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Note on Regge Method in High-Energy Physics

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Abstract

We argue that the application of Regge method to high energy scattering is not free from theoretical problems. In particular, it is shown that the absorptive part of a scattering amplitude cannot be anambigiously represented as the sum of Regge pole (and/or cut) contributions even at asymptotic energies. On the physical cut of direct channel, these contributions may be accompanied by divergent background integrals. A possible physical interpretation of this ambiguity of Regge method is presented.

The Regge method for description of amplitude asymptotics, developed in earlier 60-ties, is still widely used to parametrize asymptotic amplitudes and cross sections. This method also served as a basis for some more recent theoretical models of high energy interactions

high energy physics provides well grounded predictions for asymptotic amplitudes and cross sections in terms of Regge singularity contributions. However, we will argue below that some of this method's ambiguities might have been ignored so far.

In order to make more clear our claims, we will merely follow the conventional derivation of Regge method results[1]. We will consider hereafter the scattering of two scalar particles with an equal mass m. We have to find the elastic scattering amplitude $A(s,t)$ in the asymptotic s-channel region $s \to +\infty$, $t \leq 0$. Our interest will be focused on the absorptive part of $A(s,t)$, which dominates the real part at large s. The conventional kinematical variable

$$z = 1 + \frac{2s}{t - 4m^2} \tag{1}$$

plays the role of a scattering angle in the t-channel region. The three physical regions for the amplitude $A(s,t)$ are defined by the following conditions

$$\begin{aligned} t-region: &\quad s \leq 0,\ t \geq 4m^2,\ u \leq 0,\ -1 \leq z \leq 1, \\ s-region: &\quad s \geq 4m^2,\ t \leq 0,\ u \leq 0,\ z \leq -1, \\ u-region: &\quad s \leq 0,\ t \leq 0,\ u \geq 4m^2,\ z \geq 1. \end{aligned} \tag{2}$$

The starting point of the amplitude evoluation in the Regge approach is the partial-wave (p.w.) expansion of $A(s,t)$ in t-region,

$$A(s,t) = 16\pi \sum_{l=0}^{\infty} (2l+1) A_l(t) P_l(z), \tag{3}$$

which converges in the Lehman ellipse of this region. The t-channel partial amplitudes are defined by

$$A_l(t) = \frac{1}{32\pi} \int_{-1}^{1} dz' A(s',t) P_l(z'), \quad l = 0, 1, 2, \ldots, \tag{4}$$

where s' and z' are connected by the equation (1). Following the Sommerfeld-Watson prescription, the expansion (3) can be replaced by the contour integral

$$A(s,t) = \oint_{C_1} dl\, a(l,t)\, f(l,z), \tag{5}$$

$$f(l,z) = -\frac{16\pi(2l+1)P_l(-z)}{2i\ sin(\pi l)} \qquad (6)$$

and $a(l,t)$ is the analytic continuation in l of the partial amplitude $A_l(t)$. The contour C_1 encloses the real positive half-axis of l with poles of $1/sin(\pi l)$ on it. Our first remark concerns the choice (6) for the function $f(l,z)$. The only model-independent conditions for $f(l,z)$ are given by the values of its residues at integer l,

$$res\{f(l,z)\} = 16\pi(2l+1)P_l(z), \quad l = 0,1,2\ldots \qquad (7)$$

and by the limitation that $f(l,t)$ is regular in the vicinity of these poles. However, an infinite number of functions can fulfill these conditions because functions cannot be uniquelly reconstructed using only their pole residues. Moreover, these different functions may have a good behaviour at $|l| \to \infty$, i.e. to be less than $exp(\pi|l|)$, because the Carlson theorem is not applicable in this case. A most trivial example is given by $f'(l,z) = f(l,z) + c$ with some arbitrary constant c. Both functions f and f' may fulfill the above conditions and have good properties at $|l| \to \infty$, but $f' \neq f$. However, this freedom with the choice of $f(l,z)$ may not affect the final result for the amplitude $A(s,t)$ provided the contour integrals and all possible singularities on the complex-$|l|$ plane are properly treated.

