs and Labelings

#### Graphs:

- **B**
- **G**
- **L**
- **M**
- **P**
- **Q**

#### Sum over labelings:

<table>
<thead>
<tr>
<th>Case</th>
<th>Inequivalent Labelings (clockwise)</th>
<th>Combinatorial Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (3, 4, 2, 1), (4, 3, 1, 2)$</td>
<td>$m^3(k - m)/6$</td>
</tr>
<tr>
<td>B</td>
<td>$(1, 3, 4, 2), (3, 1, 2, 4), (2, 4, 3, 1), (4, 2, 1, 3)$</td>
<td>$m(k - m)^3/6$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1)$</td>
<td>$m^4/24$</td>
</tr>
<tr>
<td>G</td>
<td>$(1, 3, 4, 2), (2, 4, 3, 1)$</td>
<td>$(k - m)^4/24$</td>
</tr>
<tr>
<td>L</td>
<td>$(1, 2, 4, 3), (3, 4, 2, 1), (2, 1, 3, 4), (4, 3, 1, 2)$</td>
<td>$m^2/2 \cdot (k - m)^2/2$</td>
</tr>
<tr>
<td>M</td>
<td>$(1, 2, 4, 3), (2, 1, 3, 4), (1, 3, 4, 2), (3, 1, 2, 4)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>P</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2/2$</td>
</tr>
<tr>
<td>Q</td>
<td>$(1, 2, 4, 3)$</td>
<td>$m^2(k - m)^2$</td>
</tr>
</tbody>
</table>
First Test: Large $k$: Octagons

Graphs:

B  G  L  M  P  Q

All graphs consist of only octagons!
Split each octagon into two hexagons with a zero-length bridge.

Example:
First Test: Large $k$: Mirror Particles

Loop Counting:
Expand mirror measure $\mu(u) \sim e^{-\ell \widetilde{E}(u)}$ and hexagons $H$ in coupling $g 
\rightarrow n$ particles on bridge of size $\ell$: $\mathcal{O}(g^{2(n\ell+n^2)})$
All graphs consist of octagons framed by parametrically large bridges
$\rightarrow$ Only excitations on zero-length bridges inside octagons survive

Excited Octagons:
$n$ particles on a zero-length bridge $\rightarrow \mathcal{O}(g^{2n^2})$
$\rightarrow$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops

Octagon 1–2–4–3 with 1 particle:

$$M(z, \alpha) = \left[ z + \bar{z} - (\alpha + \bar{\alpha}) \frac{\alpha \bar{\alpha} + z \bar{z}}{2\alpha \bar{\alpha}} \right]$$

$$\cdot \left( g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \ldots \right)$$

For $Z = 0$: R-charge cross ratios
$$\alpha = z \bar{z} \frac{X}{Y} \text{ and } \bar{\alpha} = 1.$$
First Test: Large $k$: Match and Prediction

We are Done:
Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
⇒ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

\[
F_{k,m}^U(z, \bar{z})\bigg|_{\text{torus}} = -\frac{2k^6}{N_c^4} \left\{ 
\begin{align*}
g^2 \left[ \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right] F^{(1)} & \quad \checkmark \text{ match} \\
- 2g^4 \left[ \left( \frac{17}{6} r^4 - \frac{7}{4} r^2 + \frac{11}{32} \right) F^{(2)} + \left( \frac{29}{6} r^4 - \frac{11}{4} r^2 + \frac{15}{32} \right) \frac{t}{4} (F^{(1)})^2 \right] & \quad \checkmark \text{ match} \\
+ g^6 \left[ \ldots \right] F^{(3)} + \left[ \ldots \right] (F^{(2)}) (F^{(1)}) + \left[ \ldots \right] (F^{(1)})^3 \right) & \quad \text{prediction!} \\
+ \mathcal{O}(g^8) + \mathcal{O}(1/k) \right\}.
\]
More Tests: $k = 2, 3, 4, 5, \ldots$

Small and finite $k$:
Few propagators $\rightarrow$ Fewer bridges $\rightarrow$ Graphs with fewer edges
$\Rightarrow$ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:

Hexagonalization:
Each $2n$-gon: Split into $n - 2$ hexagons by $n - 3$ zero-length bridges.

Loop Expansion: Much more complicated!
All kinds of excitation patterns already at low loop orders
- Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- Strings of excitations wrapping around operators
Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

⇒ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:

One-strings: understood ✓
Longer strings: need to compute!
The Two-String Excitation

Has been computed for the planar five-point function.

**Very non-trivial computation:**
- 3 hexagons → 2 weight factors
- Two integrations over rapidities $u_1, u_2$
- Two infinite sums over bound states $a_1, a_2$
- A complicated matrix part $M_{a_1 a_2}$

$$
M^{(2)} = \int \frac{du_1}{2\pi} \frac{du_2}{2\pi} \sum_{a_1=1}^{\infty} \sum_{a_2=1}^{\infty} \left[ \prod_{j=1,2} \tilde{\mu}_{a_j}(u_j) e^{-i\tilde{p}_{a_j} \log |z_j|} \right] \frac{M_{a_1 a_2}}{h_{a_2 a_1}(u_2^\gamma, u_1^\gamma)}
$$
Two-String Excitation: Matrix Part

Figure 6 from Fleury/Komatsu

\[ \sum \quad \]

\[ = \quad \]

\[ = \quad \]

\[ \equiv \mathcal{F}_{ab} \]

Fleury’17
Komatsu
Two-String: Result

One-String: Can be written as

\[ M^{(1)}(z, \alpha) = m(z) + m(z^{-1}), \]

with building block

\[ m(z) = m(z, \alpha) = g^2 \left( \frac{z + \bar{z}}{2} \right) - \left( \alpha + \bar{\alpha} \right) F^{(1)}(z, \bar{z}) \]

Two-string: Despite complicated computation, simplifies to

\[ M^{(2)}(z_1, z_2, \alpha_1, \alpha_2) = m \left( \frac{z_1 - 1}{z_1 z_2} \right) + m \left( \frac{1 - z_1 + z_1 z_2}{z_2} \right) + m(z_1(1 - z_2)) - m(z_1) - m(z_2^{-1}), \]

with the same building block \( m(z) \)!
Finite $k$: Larger Strings

Larger strings: Computation will be even more complicated!
But: Can in fact bootstrap all of them by using flip invariance!

