Optics Match with Space Charge

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1. EoM

$$' = \frac{d}{dz}$$

$$x' = \frac{p_x}{p_z}$$

$$p_x = \frac{1}{v_z} F_x(x, y, z, t)$$

$$y' = \frac{p_y}{p_z}$$

$$p_y = \frac{1}{v_z} F_y(x, y, z, t)$$

$$t' = \frac{1}{v_z}$$

$$\mathcal{E}' = \frac{v}{v_z} F_{\parallel}(x, y, z, t)$$

$$X = X_r + \delta X$$

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ t \\ \mathcal{E} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t_r \\ \mathcal{E}_r \end{pmatrix} + \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta t \\ \delta \mathcal{E} \end{pmatrix}$$

$$X' = G(X, z)$$

$$\frac{d}{dz}(X_r + \delta X) = G(X_r, z) + \left(\frac{\partial}{\partial X_r} G(X_r, z) \right) \delta X + O(\delta X^2)$$

to 1st (linear) order:

- $$\frac{d}{dz} X_r = G(X_r, z)$$
- $$\frac{d\delta X}{dz} \approx \left(\frac{\partial}{\partial X_r} G(X_r, z) \right) \delta X$$

required for the following:

$$F_y(0,0,z,t) = F_y(0,0,z,t) = 0$$

$$F_z(x,y,z,t) = F_z(0,0,z,t) + O(x^2, y^2)$$

- $$\frac{dt_r}{dz} = \frac{1}{v_r}$$ $$\frac{d\mathcal{E}_r}{dz} = qE_z(0,0,z,t_r)$$

with $v_r = c\sqrt{1-(\mathcal{E}_r/\mathcal{E}_0)^{-2}}$

$p_r = mc\sqrt{(\mathcal{E}_r/\mathcal{E}_0)^2 - 1}$

- $$\frac{d\delta X}{dz} = M(z)\delta X$$

$$M = \begin{pmatrix} 0 & p_r^{-1} & 0 & 0 & 0 & 0 \\ v_r^{-1}F_{x,x} & 0 & v_r^{-1}F_{x,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_r^{-1} & 0 & 0 \\ v_r^{-1}F_{y,x} & 0 & v_r^{-1}F_{y,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_e^2/p_r^3 \\ 0 & 0 & 0 & 0 & F_{z,t} & 0 \end{pmatrix}$$

$$F_{a,b} = \frac{\partial}{\partial b} F_a(0,0,z,t_r)$$

longitudinal & transverse motion decoupled!

notation $\delta X \rightarrow X$

transport matrix:

$$X_b = T_{b\leftarrow a} X_b$$

with

$$\frac{d}{dz} T_{z\leftarrow a} = M(z) T_{z\leftarrow a}$$

here: solved by numeric integration

$$\frac{d}{dz} \det T_{z\leftarrow a} = \det T_{z\leftarrow a} \operatorname{spur} M(z) = 0 \quad \rightarrow \quad \det T_{z\leftarrow a} = \det T_{a\leftarrow a} = 1$$

trace space coordinates:

$$X \rightarrow \hat{X} = \begin{pmatrix} x & x' & y & y' & \delta s & \delta p \end{pmatrix}^t$$

with $\delta s = -v_r \delta t$, $\dfrac{\mathcal{E}_r + \delta\mathcal{E}}{\mathcal{E}_0} = \sqrt{\left(\dfrac{p_r}{m_e c}\left(1+\delta_p\right)\right)^2 + 1}$

$$X = K\hat{X}$$

$$K(a) = diag\left(1, p_r(a), 1, p_r(a), -1/v_r(a), v_r(a) p_r(a)\right)$$

$$\hat{X}_b = \hat{T}_{b\leftarrow a} \hat{X}_b$$

$$\hat{T}_{b\leftarrow a} = K(b)^{-1} T_{b\leftarrow a} K(a)$$

2. Linear Forces

drift (only space charge)

$$\rho(x,y,\delta s)=\rho(x,y)=\frac{I}{2\pi\sigma_x\sigma_y v_r}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$$

$$\mathbf{F}(x,y)\approx v_r p_r\left(xk_x^{(sc)}\mathbf{e}_x+yk_y^{(sc)}\mathbf{e}_y\right)+O\left(x^2,y^2\right)$$

with $k_{x/y}^{(sc)}=\frac{I}{I_A}\frac{2}{\gamma_r^3\beta_r^3}\frac{1}{\sigma_{x/y}\left(\sigma_x+\sigma_y\right)}$

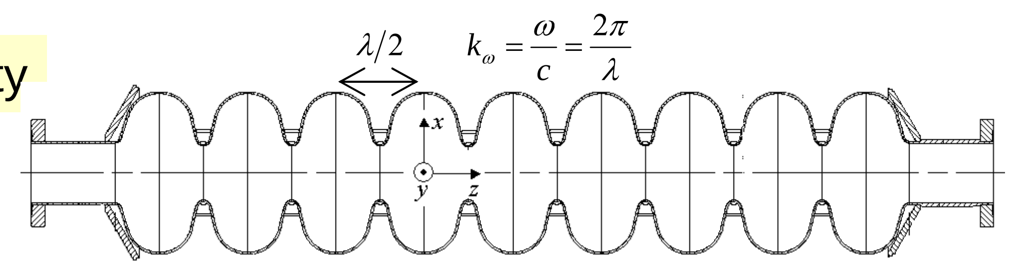
quadrupole

$$B_x(x,y)=-k\frac{p_r}{q_e}y \qquad B_y(x,y)=k\frac{p_r}{q_e}x$$

$$\mathbf{F}(x,y)=v_r p_r\left(xk_x^{(q)}\mathbf{e}_x+yk_y^{(q)}\mathbf{e}_y\right)$$

with $k_{x/y}^{(q)}=\mp k$

rf-sw-cavity



$$\mathbf{F}(x,y,\delta t)\approx v_r p_r\left(x\mathbf{e}_x+y\mathbf{e}_y\right)k^{(rf)}(z)+q_e E_z(z,t_r+\delta t)\mathbf{e}_z$$

with $k^{(rf)}(z)\approx\frac{q_e k_\omega E_0}{\left(p_r v_r\right)^2}\cos\left(\omega t_r+k_\omega z\right)$

drift (only space charge)

$$M^{(sc)}(\sigma_x,\sigma_y)=\begin{pmatrix} 0 & p_r^{-1} & 0 & 0 & 0 & 0 \\ k_x^{(sc)}p_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_r^{-1} & 0 & 0 \\ 0 & 0 & k_y^{(sc)}p_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_e^2/p_r^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

quadrupole

$$M^{(q)}\left(\cancel{\sigma_x,\sigma_y}\right)=\begin{pmatrix} 0 & p_r^{-1} & 0 & 0 & 0 & 0 \\ \left(k_x^{(q)}+\cancel{k_x^{(sc)}}\right)p_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_r^{-1} & 0 & 0 \\ 0 & 0 & \left(k_y^{(q)}+\cancel{k_y^{(sc)}}\right)p_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_e^2/p_r^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for optimization:
s.c. effects are
neglected in quads.

rf-sw-cavity

$$M^{(rf)}(\sigma_x,\sigma_y,z)=\begin{pmatrix} 0 & p_r^{-1} & 0 & 0 & 0 & 0 \\ \left(k^{(rf)}+k_x^{(sc)}\right)p_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_r^{-1} & 0 & 0 \\ 0 & 0 & \left(k^{(rf)}+k_y^{(sc)}\right)p_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -m_e^2/p_r^3 \\ 0 & 0 & 0 & 0 & F_{z,t} & 0 \end{pmatrix}$$

