Two-Cut Solutions of the Heisenberg Ferromagnet and Stability

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Work with Niklas Beisert (to appear)

Contents

Heisenberg Spin Chain, Landau-Lifshitz model, Integrability

LL model: Limit of ultrarelativistic rotating string on $\mathbb{R} \times \mathrm{S}^3$. [Kruczenski hep-th/0311203]

- Subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5 \rightarrow$ AdS/CFT correspondence.
- Quantization around certain solutions $\rightarrow$ Unstable excitation modes.

Heisenberg Spin Chain: Quantum-mechanical model for 1D magnet

- Equivalent to $\mathrm{SU}(2)$ sector of planar $\mathcal{N} = 4$ SYM (one-loop). [Minahan Zarembo]
- Solved (in principle) by the Bethe equations.
- Matches $\mathbb{R} \times \mathrm{S}^3$ sector of $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory.
- Thermodynamic limit (∞ long chain): Equivalent to the LL model $\rightarrow$ Heisenberg Chain is quantized version of LL model. [Kruczenski hep-th/0311203]

Integrability:

- LL model: Classical integrability $\rightarrow$ *Spectral curves*.
- Spin Chain: Quantum integrability $\rightarrow$ *Bethe equations*.

Aim of the Project

Goal: Find the spectrum of the Heisenberg ferromagnet in the thermodynamic limit.

- Understand the phase space of solutions to the Bethe equations.
- In particular: Investigate unstable modes of the LL model.
- Measurements on 1D-magnets?!

Contents

The Landau-Lifshitz Model

- Classical non-relativistic sigma model on the sphere S^2 with fields ϕ, θ and Lagrange function

$$L[\phi, \dot{\phi}, \theta] = -\frac{L}{4\pi} \oint \cos\theta \dot{\phi} \, \mathrm{d}\sigma - \frac{\pi}{2L} \oint (\theta'^2 + \sin^2\theta \phi'^2) \, \mathrm{d}\sigma$$

- Effective model for the exact description of strings on $\mathbb{R} \times \mathrm{S}^3$ that rotate at highly relativistic speed. [Kruczenski hep-th/0311203]
- Equations of motion:

$$\dot{\phi} = (2\pi/L)^2 \left(\cos(\theta)\phi'^2 - \csc(\theta)\theta''\right) ,$$
$$\dot{\theta} = (2\pi/L)^2 \left(2\cos(\theta)\theta'\phi' + \sin(\theta)\phi''\right) .$$

- Charges: Momentum P, spin α, energy $\tilde{E}$:

$$P = \frac{1}{2} \int \left(1 - \cos(\theta)\right) \phi' \, \mathrm{d}\sigma , \qquad \alpha = \frac{1}{4\pi} \int \left(1 - \cos(\theta)\right) \mathrm{d}\sigma ,$$
$$\tilde{E} = \frac{\pi}{2} \int \left(\theta'^2 + \sin^2(\theta)\phi'^2\right) \mathrm{d}\sigma .$$

A Simple Solution

- Vacuum: Energy minimized by string localized at a point:

$$\theta(\sigma,\tau) \equiv 0\,, \qquad P = \alpha = \tilde{E} = 0\,.$$

- A simple solution: Constant latitude θ_0, n windings:

$$\theta(\sigma,\tau) \equiv \theta_0\,, \qquad \phi(\sigma,\tau) = n\sigma + (2\pi/L)^2 n^2 \cos(\theta_0)\tau\,.$$

  Momentum P and energy $\tilde{E}$ can be expressed in terms of spin α and mode number n:

$$\alpha = \tfrac{1}{2}(1-\cos(\theta_0))\,, \qquad P = 2\pi n\alpha\,, \qquad \tilde{E} = 4\pi^2 n^2 \alpha(1-\alpha)\,.$$

- Semiclassical quantization $\rightarrow$ Contact with Heisenberg model.

Fluctuation Modes

- Add excitations with mode number k to the simple solution:

$$\theta(\sigma,\tau) = \theta_0 + \varepsilon\theta_+(\tau)e^{ik\sigma} + \varepsilon\theta_-(\tau)e^{-ik\sigma} + \varepsilon^2\theta_{\mathrm{c}}(\tau)\,,$$
$$\phi(\sigma,\tau) = \phi_0 + \varepsilon\phi_+(\tau)e^{ik\sigma} + \varepsilon\phi_-(\tau)e^{-ik\sigma} + \varepsilon^2\phi_{\mathrm{c}}(\tau)\,.$$

- Expanding $L[\phi,\dot{\phi},\theta]$ to order ε^2 yields two coupled HO's with charges

$$\delta\alpha = 1/L\,,$$
$$\delta P = 2\pi(n+k)/L\,,$$
$$\delta\tilde{E} = (2\pi)^2/L\left(n(n+2k)(1-2\alpha) + k^2\sqrt{1-4n^2\alpha(1-\alpha)/k^2}\right)\,.$$

$\Rightarrow$ The classical solution becomes unstable when $2n\sqrt{\alpha(1-\alpha)} > k = 1$, that is for large n and α.

