T-duality with $H$-flux and non-geometry: insights from generalized complex branes

Pascal Grange

II. Institut für theoretische Physik, Universität Hamburg

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Outline of the talk

1. Non-geometry from $H$-flux: non-commutative fibrations and T-folds were proposed.

2. Branes as maximally isotropic subspaces in generalized complex geometry. Branes of type A and B correspond to pure spinors of type 0 and type three.

3. Probing non-geometry by generalized complex branes: monodromies correspond to $\beta$-transforms and type-jumping phenomena.

4. Claim: *Non-commutative fibrations are the open-string version of T-folds.*

5. Generalized branes and moduli fixing.
The non-geometric T-dual of $T^3$ with uniform $H$-flux

- We are concerned with problems that occur when T-duality is performed along two directions carrying non-zero components of a $B$-field.
- Consider a flat $T^3$ as a trivial $T^2_{yz}$-fibration over $S^1_x$, equipped with $k$ units of $H$-flux,

$$H = k \, dx \wedge dy \wedge dz, \quad B := kx \, dy \wedge dz,$$

and perform T-duality along the directions $y$ and $z$.
- The resulting metric shrinks under monodromy around the base:

$$ds'^2 = dx^2 + \frac{1}{1 + (kx)^2}(dy^2 + dz^2).$$
Proposed solutions to the puzzle involve a generalization of the notion of compactification

- The **T-fold** proposal [Hull 2004]: double the fibers and use T-dualities to glue T-dual two-tori together.

- The **non-commutativity** proposal [Mathai–Rosenberg 2004]: The T-dual background is a fibration by non-commutative two-tori with non-commutativity scales $kx$.

- **More structure from generalized complex (GC) geometry.** Let us embed the three-torus into a GC six-torus and probe the T-dual space by GC branes.
Consider an \( n \)-dimensional vector space \( V \). A \( Dp \)-brane wrapping some subspace \( E \) with zero field strength turned on has an annihilator (a subspace of \( V^* \)) with equation

\[
\xi|_E = 0.
\]

The \( p \) tangent directions and the \( (n - p) \) transverse directions make for an \( n \)-dimensional subspace of \( V \oplus V^* \)

\[
E \oplus \text{Ann} E = \{X + \xi, X \in E, \xi \in V^*: \xi(X) = 0\}.
\]

\( B \)-transforms of this subspace give all possible maximally isotropic subspaces of \( V \oplus V^* \) endowed with a natural inner product.
Linear algebra on $V \oplus V^*$

- Consider the inner product on $V \oplus V^*$:

  $$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi).$$

It has signature $(n, n)$. Isotropic subspaces therefore have dimension $n$ at most.

- A representation of the Clifford algebra induced by the action of $V \oplus V^*$ is carried by $\wedge \bullet V^*$:

  $$(X + \xi).\phi = \iota_X \phi + \xi \wedge \phi,$$

  $$(X + \xi).((X + \xi).\phi) = \langle X + \xi, X + \xi \rangle \phi.$$
Symmetries of the pairing and the $B$-field transform

- The Lie algebra of $SO(V \oplus V^*)$

\[ so(V \oplus V^*) = \{ r, \langle rx, y \rangle + \langle x, ry \rangle = 0, \forall x, y \in V \oplus V^* \} \]

contains the transformations by two-forms ($B$-transforms)

\[ X + \xi \mapsto X + \xi + \iota_X B, \]

and the transformations by bivectors ($\beta$-transforms)

\[ X + \xi \mapsto X + \iota_\xi \beta + \xi. \]

- $E \oplus \text{Ann} E$ is a maximally isotropic subspace of $(V \oplus V^*, \langle , \rangle)$. So is its $B$-transform.
Generalized tangent bundle

- Given a subspace $E$ of $V$ and $F \in \Lambda^2 E^*$, one can exhibit a maximally isotropic subspace as the graph of $F$ over $E$:

$$L(E, F) := \{X + \xi \in E \oplus V^*, \xi|_E = \iota_X F\}.$$ 

Moreover every maximally isotropic subspace is of this form. A $B$-field transform acts by shifting the two-form by $B$.

- Let $M$ be a $p$-dimensional submanifold of an $n$-dimensional manifold $N$, carrying a $U(1)$-bundle with gauge curvature $F$. The generalized tangent bundle of $M$ is

$$\tau^F_M := \{X + \xi \in TM \oplus TN^*, \xi|_M = \iota_X F\}.$$
Generalized complex structures

- A generalized almost complex structure on $V \oplus V^*$ is an almost complex structure that is orthogonal w.r.t. $\langle \cdot, \cdot \rangle$. Examples are induced by a complex structure $J$ or a symplectic structure $\omega$ on $V$:

$$
\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & \omega \\ -\omega & 1 \end{pmatrix},
$$

that correspond to pure spinors $\Omega$ (type three) and $e^{i\omega}$ (type zero) respectively.

