NNNLO corrections to $t\bar{t}$ production near threshold

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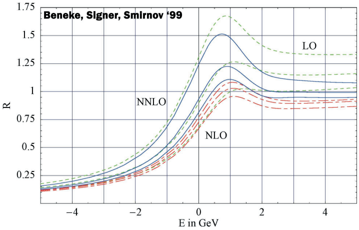
Motivation

- Measurement of
  - top quark mass with $\delta m_t \approx 100$ MeV [Martinez, Miquel '02].
  - top quark width ($\delta\Gamma_t \approx 30$ MeV), coupling constants etc.
- Study QCD effects of "toponium".

Current status: NNLO results show large uncertainties in the region of the resonance [Beneke, Signer, Smirnov; Hoang, Teubner; Melnikov, Yelkovsky; Yakovlev; Nagano et al.; Penin, Pivovarov '98-'99]:

- RG improvement ⇒ NNLL
- Fixed order ⇒ NNNLO



Fixed order approach

- (Partial) NNLL shows improved scale behaviour [Hoang et al. '02].
- Main effect comes from logs at NNNLO [Pineda, Signer '06].
- NNNLO needs inclusion of ultrasoft effects for the first time.

Cross section: $R = \frac{\sigma_{t\bar{t}X}}{\sigma_{\mu^+\mu^-}} = \frac{18\pi e_t^2}{m_t^2} \,\mathrm{Im}\, G(0,0;E+i\Gamma_t)$

Not included: axial-vector part, EW effects [Hoang, Reißer '04; Kühn et al. 90s]

Expansion in fixed order PT (in α_S and v):

$$R = v \sum_k \left(\frac{\alpha_S}{v}\right)^k \left\{ 1 + \begin{pmatrix} \alpha_s \\ v \end{pmatrix}_{NLO} + \begin{pmatrix} \alpha_s^2 \\ \alpha_s v \\ v^2 \end{pmatrix}_{NNLO} + \begin{pmatrix} \alpha_s^3 \\ \alpha_s^2 v, \alpha_s v^2 \\ v^3 \end{pmatrix}_{NNNLO} + \dots \right\}$$

Effective theory

- The expansion of v and α_s is done systematically in the framework of effective theories.

Method:

- Identify the scales in the given expansion.

$$\begin{aligned} \text{hard} \quad & l_0 \sim m, \vec{l} \sim m \\ \text{soft} \quad & l_0 \sim mv, \vec{l} \sim mv \\ \text{potential} \quad & l_0 \sim mv^2, \vec{l} \sim mv \\ \text{ultrasoft} \quad & l_0 \sim mv^2, \vec{l} \sim mv^2 \end{aligned}$$

- Integrate out the higher modes step by step.

- QCD $\Rightarrow$ NRQCD [Caswell, Lepage '86] $\Rightarrow$ PNRQCD [Pineda, Soto '97; Luke, Manohar, Rothstein '99]

Hard matching

- Matching of the QCD vector current (→ Peter Marquard's talk).

$$j^i = c_v\, \psi^\dagger \sigma^i \chi + \frac{d_v}{6m^2} \psi^\dagger \sigma^i\, \mathbf{D}^2 \chi + \dots$$

- Matching of the NRQCD Lagrangian.

Current status:

$c_v^{(2)}$ (2-loop) known [Beneke, Signer, Smirnov; Czarnecki, Melnikov '97]
$c_v^{(3)}$ (3-loop) n_f part known [Marquard et al. '06]
$d_v^{(1)}$ (1-loop) known [Luke, Savage '97]
NRQCD matching (1-loop) [Manohar '97; Wüster '03]

