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On the Scale Uncertainties in the $B \to X_s\gamma$ Decay

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Abstract

We analyze the theoretical uncertainties in $Br(B \to X_s\gamma)$ due to the choice of the high energy matching scale $\mu_W = \mathcal{O}(M_{\rm W})$ and the scale μ_t at which the running top quark mass is defined: $\overline{m}_{\rm t}(\mu_t)$. To this end we have repeated the calculation of the initial conditions confirming the final results of Adel and Yao and Greub and Hurth and generalizing them to include the dependences on μ_t and μ_W with $\mu_t \neq \mu_W$. In the leading order the μ_W and μ_t uncertainties in $Br(B \to X_s\gamma)$ turn out to be $\pm 13\%$ and $\pm 3\%$ respectively. We show analytically how these uncertainties are reduced after including next-to-leading QCD corrections. They amount to $\pm 0.4\%$ and $\pm 0.3\%$ respectively. Reanalyzing the uncertainties due to the scale $\mu_b = \mathcal{O}(m_b)$ we find that after the inclusion of NLO effects they amount to $\pm 3.2\%$ which is smaller by a factor of two than claimed in the literature. Including the uncertainties due to input parameters as well as the non-perturbative $1/m_b^2$ and $1/m_c^2$ corrections we find $Br(B \to X_s\gamma) = (3.48 \pm 0.31) \times 10^{-4}$ where the error is dominated by uncertainties in the input parameters. This should be compared with $(3.28 \pm 0.33) \times 10^{-4}$ found by Chetyrkin et al. where the error is shared evenly between the scale and parametric uncertainties.

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1. The inclusive $B \to X_s\gamma$ decay has been subject of considerable experimental and theoretical interest during the last ten years. Experimentally its branching ratio is found to be [1]:

$$Br(B \to X_s\gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}\,, \tag{1}$$

where the first error is statistical and the second is systematic. On the other hand the complete NLO analysis gives [2]

$$Br(B{\to}X_s\gamma) = (3.28 \pm 0.22\ (\mathrm{scale})\ \pm 0.25\ (\mathrm{par})) \times 10^{-4} = (3.28 \pm 0.33) \times 10^{-4}. \tag{2}$$

where the first error results from the scale uncertainty (see below) and the second error from the uncertainties in the input parameters. A similar result has been obtained in [3].

The NLO analyses presented in [2, 3] reduced by a factor of 3-4 the μ_b-uncertainties [4, 5] present in the leading order, where $\mu_b = \mathcal{O}(m_b)$ is the scale at which the relevant decay matrix element is evaluated. This reduction of the μ_b uncertainty is very welcome because in the forthcoming years much more precise measurements of $Br(B{\to}X_s\gamma)$ are expected from the upgraded CLEO detector, as well as from the B-factories at SLAC and KEK. This is also the reason why continuing efforts are being made to estimate non-perturbative corrections to the $B{\to}X_s\gamma$ decay with higher precision [6, 7, 8] as well. It appears that these latter corrections amount only to a few percent and constitute a rather small theoretical uncertainty.

2. In this letter we have repeated the numerical analysis of [2] to find that the remaining scale uncertainties are by roughly a factor 1.5-2.0 smaller than quoted by these authors and in [3]. This includes also two additional theoretical uncertainties which have not been addressed in the literature. They are related to the choice of the high energy matching scale $\mu_W = \mathcal{O}(M_{\rm W})$ and the scale $\mu_t = \mathcal{O}(m_{\rm t})$ at which the running top quark mass is defined: $\overline{m}_{\rm t}(\mu_t)$. These two scales enter the analysis of $B{\to}X_s\gamma$ in the process of calculating the initial conditions for the renormalization group running of the Wilson coefficients C_7 and C_8 of the operators

$$Q_7 = \frac{e}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu}(1+\gamma_5) b_\alpha F_{\mu\nu} \qquad Q_8 = \frac{g_s}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu}(1+\gamma_5) T^a_{\alpha\beta} b_\beta G^a_{\mu\nu}. \tag{3}$$

Here e and g_s denote the electromagnetic and strong coupling constants respectively. These initial conditions have been calculated at NLO in [9] and have been recently confirmed in [10]. These NLO corrections are necessary to remove the renormalization scheme dependence present in the renormalization group evolution from $\mu_W = \mathcal{O}(M_{\rm W})$ down to $\mu_b = \mathcal{O}(m_{\rm b})$. From our point of view the additional reason for performing these rather tedious calculations is the reduction of the uncertainties related to the choices of μ_W and μ_t. These uncertainties have not been discussed in [2, 3, 9, 10].

