A Generating Equation for Integrable Charges

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Integrable Spin Chains I

Consider a spin chain model, i.e. a tensor product

\[ \cdots \otimes V \otimes V \otimes V \otimes V \otimes V \otimes \cdots \]

of vector spaces (generalized spins), all transforming in a common representation of a symmetry algebra \( g \).

Focus on models with local and homogeneous interactions/charges:

\[ L_k = \sum_a L_k(a) = \sum_a a L_k \]

This type of spin chain finds application in the computation of anomalous dimensions of local gauge invariant operators of \( \mathcal{N} = 4 \) SYM. \cite{MinahanZarembo, Beisert}

Gauge theory spin chains are integrable, i.e. they feature an infinite set of commuting charges

\[ Q_r = \sum_k L_k, \quad r = 1, \ldots, \infty, \quad [Q_r, Q_s] = 0, \quad Q_2 = \mathcal{H}. \]
The charges of gauge theory spin chains feature a perturbative range expansion

\[ \mathcal{H} = Q_2 = \lambda + \lambda^2 + \lambda^3 \ldots, \]

where \( \lambda \) is the coupling constant.

- As an example, consider commuting charges on a spin chain with \( \mathfrak{gl}(n) \) symmetry. In this case, all symmetry invariant operators \( \mathcal{L}_k \) are permutations. These are the building blocks of the charges.
- Permutations \( \pi \in S_n \) are represented in Mathematica as

\[ \text{Perm}[\pi(1), \ldots, \pi(n)]. \]

For example, \( \text{Perm}[3, 4, 1, 2] \) maps \( \{1, 2, 3, 4\} \) to \( \{3, 4, 1, 2\} \).

**Goal:** Understand long-range integrable spin chains better!
Deforming Short-Range to Long-Range Chains

Integrable charges can be computed by brute force. *(cf. talk by Florian)*

Different approach: Deform a given set of (short-range, $\lambda = 0$) integrable charges $Q_r$ through a generating equation

\[
\frac{d}{d\lambda} Q_r(\lambda) = i [\mathcal{X}(\lambda), Q_r(\lambda)],
\]

where $\lambda \approx 0$ is the deformation parameter.

- The form of the generating equation guarantees that the algebra obeyed by the charges is invariant under the deformation. By the Jacobi identity

\[
\frac{d}{d\lambda} [Q_r(\lambda), Q_s(\lambda)] = i [\mathcal{X}(\lambda), [Q_r(\lambda), Q_s(\lambda)]].
\]

Therefore the structure constants are $\lambda$-independent,

\[
[Q_r(\lambda), Q_s(\lambda)] = f_{rst} Q_t(\lambda) \quad \frac{d}{d\lambda} f_{rst} = 0.
\]

- In particular, if the initial charges commute, $f_{rst} = 0$, also the deformed charges $Q_r(\lambda)$ commute.

$\Rightarrow$ The deformation preserves integrability.
Deformation Operators $\mathcal{X} =$ ?

Generating equation:

$$\frac{d}{d\lambda} Q_r(\lambda) = i [\mathcal{X}(\lambda), Q_r(\lambda)] ,$$

What are the required properties for the deformation operator $\mathcal{X}$?

- The commutator between $\mathcal{X}(\lambda)$ and the charges $Q_r(\lambda)$ has to be well-defined.

- The deformed charges $Q_r(\lambda)$ should again be local and homogeneous as required by gauge theory.

With suitable operators $\mathcal{X}$, long-range integrable spin chains can be constructed.