Apply recursively:

- 3-string $\simeq$ 1-strings & 2-strings
- ...iterate ...
- $n$-string $\simeq$ 1-strings & 2-strings

$\Rightarrow$ Can write all polygons in terms of only 1-strings & 2-strings.

$\Rightarrow$ All $n$-strings can be written as linear combinations of one-string building blocks $m(z)$. 
Polygon with 2n edges:
Sum over all strings inside the polygon greatly simplifies to:

\[ P_{2n}^{(1)} = \sum_{\{j,k\} \text{ non-consecutive}} m \left( z_{jk} \equiv \frac{x^2_{j,k+1} x^2_{j+1,k}}{x^2_{jk} x^2_{j+1,k+1}} \right) \]

→ Sum over \( m(z) \) evaluated in each subsquare:

Recall the one-loop building block:

\[ m(z) = g^2 \frac{(z + \bar{z}) - (\alpha + \bar{\alpha})}{2} F^{(1)}(z, \bar{z}) \]
Done! Sum over all graphs, expand all polygons to their one-loop values

<table>
<thead>
<tr>
<th>k:</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g = 0:</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>24</td>
</tr>
<tr>
<td>g = 1:</td>
<td>0</td>
<td>32</td>
<td>441</td>
<td>2760</td>
</tr>
</tbody>
</table>

Numbers of labeled graphs with assigned bridge sizes:

Result: For $k = 2, 3, 4, 5, \ldots$:
Matches the $U(N_c)$ data $F_{k,m}$, up to a copy of the planar term!

$$F_{k,m} : \quad \text{Result} = (\text{torus data}) - \frac{1}{N_c^2} (\text{planar data})$$

What does this mean?? ⇒ Puzzle.

Difference between $U(N_c)$ and $SU(N_c)$? → No
Operator normalizations? → No
Need to include planar graphs on the torus? If yes, how?
Finite $k$: Stratification

We are computing a worldsheet process.
The string amplitude involves integration over moduli space $\mathcal{M}_{g,n}$.

**Sum over graphs:** Reminiscent of moduli space integration.
This can be made more precise:
Moduli space $\Leftrightarrow$ space of *metric ribbon graphs* $\text{RGB}^\text{met}_{g,n}$.

**Metric Ribbon Graphs with labeled Boundary:**
Regular graphs, but edges at each vertex have definite ordering.
Double-line notation defines $n$ oriented boundary components (faces).
Faces define compact oriented surface of definite genus $g$.
Assign length $\ell_j \in \mathbb{R}_+$ to each edge.

**Bijection:** Via Strebel theory:

$$\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \leftrightarrow \text{RGB}^\text{met}_{g,n} = \bigsqcup_{\Gamma \in \text{RG}_{g,n}} \frac{\mathbb{R}^{e(\Gamma)}_+}{\text{Aut}_{\partial}(\Gamma)}$$
Finite $k$: Stratification

The graphs we sum over are metric ribbon graphs. The graphs in the bijection are the duals to our graphs. **Dual graphs:** Swap faces and vertices, genus is preserved.

**Translation:**
Labeled boundary components $\leftrightarrow$ Labeled operators
Edge lengths $\ell_j$ $\leftrightarrow$ bridge sizes
In our case, the bridge sizes are integer (numbers of propagators)

Via the bijection, our sum over graphs amounts to a **discretization** of the integration over the moduli space.

The bijection defines a **cell decomposition** of the moduli space.
Highest-dimensional cells: Graphs with maximal number of bridges.
Cell boundaries: Some bridge size $\ell_j \to 0$. 
**Finite \( k \): Stratification**

**Discretization:** Need to be careful at the boundaries of the space. Do not overcount/undercount. Boundary of torus moduli space: All bridges traversing a handle reduce to zero size \( \rightarrow \) handle gets pinched.

This problem has been considered before in the context of matrix models.

**Resolution:** In the sum over graphs, include planar graphs drawn on the torus. This leads to some overcounting. Compensate by subtracting planar graphs with two extra fictitious zero-size operators. *Stratification.*

\[
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\times \\
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- \quad \begin{array}{c}
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\times
\end{array} = \quad \begin{array}{c}
\times \\
\times \\
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\times
\end{array}
\]

Including these contributions indeed accounts for the \((\text{planar})/N_c^2\) term!

\[
\Rightarrow \quad \text{Now have a complete match for } k = 2, 3, 4, 5.
\]
Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_c$ expansion.

- **Sum** over free graphs, **decompose** into planar hexagons, **integrate** over mirror states.
- Large $k$: Only octagons, match at two loops, three-loop prediction
- Match for various finite $k$ $\rightarrow$ stratification

Outlook: There are many things to do that we currently explore:

- Study more examples: Higher loops/genus, more general operators
- Extract interesting data: Non-planar cups anomalous dimension?
- Understand details/implications of stratification beyond one loop
- Hexagons $\leftrightarrow$ String vertex?

Connect to recent supergravity loop computations at strong coupling?

- Promising: Large $k$ at higher genus: Only octagons. Resum $1/N_c$?
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