3. Envelope Equation

transverse EoM

$$\frac{d}{dz}\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} 0 & p_r^{-1} \\ k_x p_r & 0 \end{pmatrix}\begin{pmatrix} x \\ p_x \end{pmatrix}$$

with $k_x = k_x(z, \sigma_x, \sigma_y) = \begin{cases} \text{drift} \\ \text{quadrupole} \\ \text{cavity} \end{cases}$

$$x'' + \frac{p_r'}{p_r} x' - k_x x = 0$$

rms cross-section

coupled system

$$\sigma_x(z) = \sigma_x(a)\frac{u_x(z)}{u_x(a)}$$

with $$u'' + \frac{p_r'}{p_r} u' - \left(\frac{p_{r0}^2}{p_r^2}\frac{1}{u^3} + k_x u \right) = 0$$

$$u_x(a) = \sqrt{\beta_{x,a}}$$

$$u_x'(a) = -\frac{\alpha_{x,a}}{\sqrt{\beta_{x,a}}}$$

Twiss parameters $\varepsilon_{x,a}$ $\beta_{x,a}$ $\alpha_{x,a}$

an other approach is used here:

$$C^{(x)} = \varepsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \varepsilon_{x,a} \hat{T}^{(x)}_{z\leftarrow a} \begin{pmatrix} \beta_{x,a} & -\alpha_{x,a} \\ -\alpha_{x,a} & \gamma_{x,a} \end{pmatrix} \left(\hat{T}^{(x)}_{z\leftarrow a}\right)^t$$

$$C^{(y)} = \varepsilon_y \begin{pmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \gamma_y \end{pmatrix} = \varepsilon_{y,a} \hat{T}^{(y)}_{z\leftarrow a} \begin{pmatrix} \beta_{y,a} & -\alpha_{y,a} \\ -\alpha_{y,a} & \gamma_{y,a} \end{pmatrix} \left(\hat{T}^{(x)}_{z\leftarrow a}\right)^t$$

$$\sigma_x = \sqrt{\varepsilon_x \beta_x}$$

$$\sigma_y = \sqrt{\varepsilon_y \beta_y}$$

$$\frac{d}{dz} T^{(x)}_{z\leftarrow a} = M^{(x)}(\sigma_x, \sigma_y, z) T^{(x)}_{z\leftarrow a}$$

$$\frac{d}{dz} T^{(y)}_{z\leftarrow a} = M^{(y)}(\sigma_x, \sigma_y, z) T^{(y)}_{z\leftarrow a}$$

numerical integration:

f.i. cavity with amplitude, phase, frequency, number of cells, current

$$\left[\hat{T}_{b\leftarrow a}, t_{r,b}, \mathcal{E}_{r,b}, C_b^{(x)}, C_b^{(y)}\right] = \hat{T}\left(\text{properties}, t_{r,a}, \mathcal{E}_{r,a}, C_a^{(x)}, C_a^{(y)}\right)$$

space charge parameter

compares averaged focusing strength with s.c. force

$$x'' + \frac{p_r'}{p_r} x' - \left(k_x^{(0)} + k_x^{(sc)}\right) x = 0$$

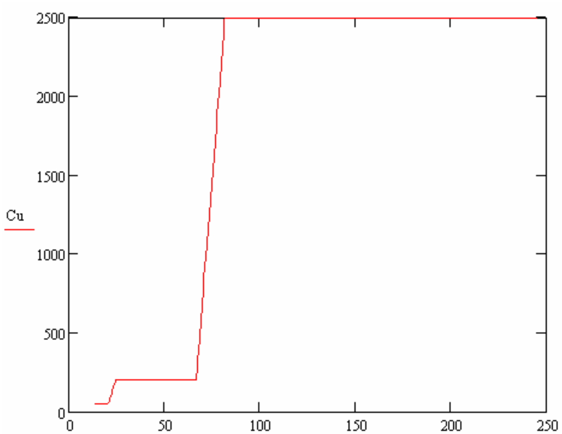
$$k_x^{(0)} \approx 1/\beta_x^2$$

$$\hat{S} = \frac{k^{(sc)}}{1/\beta^2} \qquad \boxed{\hat{S} = \frac{I}{I_A} \frac{\beta}{\gamma^2 \varepsilon_n}}$$

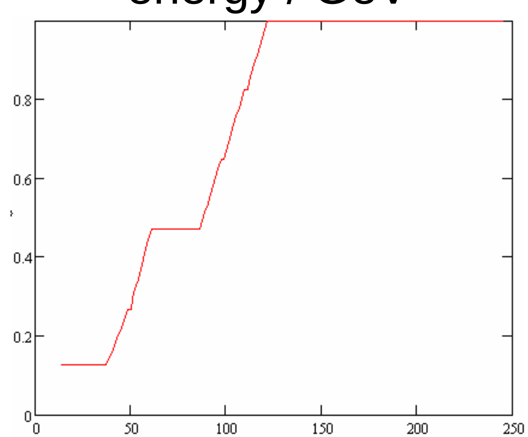
with β, $\varepsilon_n = \varepsilon\gamma$ Twiss parameters
γ Lorentz factor, $I_A = 17$ kA Alven current

