Holographic Three-Point Functions in $\mathcal{N} = 4$ super Yang–Mills Theory

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Why $\mathcal{N} = 4$ super Yang–Mills Theory?

- Simplest interacting 4d QFT
- AdS/CFT and integrability
- Can solve for and compute many observables: Spectrum, S-matrix, null polygonal Wilson loops,…
- A lot of symmetry: Superconformal $\mathfrak{psu}(2, 2|4)$, dual $\mathfrak{psu}(2, 2|4)$, Yangian $Y(\mathfrak{psu}(2, 2|4))$
- Most important properties:
  - UV finite
  - Planar limit: $N_c \to \infty$, $\lambda \sim g_{YM}^2/N_c$ fixed
  - Relation to strings on $\text{AdS}_5 \times S^5$ via AdS/CFT
Observables

• Scattering amplitudes

• Wilson loops

• Correlation functions of local gauge-invariant operators
Correlation Functions in CFT

Correlation functions of scalar conformal primary operators largely fixed by conformal symmetry:

\[
\langle \mathcal{O}_J(x_1) \mathcal{O}_K(x_2) \rangle = \frac{\delta_{JK}}{|x_{12}|^{\Delta_J + \Delta_K}}
\]

\[
\langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \mathcal{O}_K(x_3) \rangle = \frac{C_{IJK}}{|x_{12}|^{\Delta_I + \Delta_J - \Delta_K}|x_{23}|^{\Delta_J + \Delta_K - \Delta_I}|x_{31}|^{\Delta_K + \Delta_I - \Delta_J}}
\]

\[
\langle \mathcal{O}_I(x_1) \mathcal{O}_J(x_2) \mathcal{O}_K(x_3) \mathcal{O}_L(x_4) \rangle = C_{IJKL} f \left( \frac{x_{ij} x_{kl}}{x_{ik} x_{jl}}, \frac{x_{ij} x_{kl}}{x_{il} x_{jk}} \right) \prod_{i<j} \frac{1}{|x_{ij}|^{\Delta_i + \Delta_j - \Delta/3}}
\]

\[C_{IJKL}\] and \[f\] follow from \(\{\Delta_I\}\), \(\{C_{IJK}\}\) and the OPE.

**Complete information** encoded in \(\{\Delta_J\}\) and \(\{C_{IJK}\}\).
Two-Point Functions ↔ Scaling Dimensions

\( \{\Delta_I\} \) in the planar limit from AdS/CFT and integrability

- **Weak coupling, \( \lambda \ll 1 \):**
  - \( \Delta_I \): Eigenvalues of dilatation generator
  - Spin-chain picture, exploit symmetries
  - Asymptotic all-loop Bethe equations

- **Strong coupling, \( \lambda \gg 1 \):**
  - \( \Delta_I \): Energies of physical strings
  - Worldsheet scattering of excitations
  - Asymptotic Bethe equations

- **Exact, any \( \lambda \):**
  - Wrapping corrections
  - Thermodynamic Bethe ansatz
  - \( Y \)-system equations

**Upshot:** No need to compute two-point functions directly!
Three-Point Functions ↔ Structure Constants

Goal (dream): $C_{IJK}$ from integrability

But: No structure as for two-point functions known

→ For the moment, need to compute correlation functions directly.

Weak coupling:

Straightforward Feynman diagrams

More recently: Algebraic Bethe ansatz approach, label operators by Bethe roots, organize/simplify combinatorics
Correlation Functions from Holography

Essence of AdS/CFT:

\[ Z_{\text{string}}[\Phi|_{\partial \text{AdS}} = \Phi_0] = Z_{\text{CFT}}[\Phi_0] \]

More concretely,

\[ \langle e^{\Phi \cdot \mathcal{O}} \rangle_{\text{CFT}} = \langle e^{\Phi \cdot V} \rangle_{\text{string worldsheet}} \]

\[ \Phi \cdot \mathcal{O}_{\text{CFT}} = \int d^4x \Phi(x)\mathcal{O}(x), \quad \Phi \cdot V_{\text{string}} = \int d^4x \Phi(x)V(x). \]

Correlation Function:

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int \mathcal{D}X V_1 V_2 V_3 e^{-S_{\text{string}}[X]} \]

Dictionary \( \mathcal{O}_{\text{CFT}} \leftrightarrow V_{\text{string}} \) not clear!
Supergravity Approximation