Soft/potential matching

Matching of the PNRQCD Lagrangian

$$\mathcal{L}_{PNRQCD} = \psi^\dagger(x)\left(i\partial^0 + \frac{\boldsymbol{\partial}^2}{2m} + \frac{\boldsymbol{\partial}^4}{8m^3}\right)\psi(x) + \chi^\dagger(x)\left(i\partial^0 - \frac{\boldsymbol{\partial}^2}{2m} + \frac{\boldsymbol{\partial}^4}{8m^3}\right)\chi(x)$$
$$+ \int d^3\mathbf{r}\,\left[\psi^\dagger\psi\right](x+\mathbf{r})\,V(\mathbf{r})\,\left[\chi^\dagger\chi\right](x) - g_s\psi^\dagger(x)\mathbf{x}\mathbf{E}(t,\mathbf{0})\psi(x) - g_s\chi^\dagger(x)\mathbf{x}\mathbf{E}(t,\mathbf{0})\chi(x)$$
$$\widetilde{V}(\mathbf{p},\mathbf{p}') = -\frac{4\pi\alpha_s C_F}{\mathbf{q}^2}\left[\mathcal{V}_C - \mathcal{V}_{1/m}\frac{\pi^2\,|\mathbf{q}|}{m} + \mathcal{V}_{1/m^2}\frac{\mathbf{q}^2}{m^2} + \mathcal{V}_p\frac{\mathbf{p}^2+\mathbf{p}'^2}{2m^2}\right].$$

Current status:

$\mathcal{V}_C^{(2)}$ (2-loop) known [Schröder '98]

$\mathcal{V}_C^{(3)}$ (3-loop) only Padé estimates available [Chishtie, Elias '01]

$\mathcal{V}_{1/m}^{(2)}$ (2-loop) known [Kniehl et al. '01], but $O(\epsilon)$ parts missing

$\mathcal{V}_{1/m^2}^{(1)}, \mathcal{V}_p^{(1)}$ (1-loop) known [Kniehl et al. '02; Wüster '03]

Perturbative approach

Calculation of the Green function in perturbation theory:

- Perturbative treatment of the potentials:

$$
\begin{aligned}
V &= -\frac{\alpha_s C_F}{r} + \delta V_1 + \delta V_2 + \delta V_3 + \delta V_{3,c.t.} + \ldots \\
G &= G_0 - G_0 \delta V_1 G_0 - G_0 \delta V_2 G_0 + G_0 \delta V_1 G_0 \delta V_1 G_0 \\
&\quad - G_0(\delta V_3 + \delta V_{3,c.t.}) G_0 + 2 G_0 \delta V_1 G_0 \delta V_2 G_0 - G_0 \delta V_1 G_0 \delta V_1 G_0 \delta V_1 G_0 \\
&\quad + \delta G_{us} + \ldots
\end{aligned}
$$

- Coulomb corrections completed [Beneke,Kiyo,K.S. '05].
- Non-Coulomb corrections completed.
- Ultrasoft corrections completed.

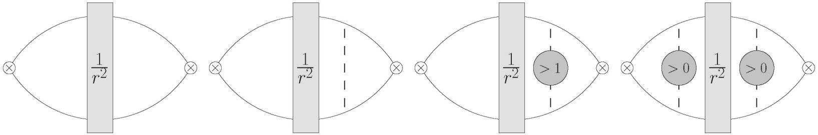
Add counter terms to the potential coefficients to make them finite and subtract c.t. back from US correction.

Potential insertions

Strategy:

- Identify the divergent structure of the Feynman diagrams.
- Divide the potential insertion into diagrams with the different divergent structures.

Example: $1/r^2$ single insertion:



$$\int \prod_{i=1}^{4} \frac{d^{d-1}\mathbf{p}_i}{(2\pi)^{d-1}} \qquad \frac{(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{p}_1-\mathbf{p}_2)}{\mathbf{p}_1^2/m - E} \frac{1}{[(\mathbf{p}_2-\mathbf{p}_3)^2]^{\frac{1}{2}+\epsilon}} \tilde{G}_C^{(>1ex)}(\mathbf{p}_3,\mathbf{p}_4)$$

$$\int \prod_{i=1}^{2} \frac{d\mathbf{p}_i}{(2\pi)} \qquad \left[\frac{1}{\epsilon} + F^0(\mathbf{p}_2) + F^1(\mathbf{p}_2)\epsilon + \ldots\right] \tilde{G}_C^{(>1ex)}(\mathbf{p}_1,\mathbf{p}_2)$$