f.i. Flash

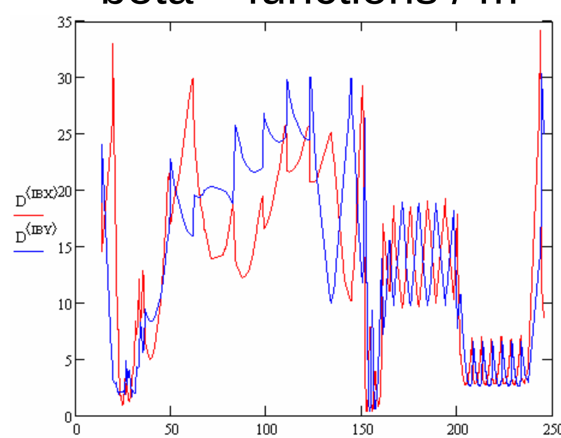
current / A



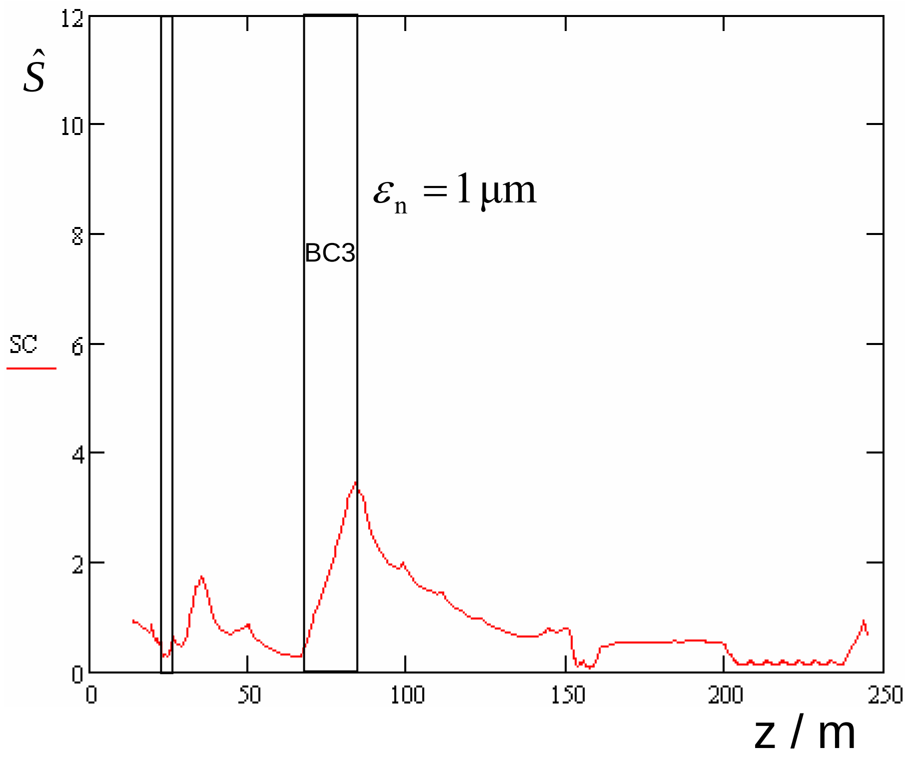
energy / GeV

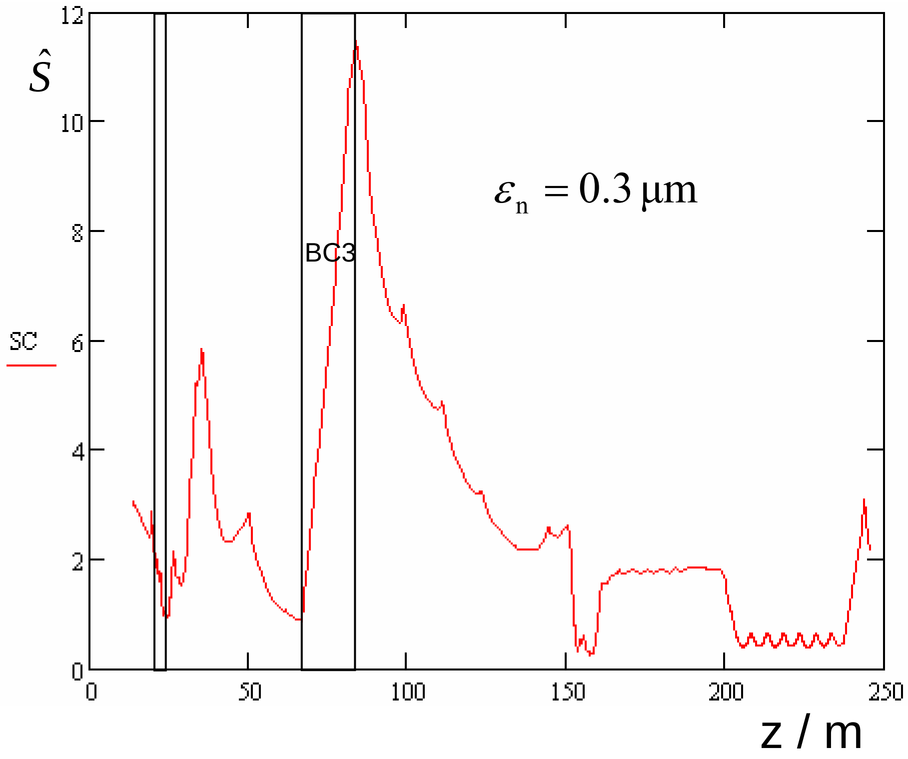


beta – functions / m



$$\hat{S} = \frac{I}{I_A} \frac{\beta}{\gamma^2 \varepsilon_n}$$

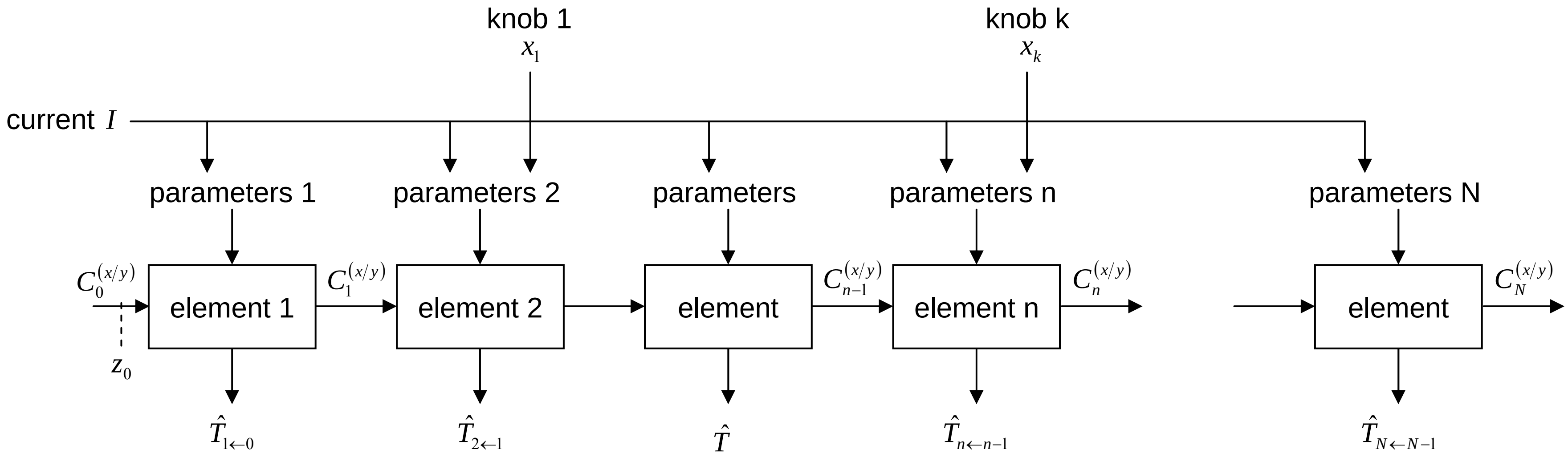