To this end we have repeated the calculation of the initial condition for the by far dominant Wilson coefficient C_7 confirming the final result in [9, 10] and generalizing it to include the dependences on μ_t and μ_W with $\mu_t \neq \mu_W$. In [9] and [10] $\mu_W = \mu_t = \mu_{Wt}$ have been used. The technical details of our calculation which differs in certain aspects from the previous ones will be presented elsewehere [11]. Here we discuss first the issue of the μ_W and μ_t uncertainties and their reduction after the inclusion of NLO corrections. Subsequently we discuss the μ_b uncertainties and we present our estimate of $Br(B \to X_s\gamma)$ in the Standard Model.

3. In the leading logarithmic approximation one has

$$\frac{Br(B \to X_s\gamma)}{Br(B \to X_c e\bar{\nu}_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} |C_7^{(0)\mathrm{eff}}(\mu_b)|^2 \,, \tag{4}$$

where

$$f(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z \quad \text{with} \quad z = \frac{m_{c,pole}^2}{m_{b,pole}^2} \tag{5}$$

is the phase space factor in $Br(B \to X_c e\bar{\nu}_e)$ and $\alpha = e^2/4\pi$.

The effective renormalization scheme independent coefficient $C_7^{(0)\mathrm{eff}}(\mu_b)$ introduced in [5] is given by

$$C_7^{(0)\mathrm{eff}}(\mu_b) = \eta^{\frac{16}{23}} C_7^{(0)}(\mu_W) + \frac{8}{3}\left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}}\right) C_8^{(0)}(\mu_W) + \sum_{i=1}^{8} h_i \eta^{a_i} \,, \tag{6}$$

where

$$\eta = \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}, \tag{7}$$

$$C_7^{(0)}(\mu_W) = \frac{3x_t^3 - 2x_t^2}{4(x_t-1)^4} \ln x_t + \frac{-8x_t^3 - 5x_t^2 + 7x_t}{24(x_t-1)^3}, \tag{8}$$

$$C_8^{(0)}(\mu_W) = \frac{-3x_t^2}{4(x_t-1)^4} \ln x_t + \frac{-x_t^3 + 5x_t^2 + 2x_t}{8(x_t-1)^3} \tag{9}$$

with

$$x_t = \frac{\overline{m}_{\mathrm{t}}^2(\mu_t)}{M_{\mathrm{W}}^2}. \tag{10}$$

The numbers a_i and h_i are given in table 1.

There are three scale uncertainties present in (4):

- The low energy scale $\mu_b = \mathcal{O}(m_b)$ at which the Wilson Coefficient $C_7^{(0)\mathrm{eff}}(\mu_b)$ is evaluated.

- The high energy scale $\mu_W = \mathcal{O}(M_{\mathrm{W}})$ at which the full theory is matched with the effective five-quark theory. In LO this scale enters only η. $C_7^{(0)}(\mu_W)$ and $C_8^{(0)}(\mu_W)$, usually denoted by $C_7^{(0)}(M_{\mathrm{W}})$ and $C_8^{(0)}(M_{\mathrm{W}})$, serve in LO as initial conditions to the renormalization group evolution from μ_W down to μ_b. As seen explicitly in (8) and (9) they do not depend on μ_W.

i	1	2	3	4	5	6	7	8
a_i	$\frac{14}{23}$	$\frac{16}{23}$	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	−0.4230	−0.8994	0.1456
h_i	2.2996	−1.0880	$-\frac{3}{7}$	$-\frac{1}{14}$	−0.6494	−0.0380	−0.0185	−0.0057
e_i	$\frac{4661194}{816831}$	$-\frac{8516}{2217}$	0	0	−1.9043	−0.1008	0.1216	0.0183
f_i	−17.3023	8.5027	4.5508	0.7519	2.0040	0.7476	−0.5385	0.0914
g_i	14.8088	−10.8090	−0.8740	0.4218	−2.9347	0.3971	0.1600	0.0225

Table 1: Magic Numbers.