Contents

The Heisenberg Spin Chain

- One of the oldest quantum mechanical models, set up by Heisenberg in 1928, describes a 1D-magnet with nearest-neighbor interaction of L spin-1/2 particles. [Heisenberg Z. Phys. A49, 619]
- Describes $\mathrm{SU}(2)$-sector of planar $\mathcal{N}=4$ SYM at one loop. [Minahan Zarembo]
- Hilbert space $\mathcal{H}$ is the tensor product of L single-spin spaces $\mathbb{C}^2$:

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = \left(\mathbb{C}^2\right)^{\otimes L}, \quad \text{e.g. } |\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\downarrow\uparrow\downarrow\rangle \in \mathcal{H}\,.$$

- The Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ is periodic

$$H = \frac{1}{2}\sum_{k=1}^{L}(1 - \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_{k+1}) = \sum_{k=1}^{L}(1 - P_{k,k+1})\,.$$

- The energy spectrum is bounded between [Hulthén 1938]
  - The ferromagnetic ground state $|\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$, energy $E=0$.
  - The antiferromangetic ground state "$|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$", $E \approx L\log 4$.

Bethe Equations

The Heisenberg spin chain is exactly solvable: [Bethe Z. Phys. 71, 205]

- Fundamental excitations: *Magnons* with definite momentum p and rapidity $u = \frac{1}{2}\cot(p/2)$.
- All eigenstates can be explicitly constructed as combinations of multiple magnons (Bethe ansatz).
- Only requirement: Constituent magnons $u_1, \ldots, u_M$ must satisfy **Bethe equations** (ensure periodicity of wave function):

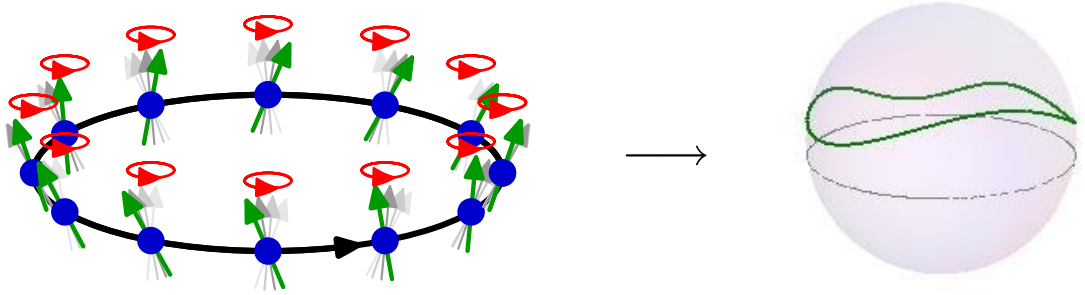
$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M\,.$$

- Momentum and Energy:

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2}\,, \qquad E = \sum_{k=1}^{M} \frac{1}{u_k^2 - 1/4}\,.$$

The Thermodynamic Limit

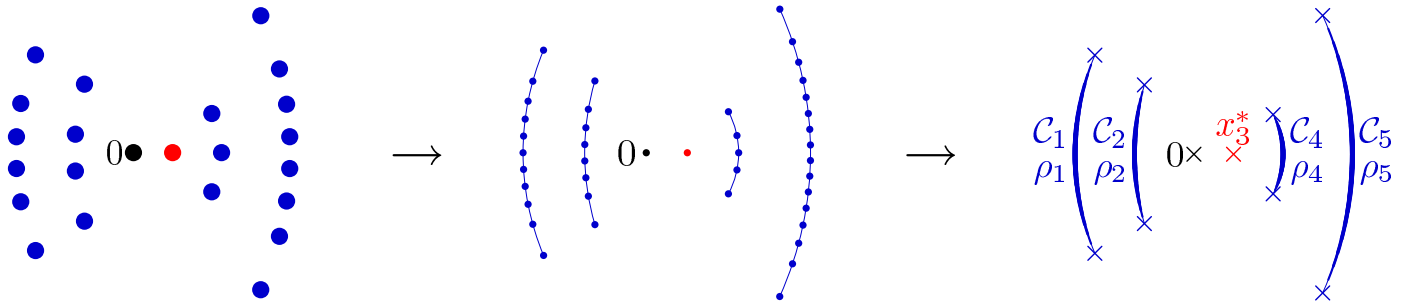
- Bethe equations hard to solve for more than a few excitations u_k.
- Problem simplifies in *thermodynamic limit*: [Sutherland Phys. Rev. Lett. 74, 816 (1995)] [Beisert, Minahan Staudacher, Zarembo]
  - Length of the chain (number of sites) $L \to \infty$.
  - Number of excitations (flipped spins) $M \to \infty$.
  - *Filling fraction* $\alpha = M/L$ fixed.
  - Keep only low-energy excitations (IR modes, coherent states), energies $E = \tilde{E}/L \sim 1/L$.
- In this limit, contact with the LL model is established: In coherent states, the expectation values of single spins form paths on S^2:



$\Rightarrow$ Classical LL model on S^2 is an effective model for the Heisenberg chain in the ferromagnetic thermodynamic limit. [Kruczenski hep-th/0311203]