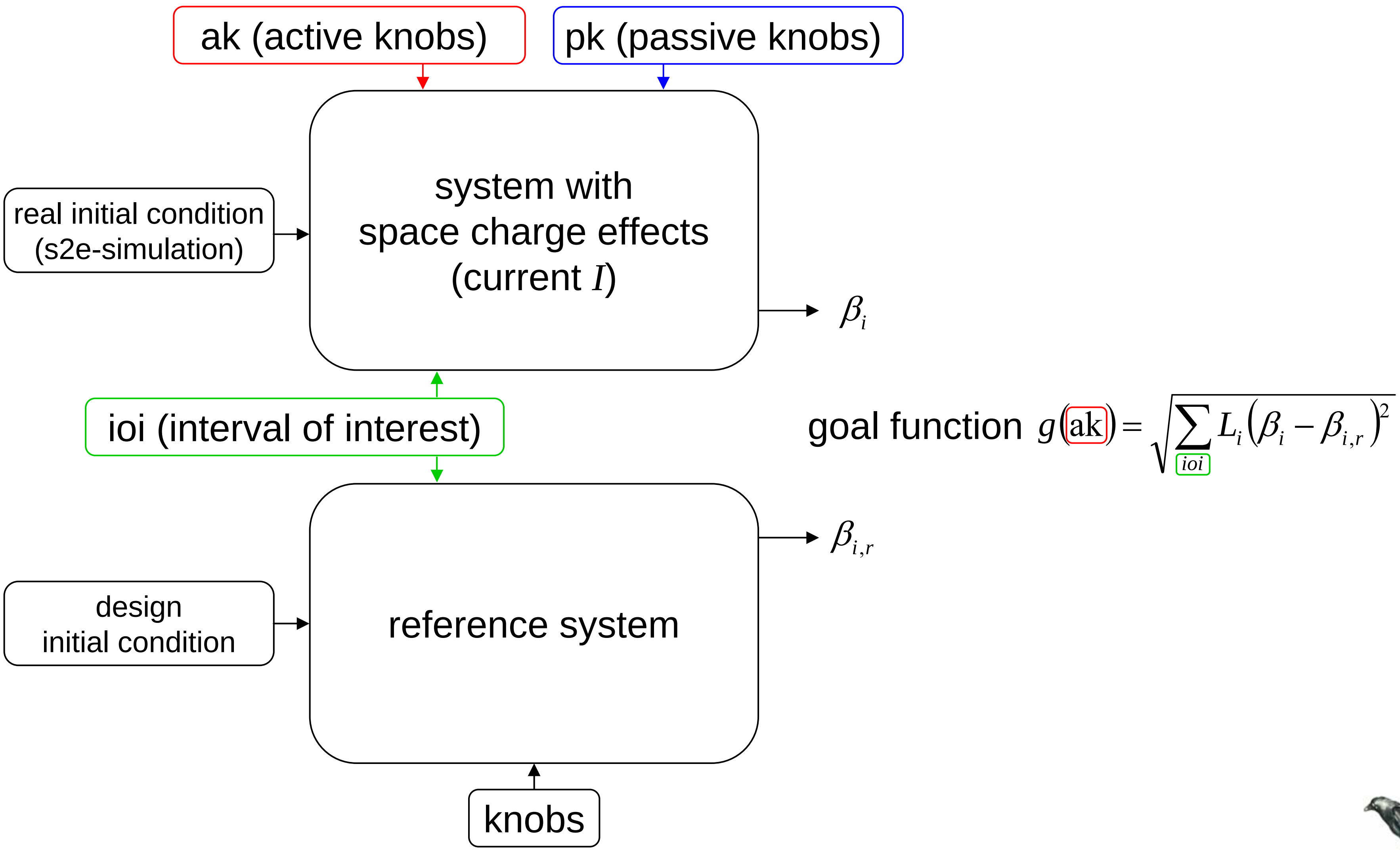
4. Optimization I

setup, knobs, goal function

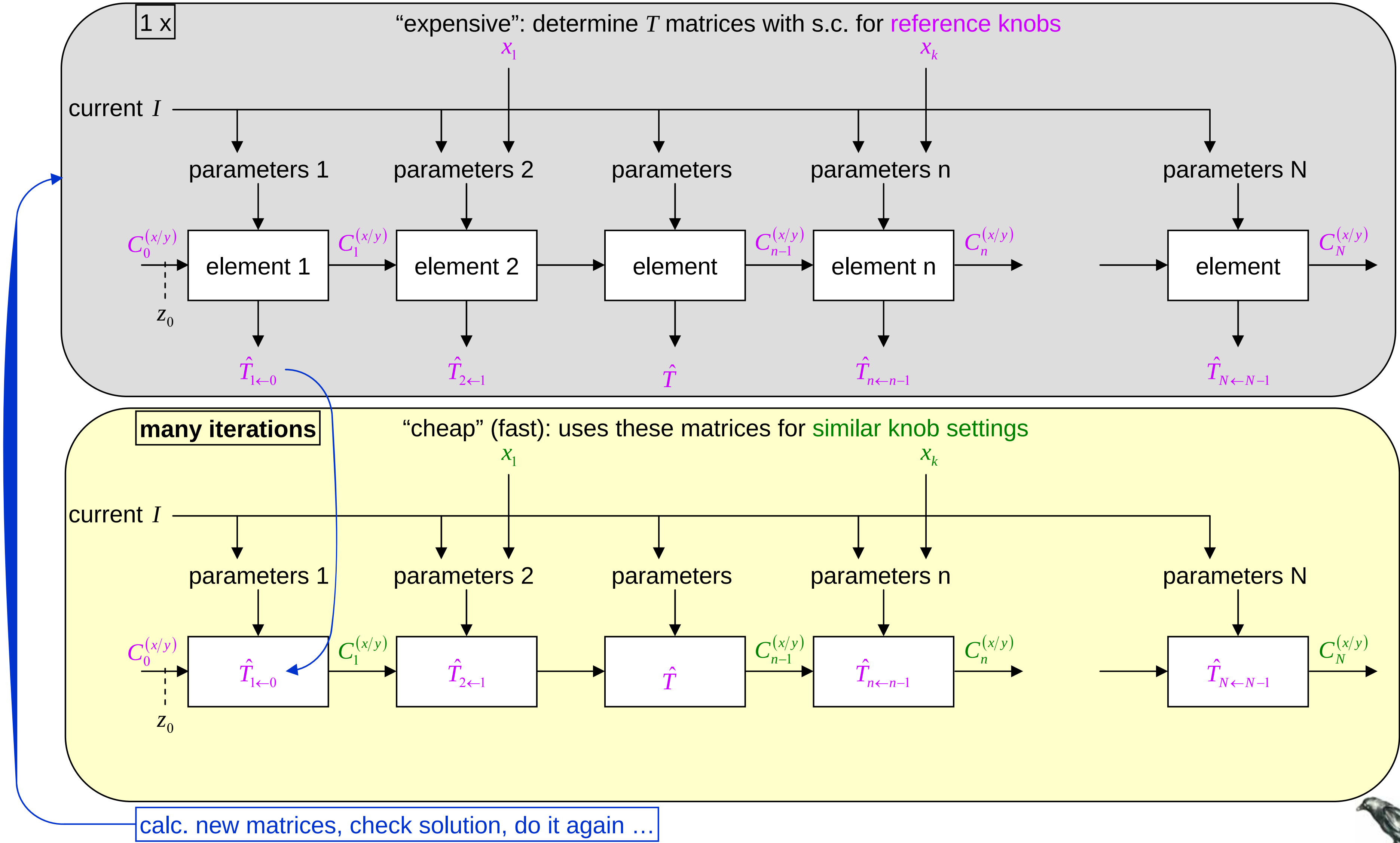
setup & knobs:



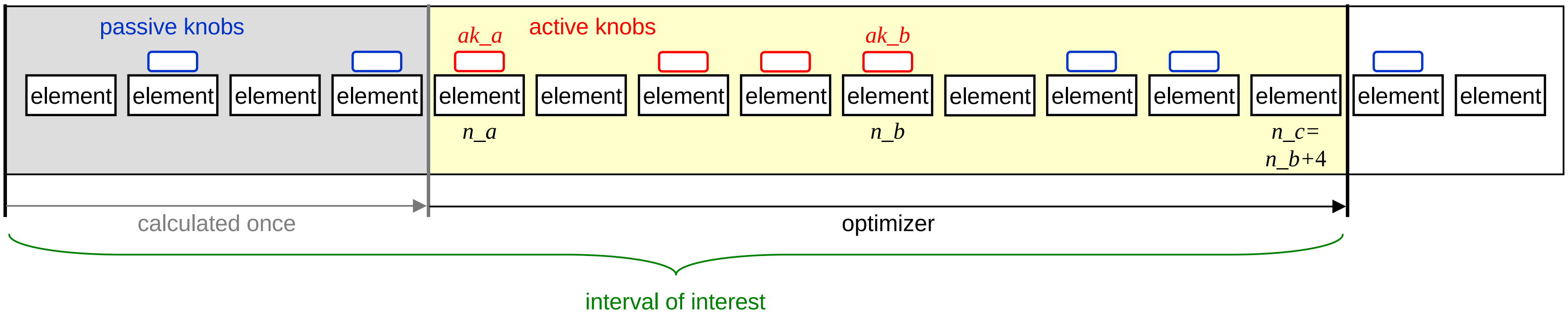
active & passive knobs, interval of interest, goal function:



cheap method:



5. Optimization II
many knobs, using an optimizer



ILO(ak_a, ak_b) = inner loop of optimization

set interval of interest ($n\_c = n\_b+1$)
calculate by "expensive method" $\rightarrow$ T matrices, initial value of goal function g_i
(*) **optimize** with cheap method $\rightarrow$ value of goal function g_c
calculate by "expensive method" $\rightarrow$ T matrices, value of goal function g_e
if $g_e < g_i$: replace initial setting by improved setting; goto (*)
otherwise: exit with last improved setting

optimization as it has been used for the following examples:

local optimization

ILO(1, 4)

ILO(2, 5)

…

ILO(K-3, K) K = number of last knob

$\rightarrow$ solution 1 (few minutes)

total optimization

ILO(1, K)

$\rightarrow$ solution 2 (few minutes)

optimization of all knobs by “expensive” method

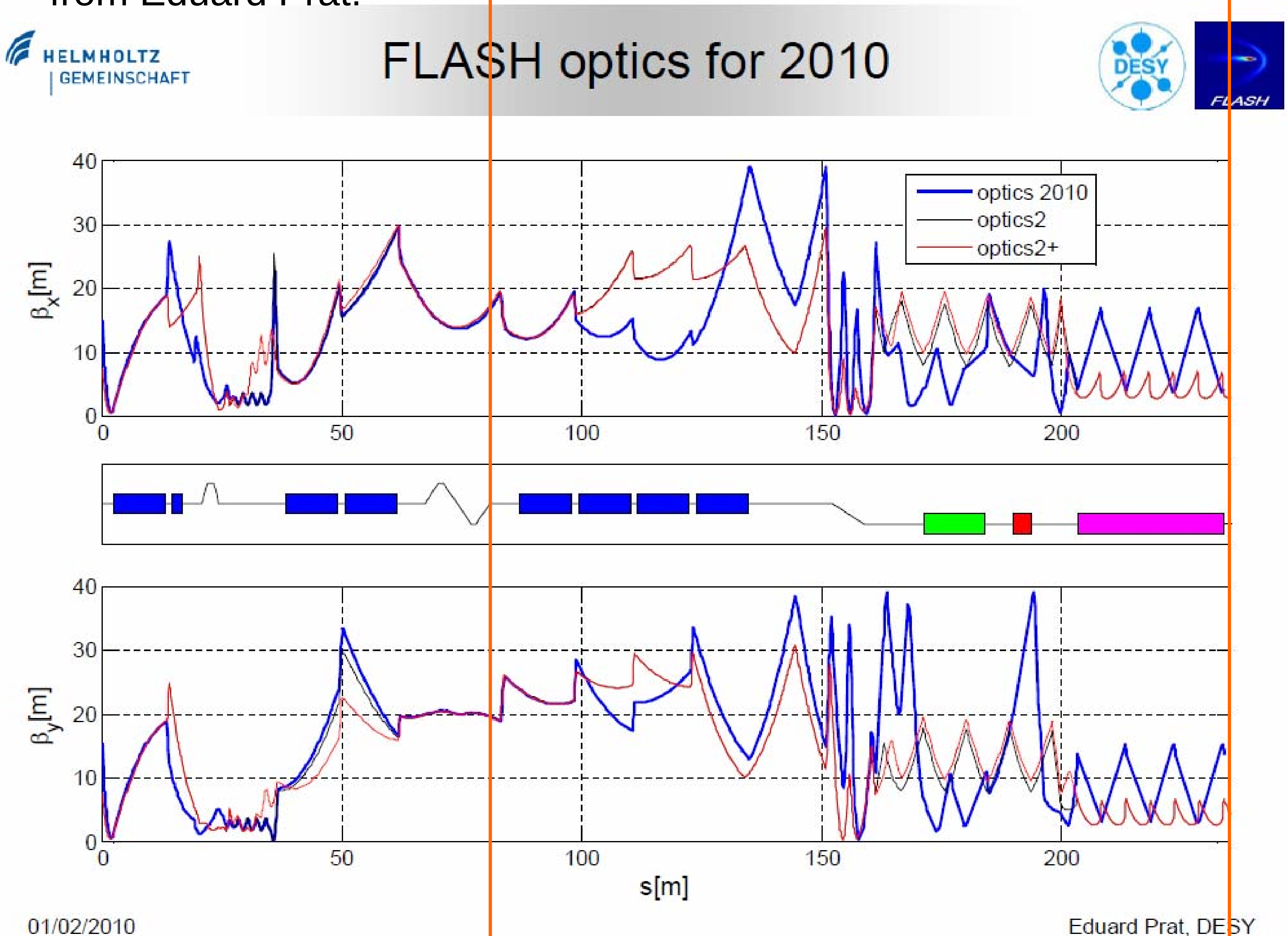
$\rightarrow$ solution 3 (some hours)

solution 2 is usually good enough

6. Examples

example 2

from Eduard Prat:

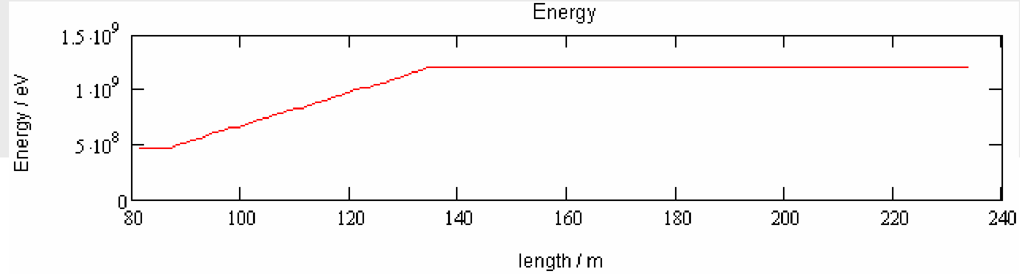


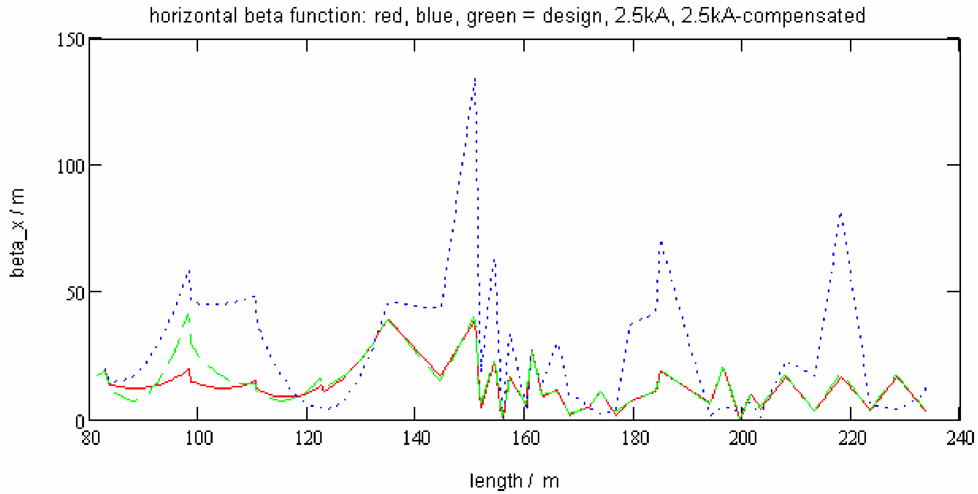
optic 2010 for 2.5kA
end of BC3 → end of undulator (46 knobs)