- The scale $\mu_t = \mathcal{O}(m_t)$ at which the running top quark mass is defined. In LO it enters only x_t in (10).

It should be stressed that μ_W and μ_t do not have to be equal. Initially when the top quark and the W-boson are integrated out, it is convenient in the process of matching to keep $\mu_t = \mu_W$. Yet one has always the freedom to redefine the top quark mass and to work with $\overline{m}_\mathrm{t}(\mu_t)$ where $\mu_t \neq \mu_W$. It is evident from the formulae above that in LO the variations of μ_b, μ_W and μ_t remain uncompensated which results in potential theoretical uncertainties in the predicted branching ratio.
In the context of phenomenological analyses of $B \to X_s\gamma$, only the uncertainty due to μ_b has been discussed [4, 5, 2, 3]. It is the purpose of this letter to analyze the uncertainties due to μ_W and μ_t and to reanalyze the μ_b-uncertainty.
It is customary to estimate the uncertainties due to μ_b by varying it in the range $m_\mathrm{b}/2 \leq \mu_b \leq 2m_\mathrm{b}$. Similarly one can vary μ_W and μ_t in the ranges $M_\mathrm{W}/2 \leq \mu_W \leq 2M_\mathrm{W}$ and $m_\mathrm{t}/2 \leq \mu_t \leq 2m_\mathrm{t}$ respectively. Specifically in our numerical analysis of μ_W and μ_t uncertainties we will consider the ranges

$$40 \ \mathrm{GeV} \leq \mu_W \leq 160 \ \mathrm{GeV} \qquad 80 \ \mathrm{GeV} \leq \mu_t \leq 320\,\mathrm{GeV} \tag{11}$$

setting $\mu_b = m_b \equiv m_{b,pole} = 4.8\,\mathrm{GeV}$.
In the LO analysis we use

$$\alpha_s(\mu) = \frac{\alpha_s(M_\mathrm{Z})}{v(\mu)} \qquad v(\mu) = 1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln\left(\frac{M_Z}{\mu}\right) \tag{12}$$

with $\alpha_s(M_\mathrm{Z}) = 0.118$ and

$$\overline{m}_\mathrm{t}(\mu_t) = \overline{m}_\mathrm{t}(m_\mathrm{t}) \left[\frac{\alpha_s(\mu_t)}{\alpha_s(m_\mathrm{t})}\right]^{\frac{4}{\beta_0}} . \tag{13}$$

Here $\beta_0 = 23/3$ and $\beta_0 = 21/3$ for $\mu \leq \mu_W$ and $\mu > \mu_W$ respectively. We set $\overline{m}_\mathrm{t}(m_\mathrm{t}) = 168$ GeV and $m_\mathrm{t} \equiv m_{t,pole} = 176$ GeV.

Varying μ_W and μ_t in the ranges (11) we find the following uncertainties in the branching ratio:

$$\Delta Br(B \to X_s\gamma) = \begin{cases} \pm 13\% & (\mu_W) \\ \pm 3\% & (\mu_t) \end{cases} \tag{14}$$

to be compared with the $\pm 22\%$ uncertainty due to the variation of the scale μ_b [4, 5]. The fact that the μ_W-uncertainty is smaller than the μ_b uncertainty is entirely due to $\alpha_s(\mu_W) < \alpha_s(\mu_b)$. Still this uncertainty is rather disturbing as it introduces an error of approximately $\pm 0.40 \cdot 10^{-4}$ in the branching ratio. The smallness of the μ_t-uncertainty is related to the weak x_t dependence of $C_7^{(0)}(\mu_W)$ and $C_8^{(0)}(\mu_W)$ which in the range of interest can be well approximated by

$$C_7^{(0)}(\mu_W) = -0.122\ x_t^{0.30} \qquad C_8^{(0)}(\mu_W) = -0.072\ x_t^{0.19} \tag{15}$$

Thus even if $161\,\mathrm{GeV} \le \overline{m}_\mathrm{t}(\mu_t) \le 178\,\mathrm{GeV}$ for μ_t in (11), the μ_t uncertainty in $Br(B \to X_s\gamma)$ is small. This should be contrasted with $B_s \to \mu\bar{\mu}$, $K_L \to \pi^0\nu\bar{\nu}$ and $B^0 - \bar{B}^0$ mixing, where μ_t uncertainties in LO have been found [12, 13] to be $\pm 13\%$, $\pm 10\%$ and $\pm 9\%$ respectively.