Simplest approximation: $\sqrt{\lambda} \to \infty$ ($\alpha' \to 0$) and $m^2 \sim \Delta$ small

Massless string modes $\leftrightarrow$ chiral primary operators

Correlation functions protected by supersymmetry

Poincaré coordinates: $ds^2 = \frac{du^2 + dx^2}{u^2}$

Boundary field $\phi_0(x)$

Bulk sugra field $\phi(u, x) = \int d^4y K(u, x; y)\phi_0(y)$

Bulk-to-boundary propagator $K(u, x; y) = \left(\frac{u}{u^2 + (x - y)^2}\right)^\Delta$

AdS/CFT: $Z_{\text{CFT}}[\phi_0] = Z_{\text{string}}[\phi] = \langle e^{-S_{\text{sugra}}[\phi]} \rangle$
Supergravity Approximation

Poincaré coordinates: 
\[ ds^2 = \frac{du^2 + dx^2}{u^2} \]

Boundary field \( \phi_0(x) \)

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AdS/CFT: \( Z_{\text{CFT}}[\phi_0] = Z_{\text{string}}[\phi] = \langle e^{-S_{\text{sugra}}}[\phi] \rangle \)

CFT: Chiral primary \( S_I = C_I^{jk_1...j_k} \text{Tr}(\phi_{j_1} \cdots \phi_{j_k}) \)
Dual sugra field contains \( h_{\alpha\beta}, a_{\alpha\beta\gamma\delta} \)

For all \( \lambda \):

\[ \langle S_I(x_1)S_J(x_2)S_K(x_3) \rangle = \frac{1}{N_c} \frac{\sqrt{\Delta_1\Delta_2\Delta_3}}{|x_{12}|^{\Delta_I+\Delta_J-\Delta_K} |x_{23}|^{\Delta_J+\Delta_K-\Delta_I} |x_{31}|^{\Delta_K+\Delta_I-\Delta_J}} \langle C_IC_JC_K \rangle \]
Semiclassical Strings

Large Noether charges: $\Delta, J, S \sim \sqrt{\lambda} \rightarrow \infty$

Classical strings good approximation, quantum fluctuations small
Access to massive string modes $\leftrightarrow$ unprotected operators
Successfully employed in spectral problem

- Classical non-linear sigma model is integrable
- Scattering of quantum fluctuations on world-sheet
- Description in terms of integrable spin-chain

Examples of semiclassical string states:

- BMN: Point-like string orbiting $S^5 \leftrightarrow \text{Tr} \ Z^J$
- GKP: Folded string spinning in $\text{AdS}_5 \leftrightarrow \text{Tr} \ Z \mathcal{D}^S Z$
- Circular spinning string: Spinning in $\text{AdS}$ and on sphere

[References]

Bena, Polchinski, Roiban, hep-th/0305116
Berenstein, Maldacena, Nastase hep-th/0202021
Gubser, Klebanov, Polyakov, hep-th/0204051
Frolov, Tseytlin, hep-th/0304255
For $\sqrt{\lambda} \to \infty$, path integral

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int \mathcal{D}X V_1 V_2 V_3 e^{-\sqrt{\lambda} \int d^2 z \mathcal{L}_{\text{string}}[X]}$$

dominated by saddle point $\to$ classical solution

For large charges $\Delta, J, S \sim \sqrt{\lambda}$, vertex operators of same order as $e^{-S}$

General structure: $V \sim e^{i \text{Charge} \cdot \text{Coordinate}} (\text{Polynomial})$

Flat-space example: $V \sim e^{i k_\mu x^\mu}$ sources tachyon modes

AdS: Poincaré $u, x$ Embedding $Y$ Global $t, \phi, \psi, \ldots$

$$ds^2 = \frac{du^2 + dx^2}{u^2} \quad \quad ds^2 = \sum_k \pm dY_k^2$$
Translating $V \sim e^{i \Delta t}$ to Poincaré/embedding coordinates, it becomes

$$V(z; x_k) \sim \left( \frac{u(z)}{u(z)^2 + (x(z) - x_k)^2} \right)^{\Delta} \sim (Y_{+,k})^{-\Delta}$$

Exactly the sugra bulk-to-boundary propagator!