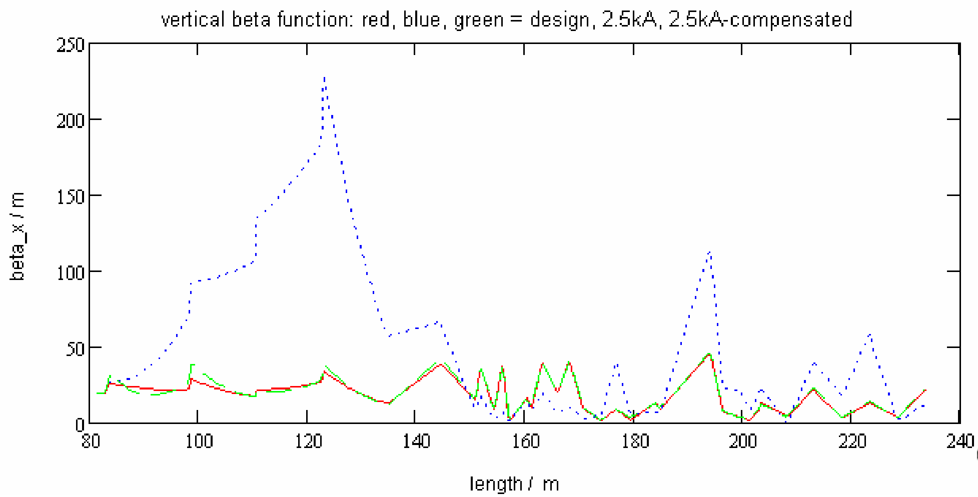
correction for:

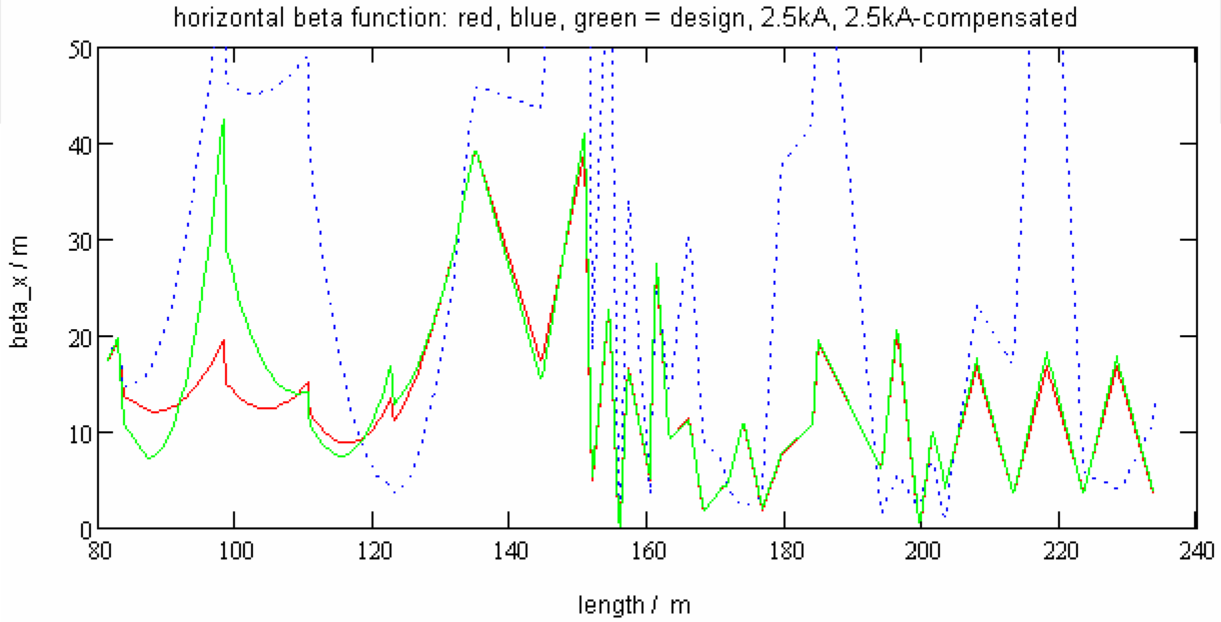
```
z_0=81.45992;
Ene_0=470E6;
emit_x=1.0E-6/(Ene_0/E_ele_eV);
Emit_y=emit_x;
alf_x=-0.5134696; bet_x=17.29299;
alf_y= 0.1942423; bet_y=19.37591;
```