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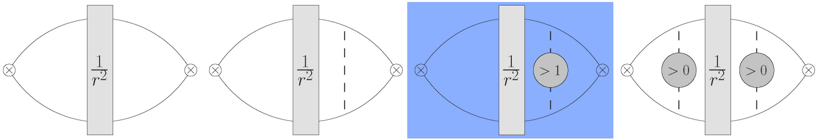

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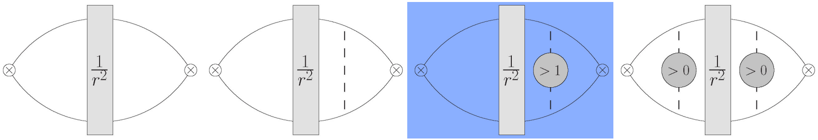

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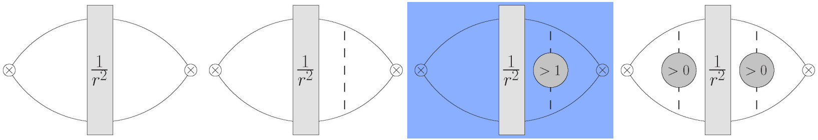

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$$\int \prod_{i=1}^{2} \frac{d^{4-1}\mathbf{p}_i}{(2\pi)^{4-1}} \qquad \left[\frac{1}{\epsilon} + F^0(\mathbf{p}_2) + F^1(\mathbf{p}_2)\epsilon + \ldots\right] \tilde{G}_C^{(>1ex)}(\mathbf{p}_1,\mathbf{p}_2)$$

The US correction

$$\delta G_{us} = \left[\tilde{\mu}^{2\epsilon}\right]^2 \int \frac{d^{d-1}\boldsymbol{\ell}}{(2\pi)^{d-1}} \frac{d^{d-1}\boldsymbol{\ell}'}{(2\pi)^{d-1}} \Bigg\{ (-\delta d_v^{\mathrm{div}}) \frac{\boldsymbol{\ell}^2 + \boldsymbol{\ell}'^{\,2}}{6m^2} \tilde{G}_C^{(s)}(\boldsymbol{\ell}, \boldsymbol{\ell}'; E)$$

$$+ \left[\tilde{\mu}^{2\epsilon}\right]^2 \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{d^{d-1}\mathbf{p}'}{(2\pi)^{d-1}} \tilde{G}_C^{(s)}(\boldsymbol{\ell}, \mathbf{p}; E) \left[\delta U + \delta \tilde{V}_{3,c.t.}\right] \tilde{G}_C^{(s)}(\mathbf{p}', \boldsymbol{\ell}'; E) \Bigg\},$$

- δU contains octet Green function.
- δU is UV finite after subtraction of $\delta \tilde{V}_{c.t.}$ and $\delta d_v^{\mathrm{div}}$.
- Then same strategy as for potential insertions is applied to calculate the (divergent) Feynman integrals.

Total result

- We calculated the cross section as well as quarkonium energy level and wave function corrections.

- Checked: divergent parts cancel once the potential and US parts and the vector current are combined.
- Result for cross section contains (sums of) Gamma and Polygamma functions and Hypergeometric functions (for potential insertions) and numerical integrals (for US correction).
- Result for wave function expressed by (nested) harmonic sums and ζ functions for arbitrary quantum number n, US constant part known only numerically.

Wave function

Wave function related to the residue Z_n of the two point function:

$$\left(q^\mu q^\nu - g^{\mu\nu} q^2\right) \Pi(q^2) = i \int d^d x\, e^{iqx} \langle \Omega | T(j^\mu(x) j^\nu(0)) | \Omega \rangle, \qquad j^\mu = \bar{Q}\gamma^\mu Q$$

$$Z_n = c_v \left[c_v - \frac{E_n}{m} \left(1 + \frac{d_v}{3} \right) + \cdots \right] |\psi_n(0)|^2$$

Z_1 is indication for the height of the $t\bar{t}$ cross section and related to the quarkonium decay (→ Antonio Pineda's talk).