The form (4) is not suitable for the analytic continuation of partial amplitudes because of bad properties of $P_l(z)$ at complex l[2]. Prior to introduce the conventional Gribov-Froissart representation for partial amplitudes, let us write down the dispersion relation for $A(s,t)$ at fixed t,

$$A(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{D_s(s',t)}{s'-s} + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \frac{D_u(u',t)}{u'-u} \equiv A_s(s,t) + A_u(s,t), \qquad (8)$$

where $D_{s,u}$ are the amplitude discontinuities in s- and u- channels, respectively. For simplicity, we ignore possible subtractions and s- and u-channel poles. From the above definition of A_s and A_u it follows that the absorptive part of $A(s,t)$ in the s-region is fully described by A_s and in the u-region - by A_u. Since we are first interested to find the absorptive part of $A(s,t)$ in s-region and because the two parts of $A(s,t)$ always remain additive, we may substitute $A_s(s,t)$ for $A(s,t)$ in what follows until the part $A_u(s,t)$ and the restoration of s-u

of the signature and to simplify the notations. We can now rewrite the equation (4) for partial amplitudes in the following Gribov-Froissart representation

$$A_l^{GF}(t) = \frac{1}{16\pi^2}\int_{z_0(t)}^{z_\infty(t)} dz' D_s(s',t)Q_l(z'), \quad l = 0,1,2,\ldots, \tag{9}$$

where $z_0(t)$ and $z_\infty(t)$ are solutions of (1) with $s = 4m^2$ and ∞, respectively. In t-region, the function $A_l^{GF}(t)$ has good properties at $|l| \to \infty$ and can be analytically continued in l. Therefore the conventional prescription for the analytic continuation in l reads

$$a(l,t) = A_l^{GF}(t) \tag{10}$$

at any complex l. In this case the Carlson theorem states that $A_l^{GF}(t)$ is the only analytic function, which is regular at $Re[l] \geq l_0$ with some arbitrary l_0, has good properties at $|l| \to \infty$ and coincides with $A_l(t)$ at integer l. Note that among the above three restrictions on $a(l,t)$ only the last one is model-independent. The two former conditions are just needed for the applicability of the Regge method. In a general case, there is no physical or mathematical argument why analytically continued partial amplitudes should have good properties at large complex l. However, this remark does not question the consistency of results, obtained within the Regge model, with conjectures of that model.

The conventional deformation of the integration contour on the complex-l plane, completed by the Cauchy theorem, allows to rewrite the equation (5) for the amplitude in the following Regge representation[1], [3]-[5]

$$A(s,t) = B(s,t) + R(s,t). \tag{11}$$

The remaining part of the contour integral, the background integral, is given by

$$B(s,t) = 8\pi i \int_{-1/2-i\infty}^{-1/2+i\infty} dl\,(2l+1)a(l,t)\frac{P_l(-z)}{sin(\pi l)}. \tag{12}$$

In case when only poles of $a(l,t)$ take place on the right half-plane of l, the Regge term is given by the following sum of Regge pole contributions

$$R(s,t) = \sum_j R_j(s,t), \quad R_j(s,t) = -16\pi^2(2\alpha_j(t)+1)a_{\alpha_j(t)}(t)\frac{P_{\alpha_j(t)}(-z)}{sin(\pi\alpha_j(t))}. \tag{13}$$

representation (11)-(13) for the amplitude can be analytically continued from t-region to s-region. However, in the latter region the function $a(l,t)$ acquires the kinematical singularity of the form $(t-4m^2)^l$ [1],[3],[4], which is not connected to physical singularities of the amplitude. In order to remove this unphysical singularity from partial amplitudes, it is common to use the so-called reduced partial amplitudes $b(l,t)$, defined by

$$b(l,t) = \frac{a(l,t)}{(t-4m^2)^l}. \tag{14}$$

The reduced partial amplitudes have only physical singularities as functions of t. The Regge pole contribution can now be rewritten more explicitely as

$$R_j(s,t) = -16\pi^2(2\alpha_j(t)+1)\frac{\beta_{\alpha_j(t)}(t)}{(t-4m^2)^{\alpha_j(t)}}\frac{P_{\alpha_j(t)}(-1-\frac{2s}{t-4m^2})}{sin(\pi\alpha_j(t))}, \tag{15}$$

where $\beta_{\alpha_j(t)}(t)$ is the residue of the pole $\alpha_j(t)$ of the function $b(l,t)$. The asymptotics of the Legendre function at large argument is given by[2]

$$P_l(z)|_{z\to\infty} \sim \frac{(2z)^l}{\sqrt{l}}, \quad Re\{l\} \geq -1/2. \tag{16}$$