What are suitable deformation operators $\mathcal{X}(\lambda)$?
One suitable type of operator: Boost operators

\[ \mathcal{L}_k = \sum_a \mathcal{L}_k(a) \implies B[\mathcal{L}_k] := \sum_a a \mathcal{L}_k(a). \]

In general, the commutator of a homogeneous charge operator \( Q_r \) with a boost \( B[\mathcal{L}_k] \) again yields a boost. However, boosts of the charges \( B[Q_k] \)

\[ \frac{d}{d\lambda} Q_r(\lambda) = i [B[Q_k(\lambda)], Q_r(\lambda)], \]

yield again homogeneous operators \( Q_r(\lambda) \) as required. This is due to the fact that the charges \( Q_r \) commute:

\[ a \cdot \mathcal{L}_k, \mathcal{L}_l = a \cdot \mathcal{L}_k + (a + 1) \cdot \mathcal{L}_l + \ldots = a \cdot \mathcal{L}_k, \mathcal{L}_l + \mathcal{L}_k \mathcal{L}_l + \ldots \]
Boost Commutator: Implementation 1/3

Boosts $B[\pi]$ on the $\mathfrak{gl}(n)$ chain are represented in Mathematica as

$$\text{PermB}[\pi(1), \ldots, \pi(n)],$$

e.g. $\text{PermB}[4, 5, 2, 1, 3]$. 

Boosts of charges can be obtained by

$$B[r_] := Q[r] \ /. \ \text{Perm} \rightarrow \text{PermB}.$$ 

In order to compute deformed charges, need a Mathematica implementation of the commutator between boosts and permutation operators.

Overview of the commutator method in Mathematica:

\[
\text{CommutePermB}[X_, Y_] :=
X /. \{X1_\text{PermB} :> (Y /. Y1_\text{Perm} :> \text{CommutePermB12}[X1, Y1])\}
\]

\[
\text{CommutePermB12}[X_\text{PermB}, Y_\text{Perm}] := \text{Plus}[
\text{CommutePerm}[X \ /. \ \text{PermB} \rightarrow \text{Perm}, \ Y] \ /. \ \text{Perm} \rightarrow \text{PermB},
\text{CommutePermB12Hom}[X, Y] \]
\]

\[
\text{CommutePermB12Hom}[X_\text{PermB}, Y_\text{Perm}] := \text{Sum}[
k \ (\text{+CombinePerm12}[X \ /. \ \text{PermB} \rightarrow \text{Perm}, \ Y, \ \text{Length}[X] + k] - \text{CombinePerm12}[Y, X \ /. \ \text{PermB} \rightarrow \text{Perm}, \ \text{Length}[Y] - k]),
\{k, 1, \ \text{Length}[Y] - 1\}\]
\]
First, make the commutator distributive:

\[
\text{CommutePermB}[X\_,\ Y\_] := \\
X /. \{X1\_PermB :> (Y /. Y1\_Perm :> \text{CommutePermB12}[X1, Y1])\}
\]

\[
\text{CommutePermB12}[X\_\text{PermB}, Y\_\text{Perm}] := \text{Plus}[ \\
\text{CommutePerm}[X /. \text{PermB} \rightarrow \text{Perm}, Y] /. \text{Perm} \rightarrow \text{PermB}, \\
\text{CommutePermB12Hom}[X, Y]\]
\]

Then, each pair of boosted and unboosted permutation yields boosts (center line) and homogeneous terms (last line),

\[
[B[\pi_1], \pi_2] = B[[\pi_1, \pi_2]] + \text{homogeneous}.
\]

\[
[a \cdot \mathcal{L}_k, \mathcal{L}_l] = a \cdot + (a + 1) \cdot + \ldots = a \cdot \mathcal{L}_k + \mathcal{L}_l + \ldots
\]

If \(\pi_1\) and \(\pi_2\) commute, the boost part (center line in the box above) vanishes.
The homogeneous part of the commutator is implemented as

\[
\text{CommutePermB12Hom}[X_{\text{PermB}}, Y_{\text{Perm}}] := \text{Sum}[
\quad k \left( +\text{CombinePerm12}[X \mapsto \text{Perm}, Y, \text{Length}[X] + k] \\
\quad \quad -\text{CombinePerm12}[Y, X \mapsto \text{Perm}, \text{Length}[Y] - k] \right) , \quad \{k, 1, \text{Length}[Y] - 1\}]
\]

\text{CommutePerm12}[P1, P2, k] (cf. Florians talk) computes the product of two overlapping permutations \(P1\) and \(P2\), where the overlap is specified by \(k\).

**Example:**

\[
\text{CommutePermB12Hom}[\text{PermB}[2,1,3], \text{Perm}[3,2,4,1]]
\]

\[
= +1 \cdot \frac{\text{Perm}[2,1,3]}{\text{Perm}[3,2,4,1]} + 2 \cdot \frac{\text{Perm}[2,1,3]}{\text{Perm}[3,2,4,1]} + 3 \cdot \frac{\text{Perm}[2,1,3]}{\text{Perm}[3,2,4,1]} \\
- 1 \cdot \frac{\text{Perm}[3,2,4,1]}{\text{Perm}[2,1,3]} - 2 \cdot \frac{\text{Perm}[3,2,4,1]}{\text{Perm}[2,1,3]} - 3 \cdot \frac{\text{Perm}[3,2,4,1]}{\text{Perm}[2,1,3]}
\]

\(k = 1\) \quad \text{\(k = 2\)} \quad \text{\(k = 3\)}
**Boundary Identifications: Spectator Legs**

On an infinite or periodic chain, we can identify terms whose action differs only at chain boundaries. This is necessary for verifying whether deformed charges commute.

E.g. recall that for local operators: \[ \mathcal{L}_k = \mathcal{L}_k. \] (cf. talk by Florian)

This implies for boosted operators

\[ \mathcal{B} (\mathcal{L}_k) = \mathcal{B} (\mathcal{L}_k) - \mathcal{L}_k. \]

![Diagram](image)

\[ \text{PermB}[1, X__] = \text{PermB}[X-1] - \text{Perm}[X-1] \]

Method that implements the identification in Mathematica:

```mathematica
IdentifyBoundaryTermsBoostLeft[P_] := P //. {
    PermB[X__ /; First[{X}] == 1 && {X} != {1}] :> PermB @@ (Drop[{X}, 1] - 1) - Perm @@ (Drop[{X}, 1] - 1)
}
```
Another suitable type of deformation operators are *bilocal operators*:

\[
\mathcal{L}_k = \sum_a \mathcal{L}_k(a), \quad \mathcal{L}_l = \sum_a \mathcal{L}_l(a) \quad \Rightarrow \quad [\mathcal{L}_k | \mathcal{L}_l] = \sum_{a \lesssim b} \frac{1}{2} \{ \mathcal{L}_k(a), \mathcal{L}_l(b) \}.
\]
In general, the commutator of a local charge operator $Q_r$ with a bilocal operator $[\mathcal{L}_k|\mathcal{L}_l]$ again is bilocal. However, bilocal operators composed of the charges $[Q_t|Q_u]$ yield again local operators $Q_r(\lambda)$ as required. Again, this is due to the fact that the charges $Q_r$ commute:

\[
\frac{d}{d\lambda} Q_r(\lambda) = i \left[ [Q_t(\lambda)|Q_u(\lambda)], Q_r(\lambda) \right],
\]

which results in $Q_r(\lambda)$ being local up to boundary conditions.

\[= 0 \text{ up to bdry} \quad \rightarrow \text{local} \]

\[= \text{local} \]
Bilocal Commutator: Implementation

Bilocal operators $[\mathcal{L}_t | \mathcal{L}_u]$ are represented in Mathematica as

$$Bi[a_1 \text{Perm}[...] + \ldots + a_{lt} \text{Perm}[...], b_1 \text{Perm}[...] + \ldots + b_{lu} \text{Perm}[...]]$$

Overview of the bilocal commutator method in Mathematica:

```mathematica
CommuteBiP[BI_, P_] := (BI // DistributeBi) /. 
    BI0_Bi :> (P /. P0_Perm -> CommuteBiP12[BI0, P0])

DistributeBi[B_] := B //. {
    Bi[0, x____] -> 0,
    Bi[x____, 0] -> 0,
    Bi[x____, a_ y_] -> a Bi[x, y],
    Bi[a_ x_, y____] -> a Bi[x, y],
    Bi[-Perm[x____], Y____] -> -Bi[Perm[x], Y],
    Bi[Y____, -Perm[x____]] -> -Bi[Y, Perm[x]],
    BO_Bi :> Distribute[B0]
}
```
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CommuteBiP[BI_, P_] := (BI // DistributeBi) /. 
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CommuteBiP12[BI_Bi, P_Perm] := Plus[
    Bi[CommutePerm12[BI[[1]], P], BI[[2]]] // DistributeBi,
    Bi[BI[[1]], CommutePerm12[BI[[2]], P]] // DistributeBi,
    CommuteBiP12loc[BI, P]
]

CommuteBiP12loc[BI_Bi, P_Perm] := Module[
    {LBI1 = Length[BI[[1]]], LBI2 = Length[BI[[2]]], LP = Length[P]},
    Sum[
        Module[{LongPerm = CombinePerm12[BI[[1]], BI[[2]], -d]},
            Sum[CombinePerm12[LongPerm, P, LBI1 + LBI2 + d + LP - s] - 
                CombinePerm12[P, LongPerm, s],
                {s, LBI1 + d + 1, LP + LBI1 - 1}]],
        {d, 0, LP - 2}]
]```

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Overview of the bilocal commutator method in Mathematica:

$$\text{CommutBiP12loc}[\text{BI}_\text{Bi}, \text{P}_\text{Perm}] := \text{Module[}$$

$$\{\text{LBI1} = \text{Length}[\text{BI}[[1]]], \text{LBI2} = \text{Length}[\text{BI}[[2]]], \text{LP} = \text{Length}[\text{P}]\},$$

$$\text{Sum[}$$

$$\text{Module[}\{\text{LongPerm} = \text{CombinePerm12}[\text{BI}[[1]], \text{BI}[[2]], -d]\},$$

$$\text{Sum[\text{CombinePerm12}[\text{LongPerm}, \text{P}, \text{LBI1} + \text{LBI2} + d + \text{LP} - s] -$$

$$\text{CombinePerm12}[\text{P}, \text{LongPerm}, s],$$

$$\{s, \text{LBI1} + d + 1, \text{LP} + \text{LBI1} - 1]\}],$$

$$\{d, 0, \text{LP} - 2\}]\}$$

$$= 0 \text{ up to bdry } \rightarrow \text{ local }$$

$$= \text{ local }$$
Generating Integrable Charges

We have found two types of operators that generate deformations of integrable charges

- Boost operators $B[Q_k]$,
- Bilocal operators $[Q_t|Q_u]$.

The charges $Q_r$ can be deformed independently by each operator:

$$Q_r = Q_r(\alpha_3, \alpha_5, \ldots; \beta_{2,3}, \beta_{2,4}, \ldots, \beta_{3,4}, \beta_{3,5}, \ldots),$$

$$\frac{d}{d\alpha_k} Q_r = i [B[Q_k], Q_r], \quad \frac{d}{d\beta_{t,u}} Q_r = i [Q_t | Q_u, Q_r].$$

- The set of deformations exhausts all non-trivial degrees of freedom previously obtained by brute force. ([Beisert, Klose])
- Specific one-dimensional deformations $Q_r(\lambda)$ can be chosen by suitably defining functions $\alpha_k(\lambda), \beta_{t,u}(\lambda)$.
- For the $g\mathfrak{l}(n)$ chain, a there exists a choice $\alpha_k(\lambda), \beta_{t,u}(\lambda)$ that reproduces the dilatation generator (anomalous dimensions) for the $su(2)$ subsector of $\mathcal{N} = 4$ SYM.
### Bethe Equations

The Bethe equations for a general symmetry group $g$ of rank $R$ for a chain of length $L$ are given by

\[
\left( \frac{u_{a,k} + \frac{i}{2} t}{u_{a,k} - \frac{i}{2} t} \right)^L = \prod_{b=1}^{R} \prod_{j=1}^{M_b} \frac{u_{a,k} - u_{b,j} - \frac{i}{2} C_{a,b}}{u_{a,k} - u_{b,j} + \frac{i}{2} C_{a,b}}.
\]

where

- $u_{a,k}$ is the rapidity of the $k$'th particle of type $a$ and
- $S_{a,b}$ is the two-particle scattering matrix for particles of type $a$, $b$. 
Bethe Equations

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The boost deformation parameters \( \alpha_k \) enter the rapidity map \( x(u) \),

\[
u = x + \sum_{k=3}^{\infty} \frac{\alpha_k}{x^{k-2}},
\]
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\]

where

- $u_{a,k}$ is the rapidity of the $k'$th particle of type $a$ and
- $S_{a,b}$ is the two-particle scattering matrix for particles of type $a$, $b$.

The boost deformation parameters $\alpha_k$ enter the *rapidity map* $x(u)$,

\[
u = x + \sum_{k=3}^{\infty} \frac{\alpha_k}{x^{k-2}},
\]

while the bilocal deformations $\beta_{t,u}$ give rise to the *dressing phase*,

\[
\theta = \sum_{u > t = 2}^{\infty} \beta_{t,u} \left( q_t(u) q_u(u') - q_u(u) q_t(u') \right).
\]
Generating Integrable $\mathfrak{gl}(n)$ Charges: Example

Deform the charges with the boost operator $B[Q_3]$, i.e.

$$\frac{d}{d\alpha_3} Q_r = i [B[Q_3], Q_r],$$

To first order in $\alpha_3$, the charges expand to

$$Q_r = Q_r^{(0)} + \alpha_3 Q_r^{(3)} + \mathcal{O}(\alpha_3^2)$$

$$= Q_r^{(0)} + \alpha_3 i [B[Q_3^{(0)}], Q_r^{(0)}] + \mathcal{O}(\alpha_3^2).$$

As a starting point, take the known short-range charges $Q_r^{(0)}$.

$$Q[2, 0] = \text{Perm}[1] - \text{Perm}[2, 1],$$
$$Q[3, 0] = -\frac{1}{2} I (\text{Perm}[2, 3, 1] - \text{Perm}[3, 1, 2]), \ldots.$$ 

The first perturbative order $Q_r^{(3)}$ of the first two charges read

In[43]:= Q[2, 3] = I CommutePermB[B[3, 0], Q[2, 0]] // IdentifyBoundaryTerms

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As a starting point, take the known short-range charges $Q_r^{(0)}$. The first perturbative order $Q_r^{(3)}$ of the first two charges read

\begin{verbatim}
In[43]:= Q[2, 3] = I CommutePermB[B[3, 0], Q[2, 0]] // IdentifyBoundaryTerms
      + 1/2 Perm[3, 1, 4, 2] - 1/2 Perm[4, 1, 2, 3]

In[44]:= Q[3, 3] = I CommutePermB[B[3, 0], Q[3, 0]] // IdentifyBoundaryTerms
      + 1/2 I Perm[2, 4, 1, 5, 3] - 1/2 I Perm[2, 5, 1, 3, 4] + 1/2 I Perm[3, 1, 4, 5, 2]
      - 1/2 I Perm[3, 1, 5, 2, 4] - 1/2 I Perm[4, 1, 2, 5, 3] + 1/2 I Perm[5, 1, 2, 3, 4]
\end{verbatim}

We can verify that the deformed charges indeed commute:

\begin{verbatim}
In[45]:= CommutePerm[Q[2, 3], Q[3, 0]] + CommutePerm[Q[2, 0], Q[3, 3]]
Out[45]= 0
\end{verbatim}
Summary

- Perturbative long-range integrable spin chains can be obtained as deformations of short-range models via a generating equation.
- Suitable deformation operators are given by boosts $B[Q_k]$ and bilocal operators $[Q_t | Q_u]$ constructed from the integrable charges $Q_r$.
- The deformation reproduces all degrees of freedom that were obtained before by brute force.
- The deformations give rise to the rapidity map $x(u)$ (boosts) and the dressing phase $\theta$ (bilocal operators).

For generic symmetry algebra $\mathfrak{g}$, all charges are defined to all orders (on an infinite chain) and are integrable by construction.