Some reduced holonomy in dimensions 7 and 8

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Examples involving $G_2$ and $\text{Sp}(2)\text{Sp}(1)$
Background
Differentials
Results
History

Q

AHS, HKLR, Galicki–Lawson, Joyce, Dancer–Swann

G2Q

Work in progress by Salamon–Gambioli–Lonegro based on that of Battaglia

G2

Bryant, Hitchin, Gukov, Atiyah–Witten–Acharya
Basic constructions

$Y^8$  \[ \text{holonomy}=\text{Sp}(2)\text{Sp}(1) \]

$X^7$  \[ \text{holonomy}=\text{G}_2 \]

$SO(4) \subset G_2$

$M^4$  Einstein+ASD
Basic constructions

\[ Y^8 = \Sigma_M \]

\[ C^2/S^1 \cong \mathbb{R}^3 \]

\[ S^1 \]

\[ X^7 = \Lambda^2 T^*M \]

\[ \mathbb{R}^3 \]

\[ M^4 \]

\[ \text{QK} \]

\[ \text{G2} \]

?
SO(5) example

\[ [e^{it}q_0, q_1, q_2] \quad * = [1, 0, 0] \]

\[ S^4 = \text{HP}^1 \]

\[ \Lambda^2 T^* S^4 \]

\[ \mathbb{H}P^2 \setminus * \]

\[ S^1 \]

\[ \mathbb{R}^3 \]
Suppose that $S^1$ acts freely on a quaternion-Kähler manifold $Y^8$ with $s \neq 0$. Then $X^7 = Y/S^1$ has a “natural” metric with holonomy $G_2$.

The “proof” relies on the QK quotient construction and an identification of normal bundles.
SU(3) example

$S^5 \rightarrow Y = \mathbb{HP}^2 \rightarrow \mathbb{CP}^2$

$S^1 \downarrow$

$\mathbb{HP}^2 / S^1 \cong S^7$

$X = \Lambda^2 T^* \mathbb{CP}^2$
Background

Differentials

Results
fixed Riemannian metric $g$
local basis $I_1, I_2, I_3$ of acs's
local basis $\omega_1, \omega_2, \omega_3$ of 2-forms

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i$$  QK 4-form
QK structure of $Y^8$

fixed Riemannian metric $g$
local basis $I_1, I_2, I_3$ of acs’s
local basis $\omega_1, \omega_2, \omega_3$ of 2-forms

$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i$ has stabilizer $\text{Sp}(2)\text{Sp}(1)$

cf. $\omega_1^2 + \omega_2^2 - \omega_3^2$ has $\text{Spin} \ 7$
QK structure of $Y^8$

fixed Einstein metric $g$

local basis $I_1, I_2, I_3$ of acs’s

local basis $\omega_1, \omega_2, \omega_3$ of 2-forms

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i$$

$\nabla \Omega \equiv 0$

cf. $\omega_1^2 + \omega_2^2 - \omega_3^2$ has Spin 7
A single $S^1$ acts on the QK manifold $Y^8$ by isometries with Killing vector field $X$.

Suppose that $s \neq 0$

\[ \mathcal{L}_X g = 0, \quad \mathcal{L}_X \Omega = 0 \]
Goal

Verify the theorem by the explicit construction of closed forms on $Y^8/S^1$ as in the case $SU(3) \sim G_2$

Difficulties:
(i) the 3-form $X^\flat \Omega$ is not of $G_2$ type
(ii) identification of $Y^8/S^1$ with $\Lambda^2 T^*$
Moment norms

Def

\[ \nabla X = \eta + \xi \in sp(1) + sp(2) \]

\[ f = \frac{1}{2} \| \eta \|^2, \quad g = \frac{1}{2} \| X \|^2 \]
\[ \nabla X = \eta + \xi \in \text{sp}(1) + \text{sp}(2) \]

\[ f = \frac{1}{2} \| \eta \|^2, \quad g = \frac{1}{2} \| X \|^2 \]

\[ I_1 = \frac{1}{\sqrt{2f}} \eta, \quad \alpha_0 = \frac{1}{\sqrt{2f}} X^b \]

\[ \frac{1}{2f} df = I_1 \alpha_0 = \alpha_1 \]
$I_1$ is a complex structure on $Y' = \{ y \in Y^8 : f(y) > 0 \}$

\[
\begin{align*}
\nabla I_1 &= \alpha_2 \otimes I_2 + \alpha_3 \otimes I_3 \\
\nabla I_2 &= -\alpha_2 \otimes I_1 + \beta \otimes I_3 \\
\nabla I_3 &= -\alpha_3 \otimes I_1 - \beta \otimes I_2
\end{align*}
\]

$\alpha_i = I_i \cdot \alpha_0$
Equivariant 4-form

\[ \hat{d} = d + x X \]

\[ \hat{\Omega} = f x^2 - \eta x + s \Omega \]

\[ \hat{d} \hat{\Omega} = 0 \]

\[ df = X[\eta], \quad d\eta = s X[\Omega] \]
### List of differential forms

<table>
<thead>
<tr>
<th>deg</th>
<th>forms on $Y^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$f, g$</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta, dg$</td>
</tr>
<tr>
<td>2</td>
<td>$\eta = \sqrt{2f}\omega_1, \omega_2, \omega_3, \xi$</td>
</tr>
</tbody>
</table>
| 3   | $\psi^+ = \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3$  
$\psi^- = \alpha_2 \wedge \omega_3 - \alpha_3 \wedge \omega_2$ |
| 4   | $\Omega$       |
Background
Differentials
Results
The twistor 2-sphere

\[ S^2 = \{ a I_1 + b I_2 + c I_3 : a^2 + b^2 + c^2 = 1 \} \]

\[
\begin{pmatrix}
0 & \alpha_2 & \alpha_3 \\
-\alpha_2 & 0 & \beta \\
-\alpha_3 & -\beta & 0
\end{pmatrix}
\]

\[ \alpha_i = I_i \cdot \alpha_0 \]
Key lemma

If $s > 0$ then

$$d\beta = \omega_1 + \alpha_2 \wedge \alpha_3$$

is a Kähler 2-form for $I_1$

Proved using both

$$\nabla I_3 = -\alpha_3 \otimes I_1 - \beta \otimes I_2$$

$$0 = d(d\omega_3) = d(-\alpha_3 \wedge \omega_1 - \beta \wedge \omega_2)$$

and fact that $\omega_3 \otimes \xi$ wedges to 0
Key lemma

If $s > 0$ then
\[ d\beta = \omega_1 + \alpha_2 \wedge \alpha_3 \]
is a Kähler 2-form for $I_1$

This is $1/2$ the curvature 2-form of the line bundle
\[ \langle \omega_2 + i\omega_3 \rangle^{\otimes 2} \cong K_1 = \Lambda^{4,0} \]
The examples

**SO(5) actions**

- $f = f(r), \ g = g(r)$
- $f = g = 0$ on $S^4$
- bad!

**SU(3) actions**

- $f = 0$ on $S^5$
- $g = 0$ on $\mathbb{C}P^2$
- good!
More SU(3) actions

$f = 0$

$S^5$

$g = 0$

$\mathbb{HP}^2$

$\mathbb{CP}^2$

$G_2(C^4)$

both Kähler and QK
Instanton twist

\[ \Lambda_2^2 T^* \mathbb{C}P^2 \]

\[ S^1 \]

\[ \mathbb{R}^3 \]

\[ \mathbb{C}P^2 \]

\[ \omega \in \Lambda_2^2 T^* \]
More SU(3) actions

\[ A \rightarrow gA g^{-1} \]

\[ S^5 \quad SU(3) \quad \frac{SU(3)}{SO(3)} \]

\[ \mathbb{CP}^2 \quad G_2/SO(4) \quad \frac{SU(3)}{SO(3)} \quad QK \]
A calibration

Consider \( \varpi = \omega_1 - f \alpha_2 \wedge \alpha_3 \)

If \( S^1 \) acts freely on \( Y^8/S^1 \) then
\[
d\varpi = (1+f)\psi^+ - 2\sqrt{fg} \alpha_1 \wedge \alpha_2 \wedge \alpha_3
\]
is a 3-form with stabilizer \( G_2 \)

Exercise: find the torsion \( d^*d\varpi \in \mathfrak{X}_2 \)
Hypersurface geometry

We work on $X^6_c = f^{-1}(c)/S^1$ with the stable exact 3-form

\[ \psi^+ = \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3 \]

\[ = d\omega_1 = -d(\alpha_2 \wedge \alpha_3) \]

\[ \psi^+ \wedge \psi^- = \frac{g}{f}(qvol) \]
A half-flat SU(3) structure

We seek a compatible 2-form:

\[ \begin{cases} 
\omega \wedge \psi^+ = 0 \\
\omega \wedge \omega \wedge \omega = c \psi^+ \wedge \psi^- \\
d(\omega \wedge \omega) = 0 
\end{cases} \]

\[ \omega = \omega_1 - \left(1 + \frac{f}{g}\right) \alpha_2 \wedge \alpha_3 \]

\[ d\omega = (2 + \frac{f}{g}) \psi^+ + \frac{f}{g^2} dg \wedge \alpha_2 \wedge \alpha_3 \]
Suppose that $S^1$ acts freely on a quaternion-Kähler manifold $Y^8$. Suppose that $\text{grad} \|X\| \in \mathbb{H}X$. Then $(\omega, \psi^+)$ makes $f^{-1}(c)/S^1$ half-flat and a neighbourhood of it is $G_2$. 