Root Condensation

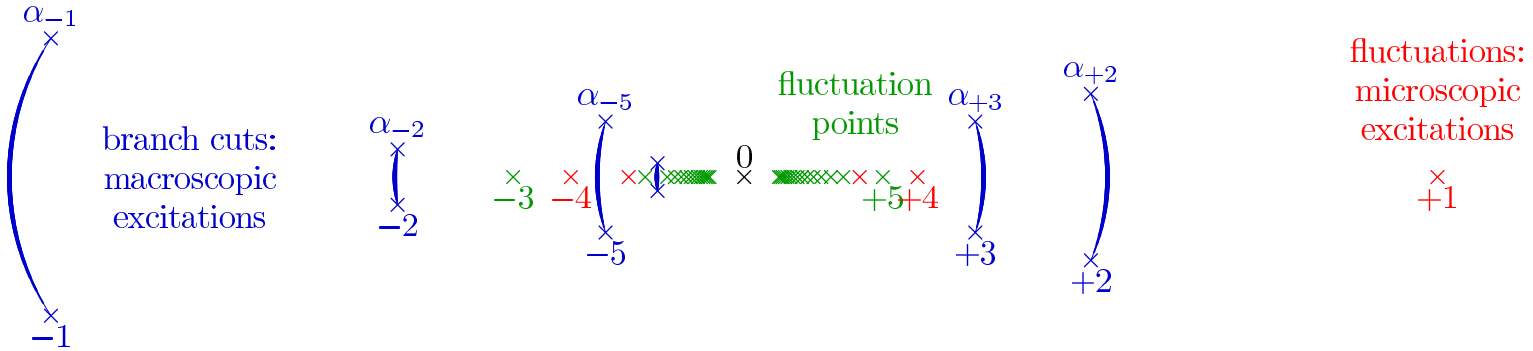
- In the thermodynamic limit, rescaled roots $x_k = u_k/L$ of coherent states condense on contours in the complex plane: [Beisert, Minahan Staudacher, Zarembo]



- In the strict limit $L \Rightarrow \infty$, the Bethe equations turn into integral eqations which describe the positions of the contours and the root density along them. [Kazakov, Marshakov Minahan, Zarembo]
  The contours constitute the branch cuts of the spectral curve, which is the solution of the LL model.

Small Filling

- General picture for small fillings α_i:



  Positions of cuts: Integer mode numbers n_i, $x_k \sim 1/n_i$.
  Small densities, weak interaction between individual cuts.

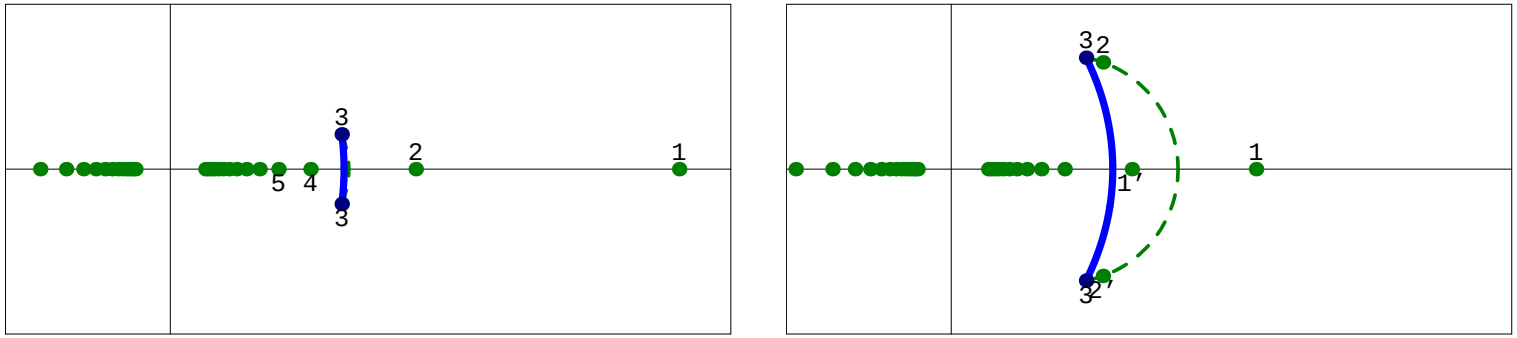
- When the filling of a cut grows, the cut gets longer and its density increases.
- **Questions**:
  - What happens when root density approaches $|\rho| = 1$?
  - Singularity in the Bethe equations; construction of the spectral curve no longer valid!?
  - Relation to unstable modes in LL model?
  - Do corresponding solutions to the Bethe equations exist for all spectral curves?