Therefore in the asymptotic s-region the Regge term can be represented as

$$R_j(s,t) \approx c_j(t)\beta_{\alpha_j(t)}(t)\frac{(-2s)^{\alpha_j(t)}}{sin(\pi\alpha_j(t))}, \quad c_j(t) = -16\pi^2\frac{(2\alpha_j(t)+1)}{\sqrt{\alpha_j(t)}}. \tag{17}$$

The Regge pole contribution has an s-cut in s-region and its discontinuity on this cut at large s is given by

$$Disc\{R_j(s,t)\} \approx -c_j(t)sin(\pi\alpha_j(t))\beta_{\alpha_j(t)}(t)(2s)^{\alpha_j(t)}. \tag{18}$$

At $t=0$, the last equation defines the Regge pole contribution to the total cross section. Following the Legendre function asymptotics (16), it was usually accepted that the background integral behaviour at large s is represented by

$$B(s,t) \sim s^{-1/2}. \tag{19}$$

Therefore the Regge term would dominate the background integral at large s. For this reason, background integrals were always ignored in the asymptotic direct-channel region.

cross section asymptotics. This conclusion and the equations (17)-(19) are the important result of Regge method in high energy physics.

Our essential remark concerns the behavior of background integral in s-region. According to (14) and (16), the background integral can be rewritten at large s as

$$B(s,t) \approx 8\pi i \int_{-1/2-i\infty}^{-1/2+i\infty} dl(2l+1)\frac{2^l}{\sqrt{l}} b(l,t)[\frac{(-s)^l}{sin(\pi l)}]. \tag{20}$$

The reduced partial amplitude $b(l,t)$ has only physical singularities in t and thus is regular in t when $t \leq 0$[3, 4]. The asymptotics of $b(l,t)$ at large complex l can be found using the following property of Legendre functions[2]

$$|Q_l(z)|_{|l|\to\infty} \sim \frac{1}{\sqrt{l}}(z+\sqrt{z^2-1})^{-l-1/2}. \tag{21}$$

From (9) and (21) it follows that in contrast to $a(l,t)$ the reduced amplitudes $b(l,t)$ do not contain a factor which might exponentially increase at $l \to \pm i\infty$[3], [4]. This factor has the form $(t-4m^2)^l$ and was canceled in equations (17) and (20) by the factor coming from the Legendre function $P_l(-z)$ (see (15) and (16)). Therefore all essential functions, which might exponentially increase or decrease under the integral (20), are located in the last factor in parenthesis. Furtheremore, the integrand in (20) is cut at real positive s due to this factor. At $s = |s| \pm i0$, we can rewrite this factor as

$$[\frac{(-s)^l}{sin(\pi l)}] = |s|^l \frac{e^{\pm i\pi l}}{e^{i\pi l} - e^{-i\pi l}}. \tag{22}$$

This equation implies that on each border of the s-channel physical cut there is one direction of the integration in (20) along which the background integral may diverge because the exponential raise of $sin(\pi l)$ in the denominator is compensated by the same raise of $(-s)^l$ in the numerator. All other factors in the integrand may not lead to integral convergence in a general case. If the background integral diverges, then the estimation (19) is no more correct and cannot be used to get further approximations. In fact, the decreasing of the integrand due to the factor $s^{-1/2}$ depends on the physical value s, which can be very large but still remains always finite in contrast to mathematical limits of the background integral (20). A divergent background integral has no any definite behaviour as a function of s.

precisely at the most important for the problem under consideration values of s and t, the Regge term may be accompanied by a divergent background integral. Therefore the Regge representation (11) for the amplitude is anambigious everywhere except the borders of the physical s-channel cut. As far as we know, this pecularity of background integral behaviour was not taken into account so far. If the background integral diverges, then the Regge representation (11) for the amplitude is practically as informative as the original partial wave expansion (3), if used outside of its convergence region. In fact, it seems to be equivalently useless to consider one particular term from the divergent p.w. expansion (3) or to consider one term (the Regge term) from the divergent r.h.s. of the representation (11).