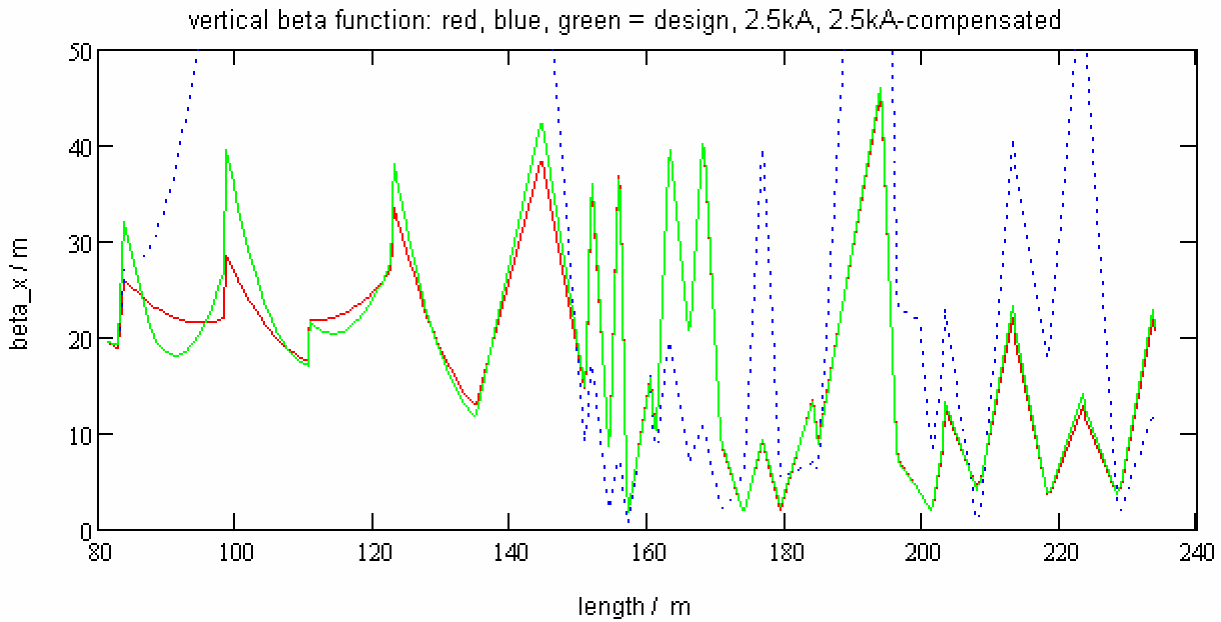
I=2500.0

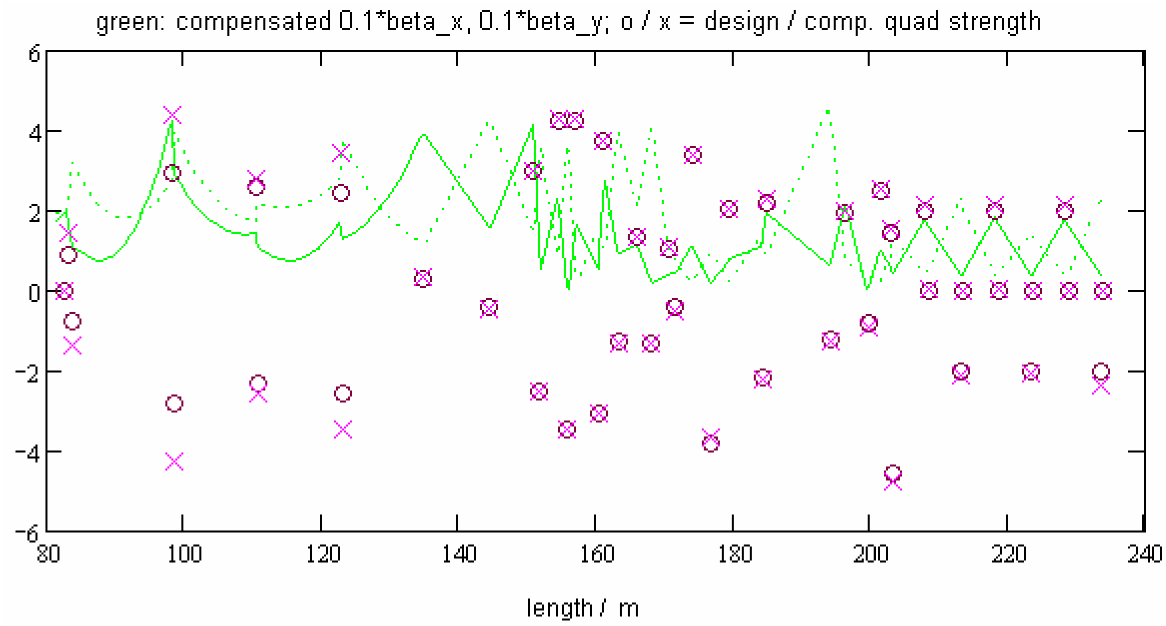







horizontal beta function: red, blue, green = design, 2.5kA, 2.5kA-compensated
beta_x / m
length / m


vertical beta function: red, blue, green = design, 2.5kA, 2.5kA-compensated
beta_x / m
length / m



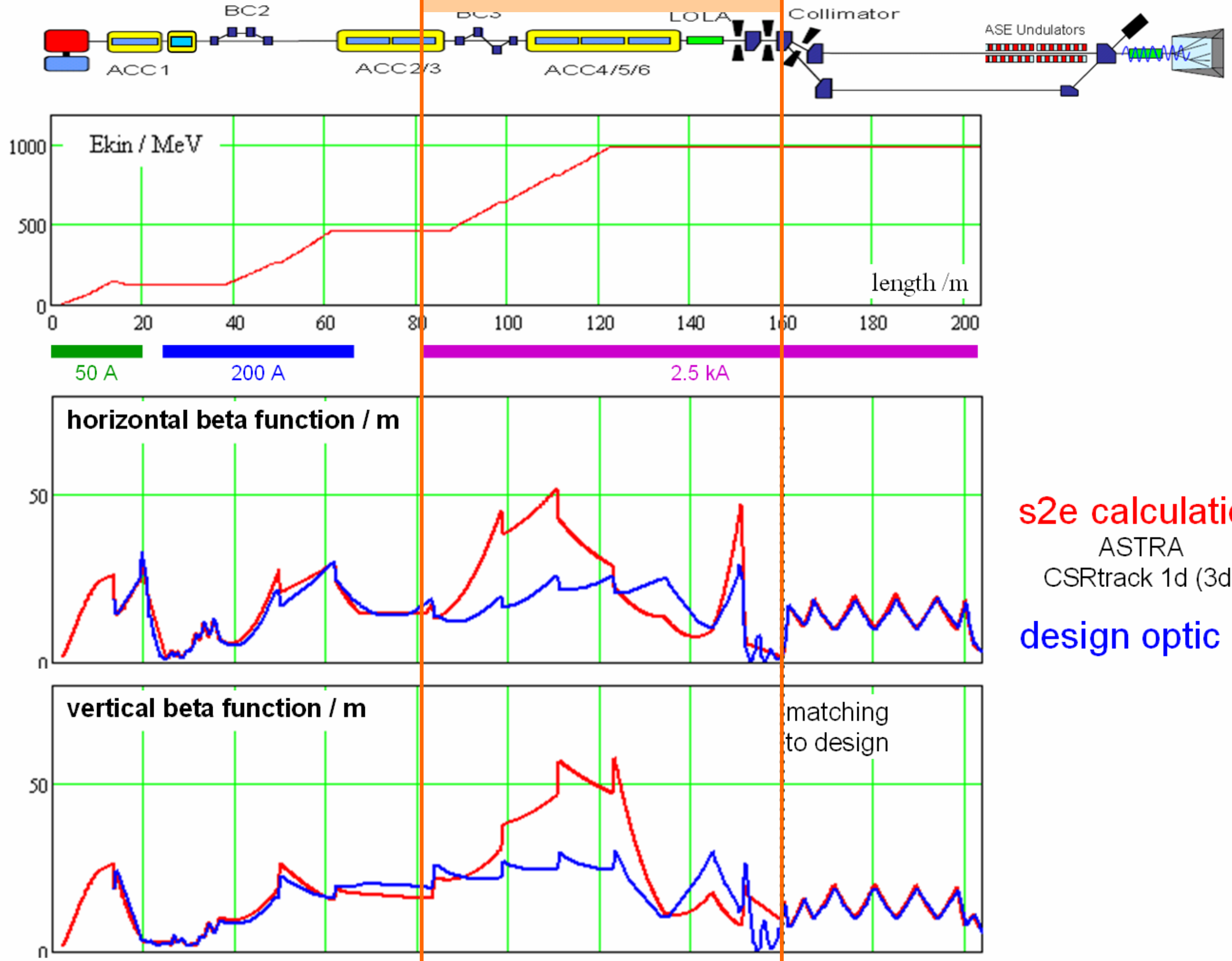
quad. strength, design

quad. strength, corrected

example 1: (from s2e seminar, 2010.02.01)