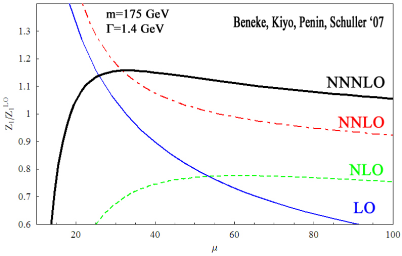
NNNLO wave function corrections completed up to unknown matching coefficients:

- Coulomb corrections [Penin, Smirnov, Steinhauser; Beneke,Kiyo,K.S. '05].
- Non-Coulomb corrections [Beneke,Kiyo,K.S. '07].
- US corrections [Beneke,Kiyo,Penin '07].

Wave function: "Toponium" 1S results

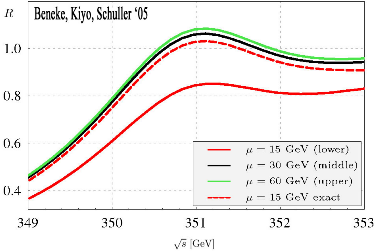
Result for Z_1 at $\mu = 30$ GeV:

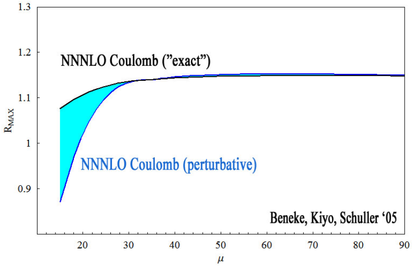
$$Z_1 = \left(\frac{m_t \alpha_s C_F}{8\pi}\right)^3 \left(1 - 2.13\alpha_s + 22.6\alpha_s^2 + [-33.0 + 37.6_{c_3,n_f}]\alpha_s^3\right)$$



- NNNLO about 1% at $\mu = 30$GeV.
- Unknown matching coefficients neglected.
- Reduced scale dependence.
- But: unstable for $\mu < 25$ GeV.

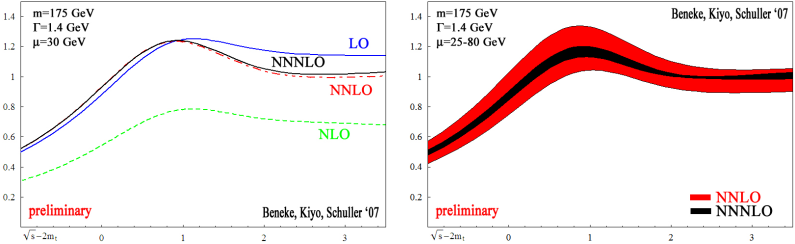
Cross section: Coulomb part





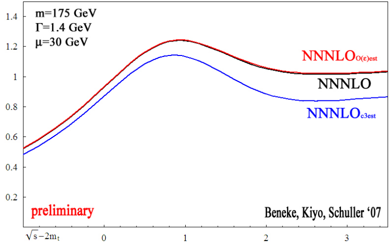
- Coulomb corrections can be calculated by numerical solution of the Schrödinger equation [Peskin, Strassler '91].
- This corresponds to resummation of multiple potential insertions (here "NNNLO exact").
- Comparison to perturbative result shows significant difference at low scales ⇒ scale behavior is an artifact of PT.
  So: Only scales $\mu > 25$ GeV are safe!

Cross section: (almost) complete NNNLO



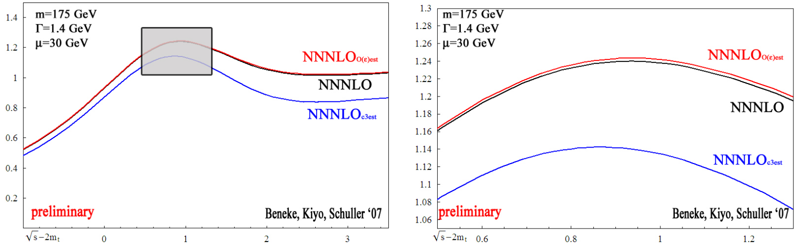
- All known parts included; unknown parts = 0.
- Good convergence! But maybe accidental due to unknown parts (see next slide).
- Scale dependence reduced significantly to about 10%.

Effect of unknown contributions



- Assume $x_{n+1}/x_n = const$ to estimate size of unknown parts.
- Effect of c_3 is about 10%.
- Effect of $O(\epsilon)$ parts of the potentials is small.

Effect of unknown contributions



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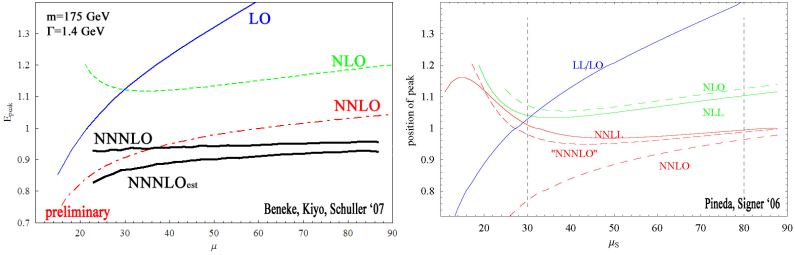
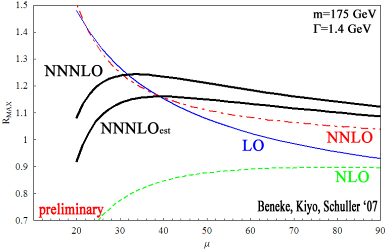
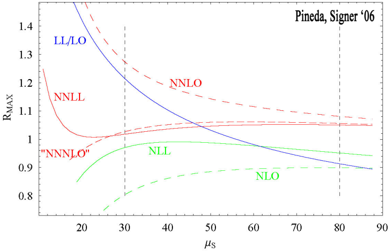
Cross Section: peak position NNNLO vs. NNLL



- Scale dependence below 100 MeV and similar to NNLL.
- But: shifted by about 100 MeV due to non-log terms.
- Contributions from c_3 are essential.
- Experimental accuracy reachable (in PS mass scheme!).

Cross Section: peak height NNNLO vs. NNLL





- Scale dependence similar to NNLL.
- Shifted by about 10-15% compared to NNLL to due to non-log terms.

Conclusion

We almost completed the NNNLO QCD quarkonium wave function and the $t\bar{t}$ QCD cross section at threshold.

- Perturbation series converges.
- Error from scale dependence reduce significantly.
- Experimental accuracy can be met.

Missing parts from QCD:

- 3-loop vector current coefficient c_3 (big impact).
- Some potential matching coefficients.

The EW and finite width effects needs further studies.

Backup slide I

$$Z_{1S(t\bar{t})} = \Big\{1 + \Big[3.66\,L - 2.13\Big]\alpha_s(\mu) + \Big[8.93\,L^2 - 6.14\,L + 10.46 - 7.26\,l_m\Big]\,\alpha_s^2(\mu)$$
$$+\Big[18.17\,L^3 - 20.26\,L^2 + (110.82 - 11.57\,l_m)\;L - 22.27\,L_{US} - 16.35\,l_m^2 - 22.65\,l_m$$
$$+(22.60 + 0.0015\,a_3 + 0.32\,\delta_\epsilon + 0.0645\,\delta c_3)\Big]\;\alpha_s^3(\mu)\;\Big\} \times |\psi^{(0)}_{1S(t\bar{t})}(0)|^2$$

$$Z_{1S(b\bar{b})} = \Big\{1 + \Big[3.98\,L - 2.00\Big]\alpha_s(\mu) + \Big[10.55\,L^2 - 6.51\,L + 11.19 - 7.44\,l_m\Big]\,\alpha_s^2(\mu)$$
$$+\Big[23.33\,L^3 - 23.12\,L^2 + (125.14 - 14.59\,l_m)\;L - 22.27\,L_{US} - 17.36\,l_m^2 - 26.61\,l_m$$
$$+(17.44 + 0.0015\,a_3 + 0.32\,\delta_\epsilon + 0.0645\,\delta c_3)\Big]\;\alpha_s^3(\mu)\;\Big\} \times |\psi^{(0)}_{1S(b\bar{b})}(0)|^2$$