- Defining condition for generalized complex submanifolds (or GC branes?) (Gualtieri 2003): stability of the generalized tangent bundle under the GC structure.
GC submanifolds for $\mathcal{J} = \mathcal{J}_\omega$ or $\mathcal{J} = \mathcal{J}_J$

- Considering the structure $\mathcal{J}_J$ gives rise to D-branes of type B, namely $E$ has to be a complex submanifold of $M$ and

$$J^*(\iota_X F) = \iota_{JX} F, \text{ i.e. } F^{(0,2)} = 0.$$ 

- Considering the structure $\mathcal{J}_\omega$ gives rise to D-branes of type A, including non-Lagrangian ones carrying non-zero field strength:

$$e^{-B} \mathcal{J}_\omega e^B = \begin{pmatrix} -\omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & B\omega^{-1} \end{pmatrix}.$$ 

- Moreover, the corresponding pure spinors and stability conditions enjoy good mirror-symmetric properties.
What happens to a pure spinor under monodromy?

- Monodromy around the circle $S^1_x$ with $H$-flux is a shift of the $B$-field,

$$g(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -kx & 1 & 0 \\
-kx & 0 & 0 & 1
\end{pmatrix},$$

which upon conjugation by the $O(2,2)$ matrix encoding T-duality along the fiber becomes a $\beta$-transform.

- Such a $\beta$-transform alters the type of a pure spinor! In particular, there can be no global choice of a holomorphic three-form. The generalized B-model is non-geometric in that sense, and meets Hull’s criterion because the choice of a pure spinor with definite type can only be made locally.
The topological B-model on GC target space admits non-commutative deformations

With a holomorphic Poisson bivector $\beta^{\mu\nu}$, the master action of the B-model is transformed into the action of the Poisson sigma model [Pestun 2006]:

$$S + \delta_\beta S = \int_{\Sigma} (\xi_\mu dz^\mu + \beta^{\mu\nu} \xi_\mu \xi_\nu),$$

whose correlation functions on the boundary of a disc involve star products derived from $\beta$ [Kontsevich, Cattaneo–Felder 1999]:

$$\int_{X(\infty)=a} DX f(X(0))g(X(1)) e^{i(S+\delta_\beta S[X])} = f \star_{\beta(a_x)} g(a).$$

This deformation is precisely the contribution of the $\beta$-transform to the master action.

So a D-brane wrapping torus fiber at point $x$ is endowed with non-commutativity scale $\theta = kx$. 
Non-commutative $T^2$-fibrations are the open-string version of T-folds: uncertainty principle for D0-branes is the manifestation of the lack of global type-three pure spinor.

The T-dual of a torus with $H$-flux along two legs of the $B$-field gives a physical realization of the jumping phenomenon exhibited by Gualtieri.
Non-commutativity and the uncertainty principle on D-branes revisited

- A D0-brane is defined by a set of holomorphic equations setting coordinates to definite values. But the non-commutativity scale prevents from localizing a point within an arbitrarily small area.

- One position modulus becomes a non-commutativity scale, and a D0-brane can only sit on the torus fiber above $x = 0$. 
Moduli fixing and pure spinors

• $\Omega$ is generically replaced by an odd polyform

$$\psi^- =: \psi^{(1)} + \psi^{(3)} + \psi^{(5)}.$$

Supersymmetry equations may be interpreted as D-flatness and F-flatness conditions [Koerber, Martucci, Smyth]. In particular, a critical point of the superpotential corresponds to the vanishing of the one-form part.

$$\psi^{(1)}|_{\text{pt}} = 0.$$

• Curiosity: mirror symmetry does not give rise to a bivector that is decomposable as the skew product of two vectors: intricate non-commutativity scale.

• A D3-brane can therefore only sit at a point where the type of the pure spinor is non-generic: this was the fiber $x = 0$ in our special case.
Conclusions and open problems

- $B$-transforms and $\beta$-transforms are exchanged by T-duality.

- T-folds and non-commutative fibrations are equivalent from the open-string viewpoint: type-changing for pure spinors by monodromy corresponds to an uncertainty principle along the non-commutative fibers.

- Moduli fixing: D3-branes sit at points where the type jumps.

- Can one define a criterion for non-geometry?

- D-branes and Hitchin functionals?

- Non-commutative fibrations à la SYZ for generalized Calabi–Yau manifolds?