Contents

From Small to Large Filling

First: Study the configuration with a single cut in detail. [TB, Beisert to appear]

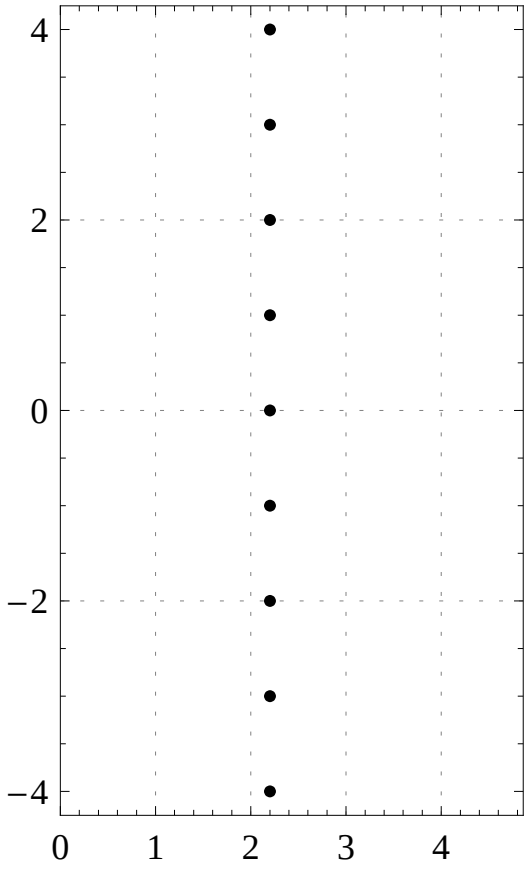
- As the cut grows, it attracts the neighboring fluctuation points:



- Expect something interesting when [Beisert, Tseytlin Zarembo] [Hernández, López Periáñez, Sierra] [Beisert Freyhult]
  - Fluctuation point collides with cut: Density reaches $|\rho| = 1$, Bethe equations singular.
  - Two successive fluctuation points collide and diverge into the complex plane. Instability? Coincides with filling where instability appears in the classical LL model.

Bethe Strings

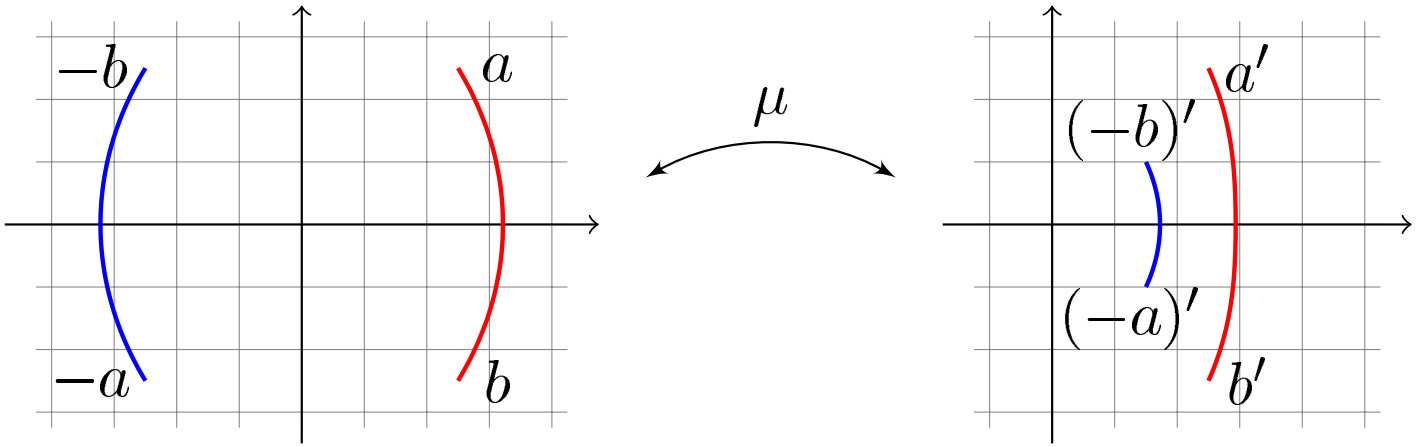
- For the a priori infinite chain, individual magnon rapidities are arbitrary (no periodicity $\longrightarrow$ no Bethe equations), but generically must be real.
- Only exception: "Bethe strings". Bound states on the infinitely long chain, complex rapidities with regular pattern.
- Guess/Expect: Bethe strings appear when density on contour reaches $|\rho| = 1$.



Contents

Construction of the Two-Cut Spectral Curve

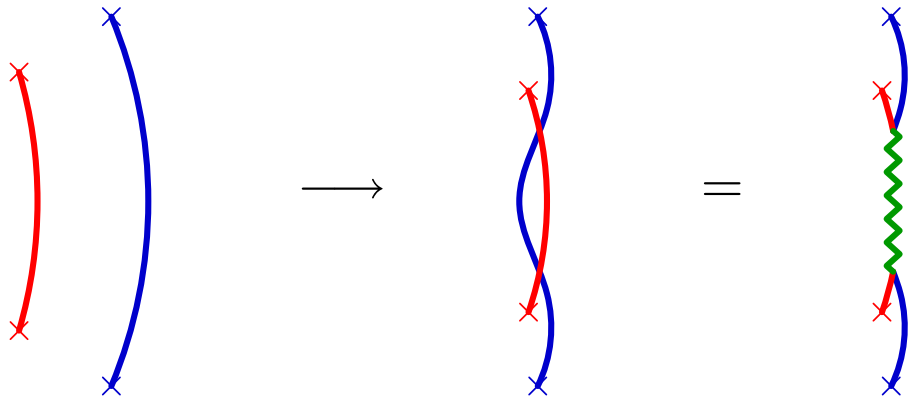
- Spectral curve that encodes the contours is a function with genus $g = 1$ (for two cuts). Find analytic expression.
- Generalize the symmetric solution [Beisert, Dippel, Staudacher]
  Map two symmetric cuts to two general cuts via Möbius transformation μ:



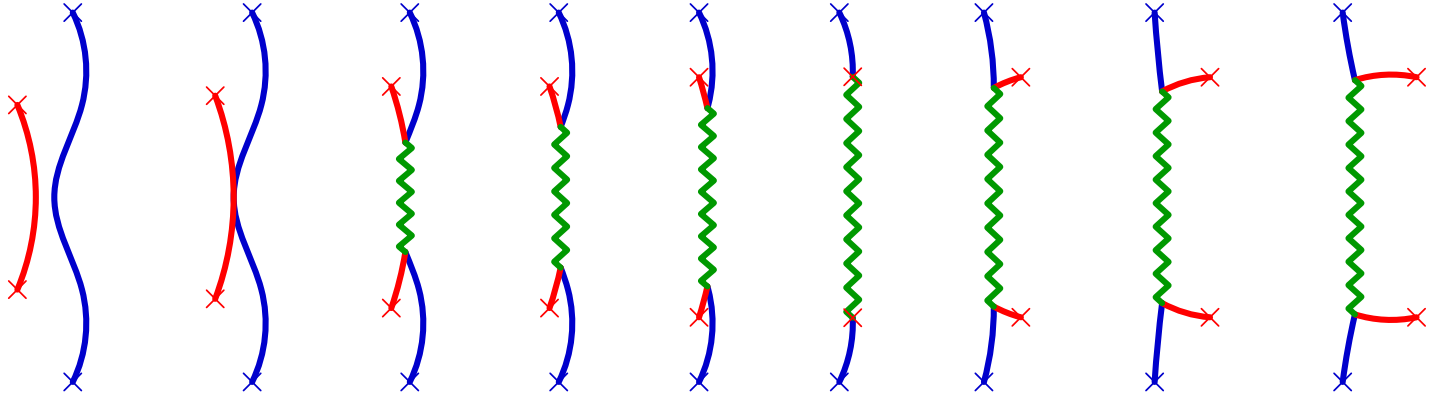
  The parameters of μ and the position of the symmetric cuts leave enough freedom for placing the cuts at will.
- Need to solve for specific solutions that obey the integral equations. Possible numerically.

Collision of Two Cuts

When the filling grows, cuts can collide and form condensates with density $|\rho| = \Delta u = 1$:



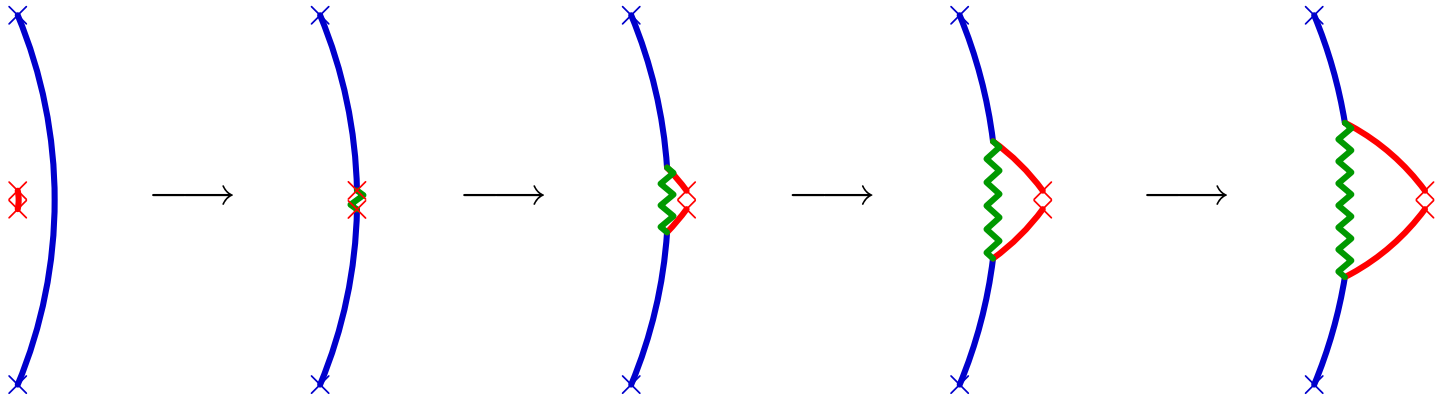
Cuts can even pass through each other:



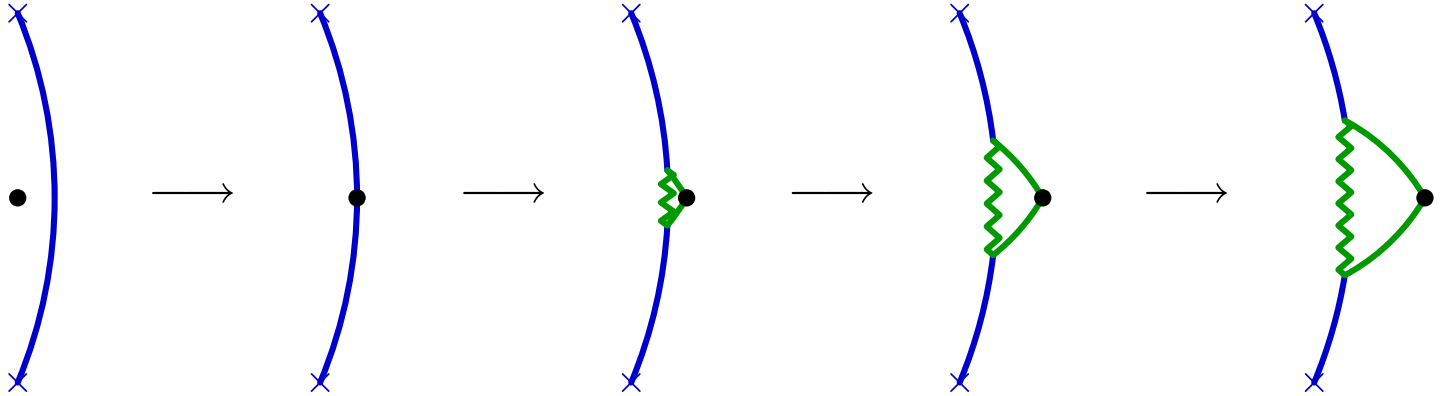
The passing cut changes its mode number: $n_2 \longrightarrow 2n_1 - n_2$.

Closed Loop Cuts

Consider a very small second cut:



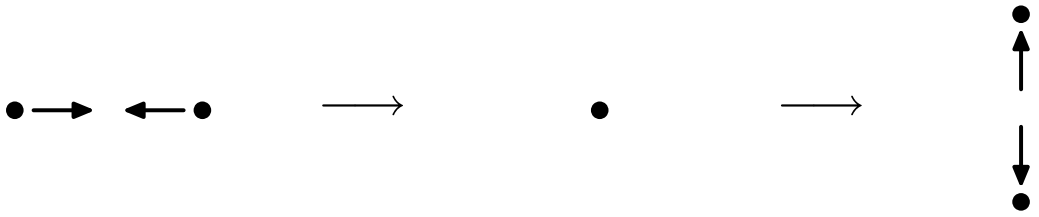
Compare this to a bare excitation point that passes through:



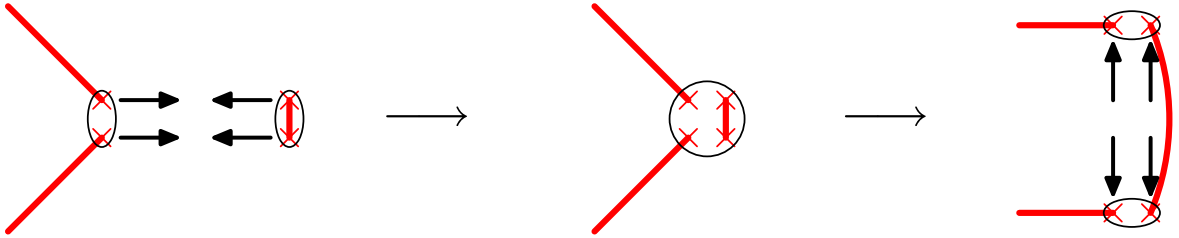
A closed loop with a condensate appears naturally.

Collision of Fluctuation Points: Instability

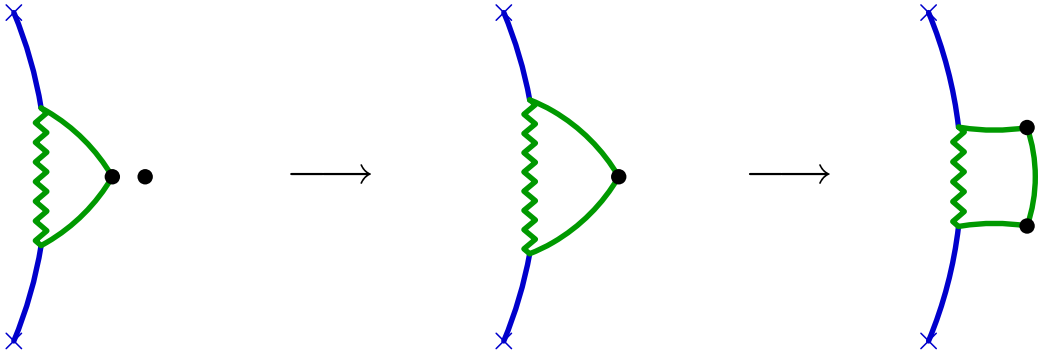
Look at the point where unstable modes in the LL model appear:



Consider again small cuts instead of bare excitation points:

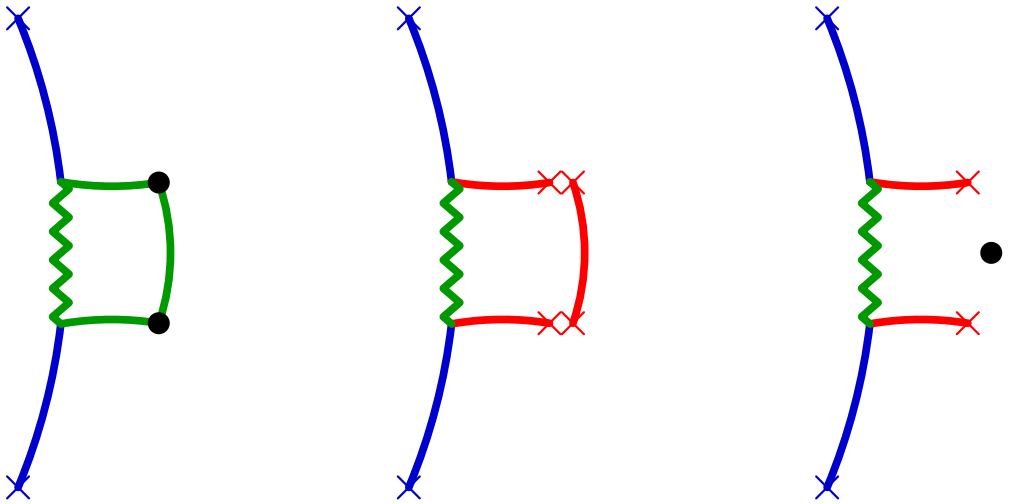


Implication for closed cut (one-cut solution):



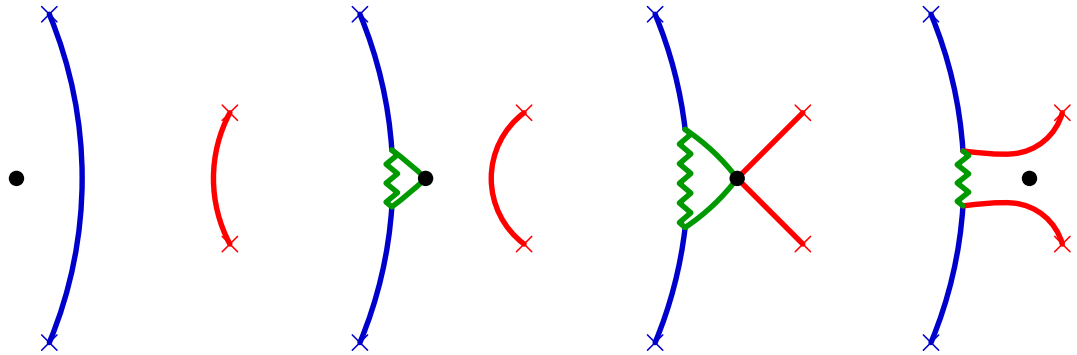
Phase transition

- Excitation of a mode means: Regular point $\longrightarrow$ Two branch points.
- Fluctuation point real: Excitation = Addition of roots.
- Fluctuation points complex with loop cut: Excitation means taking roots away.
  $\Rightarrow$ Behind instability point: Single cut + loop $\longrightarrow$ two cuts:

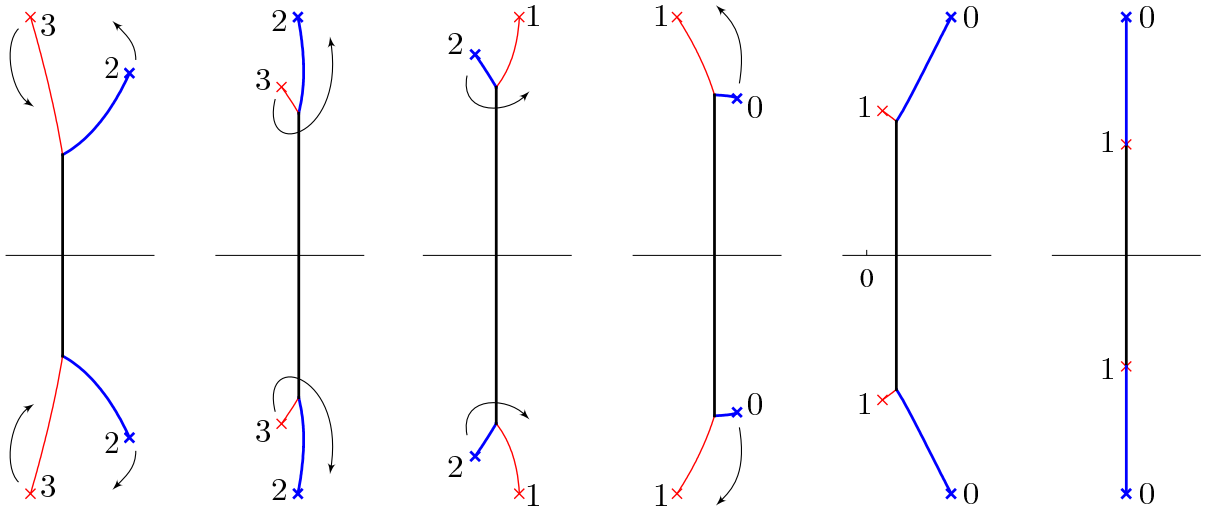


Further Features of Solutions

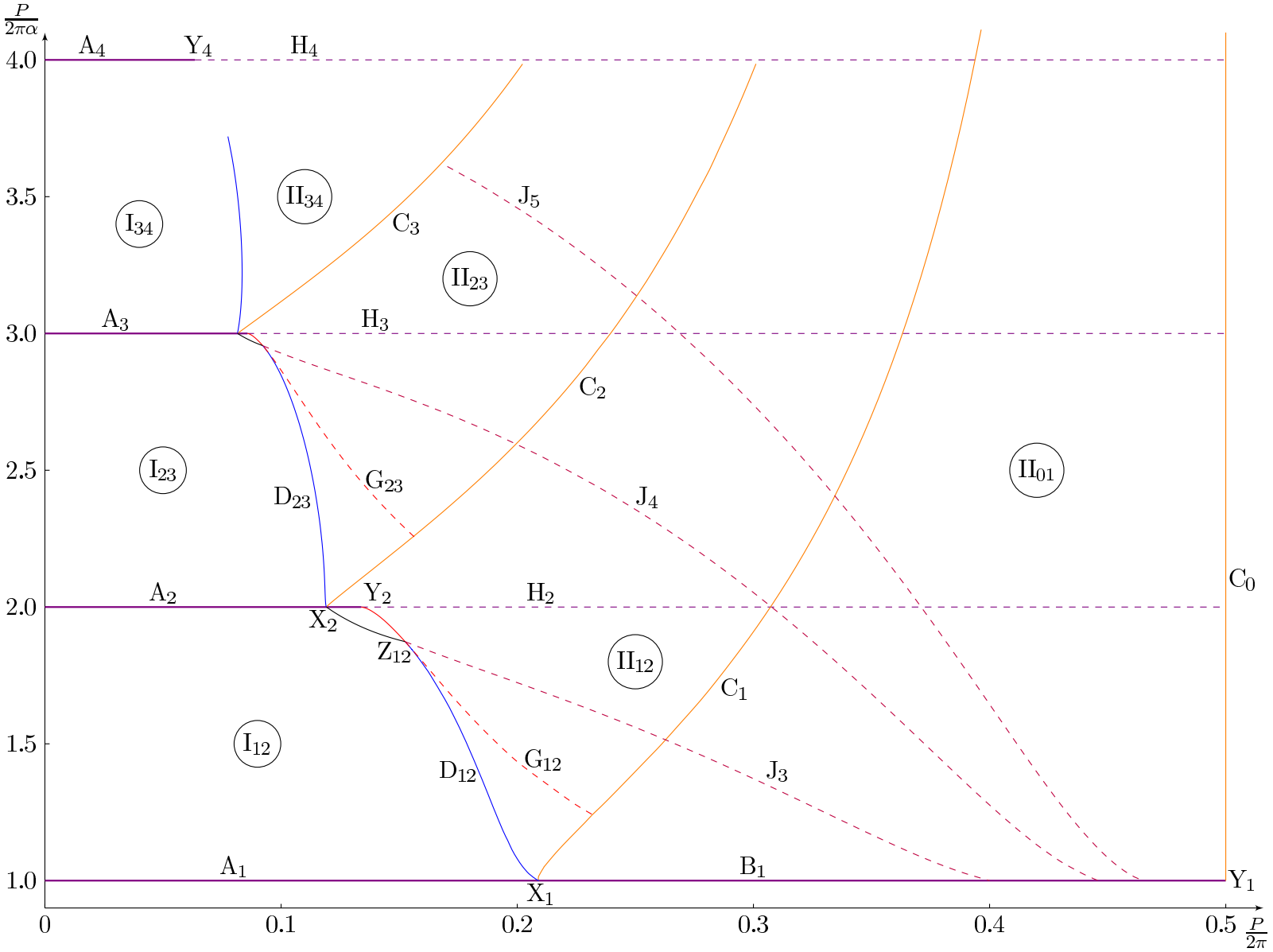
What if the fluctuation point is already excited at the instability point?



BPs can cycle around each other until they reach the imagin. line, $P = \pi$:



Phase Space of Consecutive Mode Numbers



Summary and Conclusions

- TD limit of Heisenberg Ferromagnet is equivalent to LL model.
- Macroscopic excitations are contours in complex plane.
- Apparent singularity of the Bethe equations in the TD limit is always hidden in a condensate.
- Moduli space is connected: Mode numbers can change.
- Phase space is "filled": For all values of the moduli there are corresponding solutions to the Bethe equations.
- Unstable classical solutions are degenerate three-cut solutions that "decay" into two-cut solutions.

Numerical Results

Numerical verification:

[TB, Beisert, Gromov to appear]