Strictly speaking, the convergence of background integral could, in principle, be recovered due to some specific properties of partial amplitudes $b(l,t)$ at large imaginary l. An individual contribution to $b(l,t)$ of a given Regge pole has the behaviour $\sim 1/(l-l_0(t))$ and cannot provide the integral convergence. The same is true for an individual contribution of Regge cut, having for example the behaviour $\sim 1/\sqrt{l-l_0(t)}$. Regge singularities of the form $\sim 1/(l-l_0(t))^m$ with $m>1$ would lead to essential singularities on the complex-l plane. Note that the behaviour of partial amplitudes $b(l,t)$ is connected to their singularities on the whole complex-l plane, not just at $Re\{l\} > -1/2$. Therefore only some fine combination of all Regge singularity contributions might provide (at some special values of t!) the integral convergence. However, no proof of such property of Regge singularities is known to us.

We now suggest a possible physical interpretation of the above mathematical problem of Regge method. We focus on the fact that this problem takes place only and precisely on the physical cut of the direct channel. The s-channel discontinuity of the amplitude $A(s,t)$ of elastic scattering $p_1+p_2 \to p_3+p_4$ can be represented as follows

$$Disc\{A(s,t)\} = \sum_n A_1^{(n)}(p_1,p_2,k_1,\ldots,k_n)A_2^{(n)}(k_1,\ldots,k_n,p_3,p_4)\delta(k_1^2-m_1^2)\ldots\delta(k_n^2-m_n^2), \tag{23}$$

where the blocks $A_1^{(n)}$ and $A_2^{(n)}$ describe the physical processes $p_1+p_2 \to k_1+\ldots k_n$ and $k_1+\ldots+k_n \to p_3+p_4$, respectively. Since all intermediate momenta k_i are on their mass shells,

This means that all interactions in these two blocks, as well as two pairs of corresponding external particles, are infinitely separated in space and time. According to the postulate of a short range of strong interactions [3], the discontinuity (23) may not depend on interactions between particles p_1, p_2 and particles p_3, p_4 (or corresponding antiparticles).

From the other side, it is generally accepted [1],[4],[5] to interpret a t-channel Regge pole $\alpha_i(t)$ as a family of several t-channel resonances with similar internal structure but with different masses and spins, connected by the equation

$$s_n = \alpha_i(m_n^2). \tag{24}$$

Each t-channel resonance, as well as the whole family of resonances represented by the Regge pole $\alpha_i(t)$, is a result of interactions between particles p_2 and $\overline{p}_3$ (or p_1 and $\overline{p}_4$) in the t-channel. These interactions take place due to exchange of virtual particles and thus are local in space and time. But in (23) there is no interaction between these external particles. Therefore it seems to be strange if the direct-channel discontinuity would be explicitely given by a cross-channel Regge pole residue, even at large s. The same is true for the whole amplitude on the direct-channel cut since at large s this amplitude is dominated by its absorptive part. The above argumentation becomes ungrounded outside from the direct-channel physical cut. In this case the amplitude $A(s,t)$ does not contain an absorptive part with no interaction between t-channel external particles. But we have shown that the mathematical ambiguity of Regge method also may take place precisely on the direct-channel physical cut. Might it be that this coincidence is not accidental?

The above derivations remain valid in case when there are Regge cuts in addition to Regge poles. As we mentioned, the convergence of background integrals is not provided by this type of singularities of $a(l,t)$ in a general case. Just for the sake of completeness, we now briefly discuss the restoration of the s-u crossing symmetry, broken by the assumption in the beginning of this article. So far we kept only the part $A_s(s,t)$ of the whole amplitude (see (8)). The part $A_u(s,t)$ is regular in s except the u-channel physical cut. We might use for $A_u(s,t)$ the p.w. expansion (3) with partial amplitudes $A_l^{(u)}(t)$, defined by analogy with

$$A_l^{(u)GF}(t) = \frac{1}{16\pi^2}\int_{z'_0(t)}^{z'_\infty(t)} dz' D_u(u',t)Q_l(z'),\ \ l=0,1,2\ldots \tag{25}$$