Varying the path integral

$$\langle O_1 O_2 O_3 \rangle = \int D\mathbf{X} V_1(z_1; x_1) V_2(z_2; x_2) V_3(z_3; x_3) e^{-\sqrt{\lambda} \int d^2z \mathcal{L}_{\text{string}}} ,$$

for $\Delta_k \sim \sqrt{\lambda}$, the vertex operators $V_k$ induce source terms in the classical string equations of motion:

$$\text{EOM}_{\text{string}} \rightarrow \text{EOM}_{\text{string}} + \sum_{k=1}^{3} \frac{\Delta_k}{\sqrt{\lambda}} \delta^2(z - z_k) F_k(z)$$
Translating \( V \sim e^{i\Delta t} \) to Poincaré/embedding coordinates, it becomes

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\]

\[
\text{EOM}_{\text{string}} \rightarrow \text{EOM}_{\text{string}} + \sum_{k=1}^{3} \frac{\Delta_k}{\sqrt{\lambda}} \delta^2(z - z_k) F_k(z)
\]

At the AdS boundary: \( (Y_{+,k})^{-1} \xrightarrow{u \to 0} \delta^4(x - x_k) \)

Ensures that classical solution ends on points \( x_k \) on \( \partial\text{AdS} \)
**Vertex Operators: Examples**

General form in semiclassical limit ($U$ polynomial):

\[ V = (Y_+)^{-\Delta} U(X, \partial X, \ldots), \quad X = (Y, X) \]

No complete dictionary for vertex operators

Constraints:

- $V$ has to be **marginal perturbation** of string sigma model
- $V$ has to reproduce right charges

Some examples:

- BMN (point-like string): $V_J = (Y_+)^{-\Delta}(X_1 + iX_2)^J$
- GKP (folded spinning string): $V_S = (Y_+)^{-\Delta}(\partial Y \bar{\partial} Y)^{S/2}$
- Spinning string: E.g. $V_J = (Y_+)^{-\Delta}(\partial X \bar{\partial} X)^{J/2}$

$V_k$ determine asymptotics of solution near operator insertion points $x_k$
Take a classical string solution on Minkowski cylinder \((\tau, \sigma)\)

Apply Wick rotation on worldsheet \(\tau \rightarrow i\tau\) and in target space \(t \rightarrow it\)

Map cylinder \((\tau, \sigma)\) to complex plane \(z\) by

\[
e^{\tau + i\sigma} = \frac{z - z_1}{z - z_2}
\]

Asymptotic points \(\tau = \pm \infty\) are mapped to punctures at \(z_{1,2}\)

Resulting configuration is semiclassical saddle point

- Satisfies string EOM with vertex operator insertions at \(z_{1,2}\)
- Vertex operators carry same charges as initial string solution
Heavy-Heavy-Light

Have simplified the problem to finding solution to classical string EOM with prescribed sources (from vertex operators)

Still a difficult problem!

Further simplification:

- Two asymptotic heavy states $\Delta \sim \sqrt{\lambda}$
- One massless/light state, $\Delta \sim 1$ or $\Delta \sim \sqrt[4]{\lambda}$ (sugra mode)

$\Rightarrow$ Hybrid formulation of string theory and supergravity

Path integral dominated by classical two-point solution

Leading contribution to correlator:
Supergravity mode insertion integrated over worldsheet, connected to boundary via bulk-to-boundary propagator
Hybrid formulation of string theory and supergravity

Path integral dominated by classical two-point solution

Leading contribution to correlator:
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Results:

- Folded spinning string + CPO dual
- Various strings + dilaton. Also gauge theory.

Activity: Spinning strings, folded strings, giant magnon, giant graviton, finite size ...
AdS Geodesics

Simplest case beyond heavy-heavy-light:
Large spins only on the sphere, $J \sim \Delta$

- Asymptotic states are point-like geodesics in AdS
- Three geodesics glued together at a point
- Saddle-point found by extremizing action w.r.t. interaction point
- Contribution of the sphere also included

\[ \text{[Klose, McLoughlin 1106.0495]} \]
Wanted: Heavy-Heavy-Heavy

Correlator of three states with large AdS charges
Qualitatively different from

- Three geodesics: Non-trivial worldsheet
- Heavy-heavy-light: Worldsheet is not a cylinder

Want: Solution to classical string EOM with

- Vertex operator sources
- Prescribed two-point asymptotics at three points

\[
\text{AdS}/\mathbb{H}: 0 = \partial \bar{\partial} Y - (\partial Y \cdot \bar{\partial} Y) Y \\
\text{Sphere}: 0 = \partial \bar{\partial} X + (\partial X \cdot \bar{\partial} X) X \\
\text{Virasoro}: (\partial Y)^2 + (\partial X)^2 = 0, \quad (\bar{\partial} Y)^2 + (\bar{\partial} X)^2 = 0
\]
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\[
\text{AdS/}\mathbb{H}: \quad 0 = \partial \bar{\partial} Y - (\partial Y \cdot \bar{\partial} Y) Y + \sum_k \delta^{(2)}(z - z_k) F_k
\]
\[
\text{Sphere}: \quad 0 = \partial \bar{\partial} X + (\partial X \cdot \bar{\partial} X) X + \sum_k \delta^{(2)}(z - z_k) G_k
\]
\[
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AdS/$\mathbb{H}$: $0 = \partial \bar{\partial} Y - (\partial Y \cdot \bar{\partial} Y) Y + \sum_k \delta^{(2)}(z - z_k) F_k$

Sphere: $0 = \partial \bar{\partial} X + (\partial X \cdot \bar{\partial} X) X + \sum_k \delta^{(2)}(z - z_k) G_k$

Virasoro: $(\partial Y)^2 + (\partial X)^2 = 0$, \quad $(\bar{\partial} Y)^2 + (\bar{\partial} X)^2 = 0$
Use Integrability?

Find explicit solution and evaluate action. Looks difficult!
But non-linear sigma model on symmetric space is integrable
Has been very successfully exploited in
- Spectral problem: Algebraic curve
- Scattering amplitudes: Find surface, evaluate action

Can we use integrability to find a solution / evaluate the action?
Split AdS/Sphere

Saddle point of the path integral factorizes:

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \approx V_1 V_2 V_3 e^{-S} \bigg|_{\text{saddle point}} = G_{\text{AdS}} G_{\text{Sphere}} \]

Concentrate on AdS part

Virasoro constraints link AdS and sphere:

\[ (\partial Y)^2 + (\partial X)^2 = 0, \quad (\bar{\partial} Y)^2 + (\bar{\partial} X)^2 = 0 \]

Rewrite as

\[ \partial Y(z) \cdot \partial Y(z) = -T(z), \quad \bar{\partial} Y(z) \cdot \bar{\partial} Y(z) = -\bar{T}(z) \]

with “energy-momentum tensor” \( T, \bar{T} \).
The Energy-Momentum Tensor

$T, \bar{T}$ a priori depend on the dynamics on the sphere

$$T(z) = -\partial Y \cdot \partial Y = +\partial X \cdot \partial X$$

$$\bar{T}(z) = -\bar{\partial} Y \cdot \bar{\partial} Y = +\bar{\partial} X \cdot \bar{\partial} X$$

But can be constrained by

- Two-point asymptotics at the insertion points
- Transformation under conformal inversions:

$$T(z; z_k) = z^4 T(1/z; 1/z_k)$$

For three states without large AdS spins, $T$ completely fixed

$$T(z) \xrightarrow{z \to z_k} \frac{\Delta_k^2}{4\lambda (z - z_k^2)} , \quad T(z) = \frac{P(z)}{(z - z_1)^2 (z - z_2)^2 (z - z_3)^2}$$

Reasonable to assume that this is true also for other states
Pohlmeyer Reduction

Reduction of EOM to generalized complex sinh-Gordon:

**AdS**$_3$: Basis of embedding space $\{Y, \partial Y, \bar{\partial} Y, N\}$

$$
cosh \alpha := \frac{\partial \bar{Y} \cdot \bar{\partial} Y}{\sqrt{T \bar{T}}} , \quad p := \frac{\vec{N} \cdot \partial^2 \bar{Y}}{T} , \quad \bar{p} := \frac{\vec{N} \cdot \bar{\partial}^2 \bar{Y}}{\bar{T}}
$$

Take derivatives, apply EOM for $Y$, expand in basis:

$$
\partial \bar{\partial} \alpha = \sqrt{T \bar{T}} \left( \sinh \alpha + \frac{p \bar{p}}{\sinh \alpha} \right) , \quad \bar{\partial} p = -\frac{\sqrt{\bar{T}} \bar{p} \partial \alpha}{\sqrt{T} \sinh \alpha} , \quad \partial \bar{p} = -\frac{\sqrt{T} p \bar{\partial} \alpha}{\sqrt{\bar{T}} \sinh \alpha}
$$
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\]

Take derivatives, apply EOM for $Y$, expand in basis:

\[
\partial \bar{\partial} \alpha = \sqrt{T\bar{T}} \left( \sinh \alpha + \frac{pp}{\sinh \alpha} \right) , \quad \bar{\partial} p = -\frac{\sqrt{T} \bar{p} \partial \alpha}{\sqrt{T} \sinh \alpha} , \quad \partial \bar{p} = -\frac{\sqrt{T} p \bar{\partial} \alpha}{\sqrt{T} \sinh \alpha}
\]

Integrable system!

Known solutions and techniques

But: Surface only implicit, reconstruction difficult
Pohlmeyer Reduction

Reduction of EOM to generalized complex sinh-Gordon:

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\cosh \alpha := \frac{\partial \bar{\partial} Y \cdot \bar{\partial} Y}{\sqrt{T \bar{T}}}, \quad p := \frac{\bar{N} \cdot \partial^2 Y}{T}, \quad \bar{p} := \frac{\bar{N} \cdot \bar{\partial}^2 Y}{\bar{T}}
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Reconstruction of the Surface

Take basis \( q_\mu = \{ Y, \partial Y, \bar{\partial} Y, N \} \) and expand derivatives

\[
\partial q_\mu = q_\nu B^{\nu}_{\ \mu} , \quad \bar{\partial} q_\mu = q_\nu \bar{B}^{\nu}_{\ \mu} \quad \text{(Gauss-Weingarten)}
\]

where \( B = B(\sqrt{T}, p, \alpha, \partial \alpha) \) and \( \bar{B} \) are \( 4 \times 4 \) matrices.

Write \( q \) in spinor indices, and split

\[
q^m_\mu = q_{\alpha \dot{\alpha}, a \dot{\alpha}} \sigma^m_{\mu} \sigma^{\alpha, a \dot{\alpha}} , \quad q_{\alpha \dot{\alpha}, a \dot{\alpha}} = \psi^L_{\alpha, a} \psi^R_{\dot{\alpha}, \dot{a}}
\]

Left and right \( 2 \times 2 \) auxiliary linear problem

\[
\partial \psi^L_{\alpha, a} + (B^L_z)_{\alpha}^{\beta} \psi^L_{\beta, a} = 0 , \quad \partial \psi^R_{\dot{\alpha}, \dot{a}} + (B^R_z)_{\dot{\alpha}}^{\dot{\beta}} \psi^R_{\dot{\beta}, \dot{a}} = 0 , \\
\bar{\partial} \psi^L_{\alpha, a} + (B^L_{\bar{z}})_{\alpha}^{\beta} \psi^L_{\beta, a} = 0 , \quad \bar{\partial} \psi^R_{\dot{\alpha}, \dot{a}} + (B^R_{\bar{z}})_{\dot{\alpha}}^{\dot{\beta}} \psi^R_{\dot{\beta}, \dot{a}} = 0 .
\]

Compatibility condition: Connections \( B^{L,R} \) are flat

\[
\partial B^{L,R} - \bar{\partial} B^{L,R} + [B^{L,R}, \bar{B}^{L,R}] = 0 . \quad \text{(Gauss-Codazzi)}
\]
Reconstruction of the Surface

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\]
Explicit Construction?

Have split the problem into two pieces:

- Generalized sinh-Gordon equations $\partial \bar{\partial} \alpha = \ldots$, $\partial \bar{p} = \ldots$,
- Auxiliary problem $\partial \psi + B \psi = 0$

Still difficult to solve

One approach: **Find explicit solution**

“N-point-like” surfaces in $\mathbb{H}_3$ (N-noids) known

![Diagram of N-noids]

But: Not exactly right (no $T$, $\bar{T}$)

→ No luck so far.
Try to compute similar to Alday & Maldacena:

- Find saddle-point action without knowing the exact solution
- Exploit the integrability

Partially done for three folded spinning (GKP) strings
And for BMN-like strings (geodesics)
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Partially done for three folded spinning (GKP) strings
And for BMN-like strings (geodesics)
Summary and Outlook

We are still at the beginning

Classical strings on $\text{AdS}_5 \times S^5$ integrable
Adapt amplitude methods for semiclassical strings
Qualitative differences:
- Multiple asymptotic regions
- Logarithmic branch cuts
Can saddle-point action be recovered from asymptotic data?
Regularization/cancellation of divergences with vertex operators?
What about the sphere contribution?

Description of asymptotic states in terms of algebraic curves useful?