4. We will next investigate how much the uncertainties in (14) are reduced after including NLO corrections.

The formula (4) modifies after the inclusion of NLO corrections as follows [2]:

$$\frac{Br(B \to X_s\gamma)}{Br(B \to X_c e\bar{\nu}_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} F\left(|D|^2 + A\right), \tag{16}$$

where

$$F = \frac{1}{\kappa(z)} \left(\frac{\bar{m}_b(\mu = m_b)}{m_{b,\mathrm{pole}}}\right)^2 = \frac{1}{\kappa(z)} \left(1 - \frac{8}{3}\frac{\alpha_s(m_b)}{\pi}\right), \tag{17}$$

with $\kappa(z)$ being the QCD correction to the semileptonic decay [14] and given to a good approximation by [15]

$$\kappa(z) = 1 - \frac{2\alpha_s(\bar{\mu}_b)}{3\pi}\left[(\pi^2 - \frac{31}{4})(1-\sqrt{z})^2 + \frac{3}{2}\right]. \tag{18}$$

An exact analytic formula for $\kappa(z)$ can be found in [16]. Here $\bar{\mu}_b = \mathcal{O}(m_b)$ is a scale in the calculation of QCD corrections to the semi-leptonic rate which is generally different from the one used in the $b \to s\gamma$ transition. In this respect we differ from Greub et al. [3] who set $\bar{\mu}_b = \mu_b$. In [2] the choice $\bar{\mu}_b = m_b$ has been made. We will return to this point below.

Next

$$D = C_7^{(0)\mathrm{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi}\left\{C_7^{(1)\mathrm{eff}}(\mu_b) + \sum_{i=1}^{8} C_i^{(0)\mathrm{eff}}(\mu_b)\left[r_i + \frac{\gamma_{i7}^{(0)\mathrm{eff}}}{2}\ln\frac{m_b^2}{\mu_b^2}\right]\right\} \tag{19}$$

where $C_7^{(1)\mathrm{eff}}(\mu_b)$ is the NLO correction to the effective Wilson coefficients of Q_7:

$$C_7^{\mathrm{eff}}(\mu_b) = C_7^{(0)\mathrm{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_7^{(1)\mathrm{eff}}(\mu_b). \tag{20}$$

Generalizing the formula (21) of [2] to include μ_t and μ_W dependences we find

$$\begin{aligned}
C_7^{(1)eff}(\mu_b) &= \eta^{\frac{39}{23}} C_7^{(1)eff}(\mu_W) + \frac{8}{3}\left(\eta^{\frac{37}{23}} - \eta^{\frac{39}{23}}\right) C_8^{(1)eff}(\mu_W) \\
&+ \left(\frac{297664}{14283}\eta^{\frac{16}{23}} - \frac{7164416}{357075}\eta^{\frac{14}{23}} + \frac{256868}{14283}\eta^{\frac{37}{23}} - \frac{6698884}{357075}\eta^{\frac{39}{23}}\right) C_8^{(0)}(\mu_W) \\
&+ \frac{37208}{4761}\left(\eta^{\frac{39}{23}} - \eta^{\frac{16}{23}}\right) C_7^{(0)}(\mu_W) + \sum_{i=1}^{8} (e_i \eta E(x_t) + f_i + g_i \eta)\eta^{a_i},
\end{aligned} \tag{21}$$

where in the $\overline{MS}$ scheme

$$C_7^{(1)eff}(\mu_W) = C_7^{(1)eff}(M_W) + 8x_t \frac{\partial C_7^{(0)}(\mu_W)}{\partial x_t} \ln \frac{\mu_t^2}{M_{\rm W}^2} + \left(\frac{16}{3}C_7^{(0)}(\mu_W) - \frac{16}{9}C_8^{(0)}(\mu_W) + \frac{\gamma_{27}^{(0)\rm eff}}{2}\right) \ln \frac{\mu_W^2}{M_{\rm W}^2} \tag{22}$$

$$C_8^{(1)eff}(\mu_W) = C_8^{(1)eff}(M_W) + 8x_t \frac{\partial C_8^{(0)}(\mu_W)}{\partial x_t} \ln \frac{\mu_t^2}{M_{\rm W}^2} + \left(\frac{14}{3}C_8^{(0)}(\mu_W) + \frac{\gamma_{28}^{(0)\rm eff}}{2}\right) \ln \frac{\mu_W^2}{M_{\rm W}^2} \tag{23}$$

Here $(x = x_t)$

$$\begin{aligned}
C_7^{(1)eff}(M_W) &= \frac{-16x^4 - 122x^3 + 80x^2 - 8x}{9(x-1)^4} {\rm Li}_2\left(1 - \frac{1}{x}\right) + \frac{6x^4 + 46x^3 - 28x^2}{3(x-1)^5} \ln^2 x \\
&+ \frac{-102x^5 - 588x^4 - 2262x^3 + 3244x^2 - 1364x + 208}{81(x-1)^5} \ln x \\
&+ \frac{1646x^4 + 12205x^3 - 10740x^2 + 2509x - 436}{486(x-1)^4}
\end{aligned} \tag{24}$$

$$\begin{aligned}
C_8^{(1)eff}(M_W) &= \frac{-4x^4 + 40x^3 + 41x^2 + x}{6(x-1)^4} {\rm Li}_2\left(1 - \frac{1}{x}\right) + \frac{-17x^3 - 31x^2}{2(x-1)^5} \ln^2 x \\
&+ \frac{-210x^5 + 1086x^4 + 4893x^3 + 2857x^2 - 1994x + 280}{216(x-1)^5} \ln x \\
&+ \frac{737x^4 - 14102x^3 - 28209x^2 + 610x - 508}{1296(x-1)^4}
\end{aligned} \tag{25}$$

and

$$E(x) = \frac{x(18 - 11x - x^2)}{12(1-x)^3} + \frac{x^2(15 - 16x + 4x^2)}{6(1-x)^4} \ln x - \frac{2}{3} \ln x. \tag{26}$$

The formulae for $C_{7,8}^{(1)eff}(M_W)$ given above and presented in [2] are obtained from the results in [9, 10] by using the general formulae for the effective coefficient functions [5]. The formula for $C_7^{(1)eff}(M_W)$ has been confirmed by us [11].

The numbers e_i–g_i are given in Table 1. We have confirmed these numbers as well as the numerical coefficients in (21) using the anomalous dimension matrices in [2]. Next the η in (7) should now be calculated using the NLO expression

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)} \left[1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(M_Z)}{4\pi} \frac{\ln v(\mu)}{v(\mu)}\right], \tag{27}$$

where $v(\mu)$ is given in (12). $\beta_1 = \frac{116}{3}$ and $\beta_1 = 26$ for $\mu \leq \mu_W$ and $\mu > \mu_W$ respectively.

The constants r_i resulting from the calculations of NLO corrections to decay matrix elements [3] are collected in [2] where also explicit formulae for $C_i^{(0)\mathrm{eff}}(\mu_b)$ with $i = 1-6, 8$ and the values of $\gamma_{i7}^{(0)\mathrm{eff}}$ can be found. It should be stressed that the basis of the operators with $i = 1-6$ used in [2] differs from the standard basis used in the literature [19]. For the discussion below it will be useful to have [20]

$$\gamma_{27}^{(0)\mathrm{eff}} = \frac{416}{81} \qquad \gamma_{28}^{(0)\mathrm{eff}} = \frac{70}{27} \tag{28}$$

which enter (22) and (23) respectively.

Finally the term A in (16) originates from the bremsstrahlung corrections and the necessary virtual corrections needed for the cancellation of the infrared divergences. These have been calculated in [17, 18] and are also considered in [2, 3] in the context of the full analysis. The explicit formula for A, which we use in our numerical analysis, can be found in equation (32) of [2].

Setting $\mu_W = \mu_t = \mu_{Wt}$, replacing $\gamma_{27}^{(0)\mathrm{eff}}$ by its value in the NDR scheme $\gamma_{27}^{(0)\mathrm{NDR}} = 232/81$ and adding all μ_i dependent terms in (22) we recover the μ_{Wt} dependence of $C_7^{(1)\mathrm{NDR}}(\mu_{Wt})$ found in [10]. Similarly replacing $\gamma_{28}^{(0)\mathrm{eff}}$ by $\gamma_{28}^{(0)\mathrm{NDR}} = 76/27$ in (23) we recover the μ_{Wt} dependence of $C_8^{(1)\mathrm{NDR}}(\mu_{Wt})$ given in [10]. For $\mu_W = \mu_t = M_\mathrm{W}$ the formulae above reduce to the ones given in [2].

5. Before entering the numerical analysis let us demonstrate analytically that the μ_t and μ_W dependences present in $C_7^{(0)\mathrm{eff}}(\mu_b)$ are indeed cancelled at $\mathcal{O}(\alpha_s)$ by the explicit scale dependent terms in (22). The scale dependent terms in (23) do not enter this cancellation at this order in α_s in $B \to X_s\gamma$. On the other hand they are responsible for the cancellation of the scale dependences in $C_8^{(0)\mathrm{eff}}(\mu_b)$ relevant for the $b \to s$ gluon transition.

Expanding the three terms in (6) in α_s and keeping the leading logarithms we find:

$$\eta^{\frac{16}{23}} C_7^{(0)}(\mu_W) = \left(1 + \frac{\alpha_s}{4\pi}\frac{16}{3}\ln\frac{\mu_b^2}{\mu_W^2}\right) C_7^{(0)}(\mu_W) \tag{29}$$

$$\frac{8}{3}\left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}}\right) C_8^{(0)}(\mu_W) = -\frac{\alpha_s}{4\pi}\frac{16}{9}\ln\frac{\mu_b^2}{\mu_W^2} C_8^{(0)}(\mu_W) \tag{30}$$

$$\sum_{i=1}^{8} h_i \eta^{a_i} = \frac{\alpha_s}{4\pi}\frac{23}{3}\ln\frac{\mu_b^2}{\mu_W^2}\sum_{i=1}^{8} h_i a_i = \frac{208}{81}\frac{\alpha_s}{4\pi}\ln\frac{\mu_b^2}{\mu_W^2} \tag{31}$$

respectively. In (31) we have used $\sum h_i = 0$. Inserting these expansions into (19), we observe that the μ_W dependences in (29), (30) and (31) are precisely cancelled by the three explicit logarithms in (22) involving μ_W, respectively. Similarly one can convince oneself that the μ_t-dependence of $C_7^{(0)\mathrm{eff}}(\mu_b)$ is cancelled at $\mathcal{O}(\alpha_s)$ by the $\ln \mu_t^2/M_\mathrm{W}^2$ term in (22).

Clearly there remain small μ_t and μ_W dependences in (16) which can only be reduced by going beyond the NLO approximation. They constitute the theoretical uncertainty which should be taken into account in estimating the error in the prediction for $Br(B \to X_s\gamma)$.
Using the well known two-loop generalization of (13) and varying μ_W and μ_t in the ranges (11) we find that the respective uncertainties in the branching ratio after the inclusion of NLO corrections are negligible:

$$\Delta Br(B \to X_s\gamma) = \begin{cases} \pm 0.4\% & (\mu_W) \\ \pm 0.3\% & (\mu_t) \end{cases} \tag{32}$$

6. We have next performed the NLO analysis of the μ_b dependence. Varying μ_b in the range $2.5\,\text{GeV} \leq \mu_b \leq 10\,\text{GeV}$ we find

$$\Delta Br(B \to X_s\gamma) = \pm 3.2\% \quad (\mu_b) \tag{33}$$

This reduction of the μ_b-uncertainty by roughly a factor of seven relative to $\pm 22\%$ in LO is caused by the presence of the explicit logarithm $\ln m_b^2/\mu_b^2$ in (19). We note that our result in (33) differs from the μ_b- uncertainty of $\pm 6.6\%$ quoted in [2]. A discussion with the latter authors confirmed our result.

Next we would like to comment on the uncertainty due to variation of $\bar{\mu}_b$ in $\kappa(z)$ given in (18). In [3] the choice $\bar{\mu}_b = \mu_b$ has been made. Yet in our opinion such a treatment is not really correct, since the scale $\bar{\mu}_b$ in the semi-leptonic decay has nothing to do with the scale μ_b in the renormalization group evolution in the $B \to X_s\gamma$ decay. Varying $\bar{\mu}_b$ in the range $2.5\,\text{GeV} \leq \mu_b \leq 10\,\text{GeV}$ we find

$$\Delta Br(B \to X_s\gamma) = \pm 1.8\% \quad (\bar{\mu}_b) \tag{34}$$

Since the μ_b and $\bar{\mu}_b$ uncertainties are uncorrelated we can add them in quadrature finding $\pm 3.7\%$ for the total scale uncertainty due to μ_b and $\bar{\mu}_b$. This is smaller by roughly 50% than the case in which $\bar{\mu}_b = \mu_b$ is used. The addition of the uncertainties in μ_t and μ_W in (32) does not modify this result and the total scale uncertainty in $Br(B \to X_s\gamma)$ amounts then to

$$\Delta Br(B \to X_s\gamma) = \pm 3.7\% \quad (\text{scale}) \tag{35}$$

which is roughly by a factor of 1.5-2.0 smaller than quoted in [2, 3].

It should be stressed that this pure theoretical uncertainty related to the truncation of the perturbative series should be distinguished from parametric uncertainties related α_s, the quark masses etc. discussed below.

In our numerical calculations we have included all corrections in the NLO approximation. To work consistently in this order, we have in particular expanded the various factors in (16) in

α_s and discarded all NNLO terms of order α_s^2 which resulted in the process of multiplication. This treatment is different from [2, 3], where the α_s corrections in (17) have not been expanded in the evaluation of (16) and therefore some higher order corrections have been kept. Different scenarios of partly incorporating higher order corrections by expanding or not expanding various factors in (16) affect the branching ratio by $\Delta Br(B \to X_s\gamma) \approx \pm 3.0\%$. This number indicates that indeed the scale uncertainty in (35) realistically estimates the magnitude of yet unknown higher order corrections.

The remaining uncertainties are due to the values of the various input parameters. In order to obtain the final result for the branching ratio we have used the same parameters as in [2]. In addition we have included small $1/m_b^2$ corrections as in [2] and also a 3% enhancement from $1/m_c^2$ corrections [8] which were not known at the time of the analysis [2]. The relative importance of various uncertainties is shown in Table. 2. Comparing this table with the first row in the corresponding table in [2] we observe that except for the scale uncertainties discussed above, our error analysis agrees well with the one presented in [2].

Scales	$\alpha_s(M_Z)$	$m_{t,pole}$	$m_{c,pole}/m_{b,pole}$	$m_{b,pole}$	α_{em}	CKM angles	$B \to X_c e\bar{\nu}_e$
$\pm 3.7\%$	$\pm 2.8\%$	$\pm 1.8\%$	$\pm 5.3\%$	$\pm 0.8\%$	$\pm 1.8\%$	$\pm 2.1\%$	$\pm 3.8\%$

Table 2. Uncertainties in $Br(B \to X_s\gamma)$ due to various sources.

Adding all the uncertainties in quadrature we find

$$Br(B \to X_s\gamma) = (3.48 \pm 0.13 \text{ (scale) } \pm 0.28 \text{ (par)}) \times 10^{-4} = (3.48 \pm 0.31) \times 10^{-4} \qquad (36)$$

where we show separately scale and parametric uncertainties.

Comparing this result with the one of [2] as given in (2) we observe that in spite of the smaller scale uncertainties in (36) our final result is compatible with the one of [2] and the one given in [3]. This is due to the parametric uncertainties which dominate the theoretical error at present. Once these parametric uncertainties will be reduced in the future the smallness of the scale uncertainties achieved through very involved QCD calculations, in particular in [2, 3, 17, 18, 9, 10, 11], can be better appreciated. This reduction of the theoretical error in the Standard Model prediction for $Br(B \to X_s\gamma)$ could turn out to be very important in the searches for new physics. To this end also a better understanding of non-perturbative corrections [7] beyond those considered here should be improved.

The theoretical estimate in (36) is somewhat higher than the CLEO result in (1) but in view of the remaining theoretical and in particular experimental uncertainties, the Standard Model value is compatible with experiment. It will be exciting to watch the improvements in the theoretical estimate and in the experimental value in the coming years.

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