with $z'_0(t) = -(t+4m^2)/(t-4m^2)$ and $z'_\infty(t) = \infty \times sign\{4m^2-t\}$. Since these integration limits are negative in t-region, the representation (25) is not good for the analytic continuation in l. Instead of introducing the signature, let us define the new partial amplitudes $\tilde{A}_l^{(u)GF}$ by the equation

$$\tilde{A}_l^{(u)GF}(t) = \frac{1}{16\pi^2}\int_{z_0(t)}^{z_\infty(t)} dz' D_u(u',t)Q_l(z'),\ \ l=0,1,2\ldots \tag{26}$$

with integration limits as in equation (9). In contrast to (25), the integration limits in (26) are positive in t-region and the function (26) may be continued in l. The p.w. expansion of $A_u(s,t)$ in terms of partial amplitudes $\tilde{A}_l^{(u)GF}(t)$ reads

$$A_u(s,t) = 16\pi \sum_{l=0,1,2,\ldots}^{\infty} (2l+1)\tilde{A}_l^{(u)GF}(t)P_l(-z). \tag{27}$$

Analytic properties of $\tilde{a}(l,t) \equiv \tilde{A}_l^{(u)GF}(t)$ at arbitrary complex l and t are similar to properties of $a(l,t)$, defined by (10) and (9). Taking into account the different signs in front of z in the p.w. expansions (27) and (3), we can see that our previous results for the amplitude $A_s(s,t)$, given by the equations (11)-(17), remain valid for $A_u(s,t)$ provided partial amplitudes are replaced by tilded partial amplitudes and $-z$ $(-s)$ is substituted for z (s) in equations (11)-(17). This allows us to conclude that a Regge pole contribution to $A_u(s,t)$ has an $u-$cut at $u \geq 4m^2$, but the corresponding background integral diverges on both borders of this cut. In all other regions, both the Regge term coming from $A_u(s,t)$ and the corresponding background integral are regular in u (or s) and may be anambigiously calculated. Therefore the Regge representation (11) for the amplitude $A_u(s,t)$ is well defined everywhere except the physical u-channel cut. We are not discussing symmetry properties of $A_s(s,t)$ and $A_u(s,t)$ and of the correponding partial amplitudes because these properties are unimportant for our conclusions and can be found elsewhere. All above conclusions remain valid if the Mandelstam-Sommerfeld-Watson representation[3] for the amplitude will be used instead of the Sommerfeld-Watson representation (5).

can be used to describe $A(s,t)$ by the Regge representation (11) everywhere except the physical cuts of two other channels. Divergent background integrals prevent an anambigious application of Regge method on these cuts even in the asymptotic regions. The fact that total cross section asymptotics or some properties of asymptotic amplitudes can be described by pomeron or by any other Regge singularity cannot serve as a proof of background integral convergence. Similar properties of asymptotic amplitudes and cross sections can be obtained without use of complex-l plane and Regge method, for example by summation of perturbation theory diagrams. In fact, the Regge-like asymptotics of direct-channel amplitudes

$$A(s,t) \sim (-s)^{\alpha(t)} \tag{28}$$

was obtained in several approaches within the perturbation theory [6],[7],[8]. The realistic result (28) of these approaches is not explicitely based on properties of partial amplitudes on the complex-l plane, but it is a net result of perturbative diagram combinatorics. Already from the initial p.w. expansion (3) we can see that the s-channel asymptotics (28) looks like if there would be something with spin $\alpha(t)$ exchanged in t-channel. However, this might be nothing but a similarity. We have shown that it is ambigious to implicate the quantum number of t-channel spin in expressions for the absorptive part of amplitudes in s- and u-regions. Therefore the reggeization of both empirical cross sections and asymptotic amplitudes in perturbation theory does not automatically confirm the reggeization of asymptotic *physical* amplitudes in the Regge method. However, in the BFKL model[9] it was explicitely shown that the asymptotics (28) of physical amplitudes may be provided by contributions of Regge singularities of partial amplitudes, calculated within the perturbative QCD. The discussion of BFKL model assumptions is beyond the scope of this paper. We only notice that the BFKL solution for partial amplitudes evidently leads to divergent background integrals. As we mentioned, the result (28) can be obtained by the same perturbative methods as in Ref.[9] but without use of complex-l plane and Regge singularities[8].

By this note we do not question the fact that elementary particles may be described by Regge trajectories. We have only argued that the possibility to describe direct-channel phy-

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