Transverse Dynamic

end of BC3 →
start of collimator
2.5 kA, 13 knobs

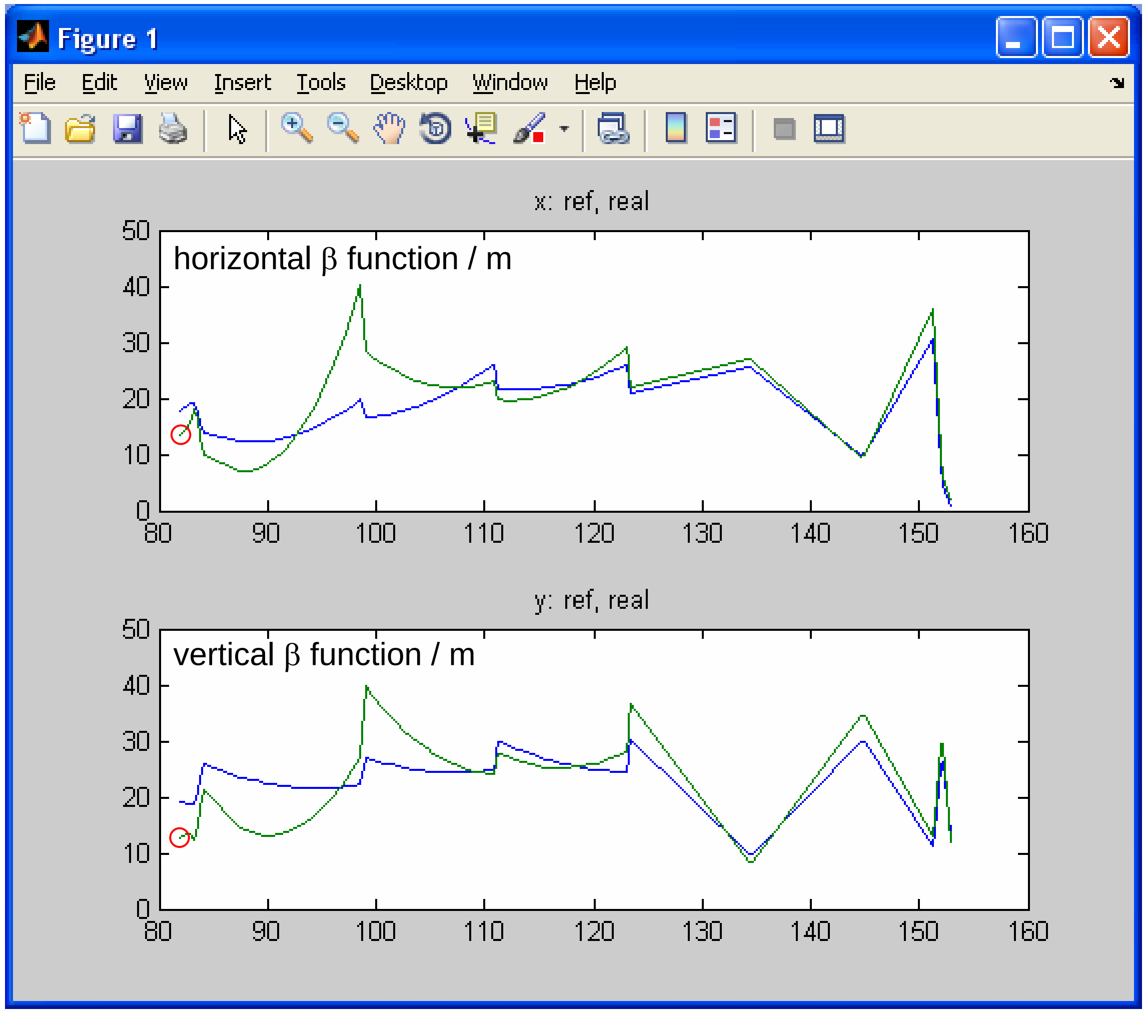


s2e calculation
ASTRA
CSRtrack 1d (3d)

design optic (2+)

design optic

optic with s.c.



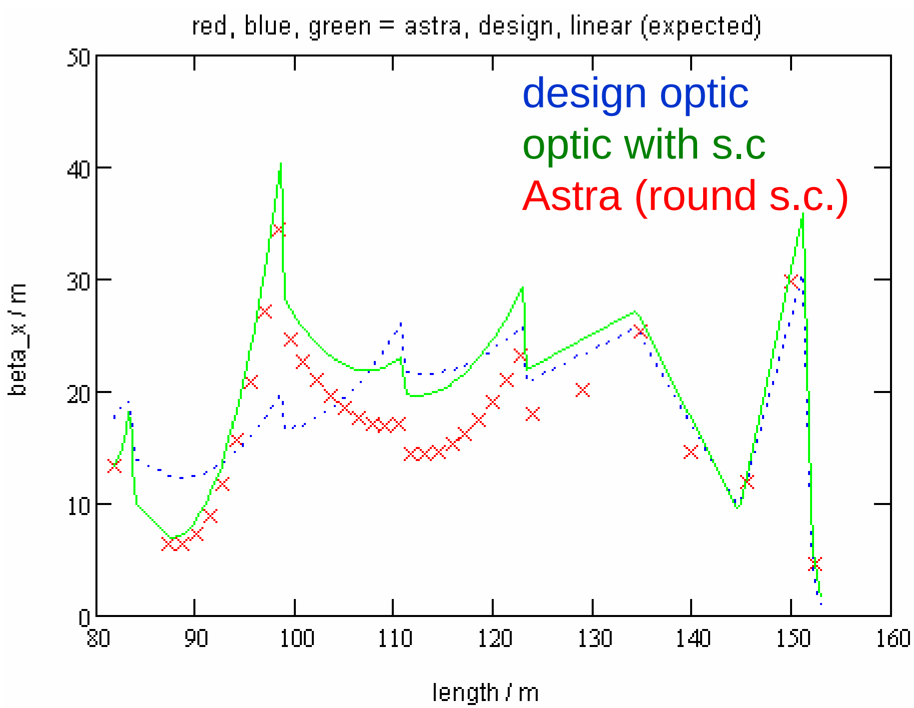
optics after BC3
(s2e simulation)
deviates from design

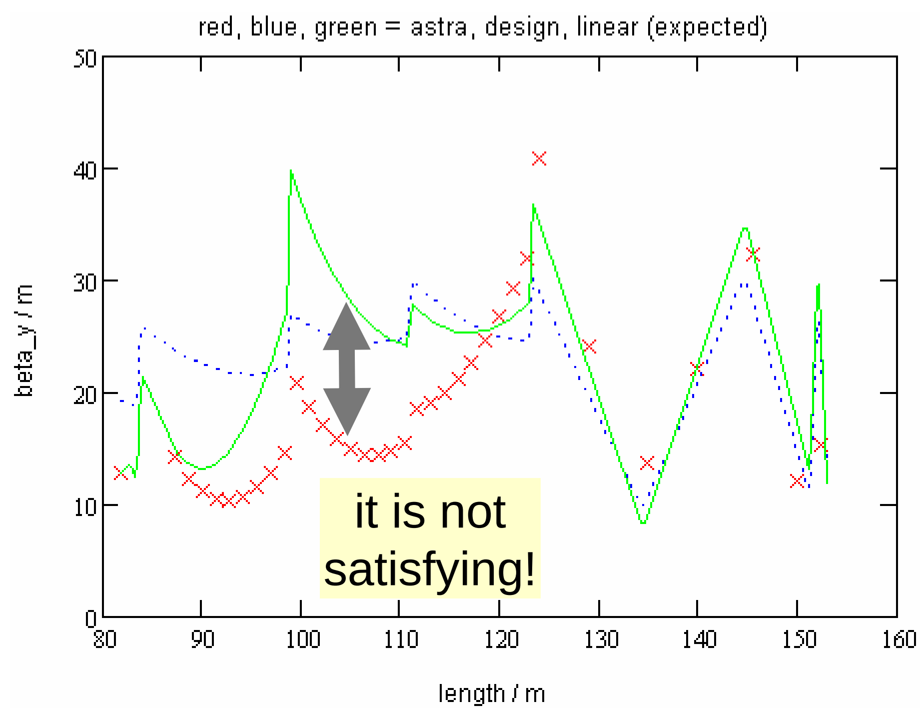
7. Verification & work to be done

how good is the model of linear space charge forces?

$$\mathbf{F}(x,y) \approx v_r p_r \left(x k_x^{(sc)} \mathbf{e}_x + y k_y^{(sc)} \mathbf{e}_y \right) + O\left(x^2, y^2\right)$$

for example 1





possible reasons: non linearity, non gaussian distribution, round Astra model

but it is an improvement
compare again with s2e seminar, 2010.02.01: