# $\mathcal{O}(\alpha_s^2)$ Corrections to Top Quark Production at $e^+e^-$ Colliders

R. Harlander[a] and M. Steinhauser[b]

[a] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany,*

[b] *Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, D-80805 Munich, Germany.*

**Abstract**

In this letter we evaluate mass corrections up to $\mathcal{O}((m^2/q^2)^6)$ to the three-loop polarization function induced by an axial-vector current. Special emphasis is put on the evaluation of the singlet diagram which is absent in the vector case. As a physical application $\mathcal{O}(\alpha_s^2)$ corrections to the production of top quarks at future $e^+e^-$ colliders is considered. It is demonstrated that for center of mass energies $\sqrt{s} \gtrsim 500$ GeV the inclusion of the first seven terms into the cross section leads to a reliable description.

In the total cross section $\sigma(e^+e^- \to \text{hadrons})$ corrections arising from the finite mass, $m$, of the produced quarks may often be neglected. Concerning precision measurements around the $Z$ resonance first order mass corrections, known up to $\mathcal{O}(\alpha_s^3)$ [1, 2], are usually adequate. However, having in mind top quark production at future colliders like the NLC with a center of mass energy of $\sqrt{s} = 500$ GeV higher order terms in $m^2/s$ may become important. The velocity of the produced particles is then $v \approx 0.7$ which means that on one side threshold effects are not important and on the other side we are not in the region of very high energies.

In Ref. [3] the contribution of the photon to the production of top quarks was considered. In this letter also the exchange of the $Z$ boson is included. Hence, in a first step results for the axial-vector polarization function up to $\mathcal{O}(\alpha_s^2)$ are considered. The imaginary part in combination with the recently evaluated rate for the vector case [4] directly leads to the cross section $\sigma(e^+e^- \to t\bar{t}(g(g)))$ mediated by a virtual $Z$ boson. The $\mathcal{O}(\alpha_s)$ corrections to this process were considered in [5].

To be more precise let us define the axial-vector current correlator as:

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi^a(q^2) + q_\mu q_\nu \, \Pi_L^a(q^2) \;\; = \;\; i \int dx \, e^{iqx} \langle 0|T j_\mu^a(x) j_\nu^a(0)|0\rangle \tag{1}$$

with $j_\mu^a = \bar{\psi}\gamma_\mu\gamma_5\psi$. We will only present results for $\Pi^a(q^2)$ as the longitudinal part, $\Pi_L^a(q^2)$, is directly connected to the pseudo-scalar polarization function $\Pi^p(q^2)$ for which the high energy expansion was considered in [6], via the axial Ward identity. It is convenient to write

$$\begin{aligned}
\Pi^a(q^2) &= \Pi^{(0),a}(q^2) + \frac{\alpha_s(\mu^2)}{\pi} C_F \Pi^{(1),a}(q^2) + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \Pi^{(2),a}(q^2) + \dots\ , \\
\Pi^{(2),a} &= C_F^2 \Pi_A^{(2),a} + C_A C_F \Pi_{NA}^{(2),a} + C_F T n_l \Pi_l^{(2),a} + C_F T \Pi_F^{(2),a},
\end{aligned} \tag{2}$$

with the $SU(3)$ colour factors $C_F = 4/3, C_A = 3$ and $T = 1/2$. $\Pi_A^{(2),a}$ is the abelian contribution already present in QED and $\Pi_{NA}^{(2),a}$ originates from the non-abelian structure specific for QCD. The polarization functions containing a second massless or massive quark loop are denoted by $\Pi_l^{(2),a}$ and $\Pi_F^{(2),a}$, respectively. $\Pi^a$ represents the so-called non-singlet part. However, for external axial-vector currents already at $\mathcal{O}(\alpha_s^2)$ there exists also a singlet or double-triangle contribution:

$$\Pi_S^a(q^2) = \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 C_F T\, \Pi_S^{(2),a}(q^2). \tag{3}$$

As $\Pi_S^a$ depends on the properties of both members of the fermion doublet we will from now on specify to the top-bottom case. The generalization to other quark flavours is obvious. For this contribution it is convenient to replace the current $j_\mu^a$ in Eq. (1) by $j_{S,\mu}^a = \bar{t}\gamma_\mu\gamma_5 t - \bar{b}\gamma_\mu\gamma_5 b$ because in this combination the axial anomaly cancels.

Similar relations as in Eqs. (2) and (3) also hold for $R^a(s)$ and $R_S^a(s)$, respectively, defined through

$$R_{(S)}^a(s) = 12\pi\, \mathrm{Im}\, \Pi_{(S)}^a(q^2 = s + i\epsilon), \tag{4}$$

so that the cross section for the inclusive production of top quarks may be written as

$$\begin{aligned}
R_t(s) &= \frac{\sigma(e^+e^- \to t\bar{t}(g(g)))}{\sigma_{pt}} \\
&= (v_e^2 + a_e^2) a_t^2 \left(\frac{s}{s - M_Z^2}\right)^2 \left(R^a(s) + R_S^a(s) - \left(\frac{\alpha_s}{\pi}\right)^2 C_F T R_{Sb}^{(2),a}(s)\right) \\
&\quad + \left[Q_e^2 Q_t^2 + 2 Q_e v_e Q_t v_t \frac{s}{s - M_Z^2} + (v_e^2 + a_e^2) v_t^2 \left(\frac{s}{s - M_Z^2}\right)^2\right] R^v(s),
\end{aligned} \tag{5}$$

with $\sigma_{pt} = 4\pi\alpha^2/3s$, $v_f = (I_3^f - 2Q_f s_\theta^2)/(2s_\theta c_\theta)$, $a_f = I_3^f/(2s_\theta c_\theta)$, $Q_e = -1, Q_t = 2/3, I_3^e = -1/2$ and $I_3^t = 1/2$. Furthermore we have $c_\theta^2 = 1 - s_\theta^2$ with $s_\theta$ being the sine of the weak mixing angle. $R^v(s)$ is given in [4] and both $R^a(s)$ and $R_S^a(s)$ are considered in this work. $R_{Sb}^{(2),a}$ is the contribution from cuts of the singlet diagram that do not involve top quarks [7]. Contributions where the photon or $Z$ boson couples to a light quark flavour and the top quarks are produced via gluon splitting [8] will be neglected as their numerical values are tiny [3].

Using the large momentum procedure the first seven terms in the $m^2/q^2$-expansion of $\Pi^a(q^2)$ have been evaluated. We refrain from listing the results and only show the figures

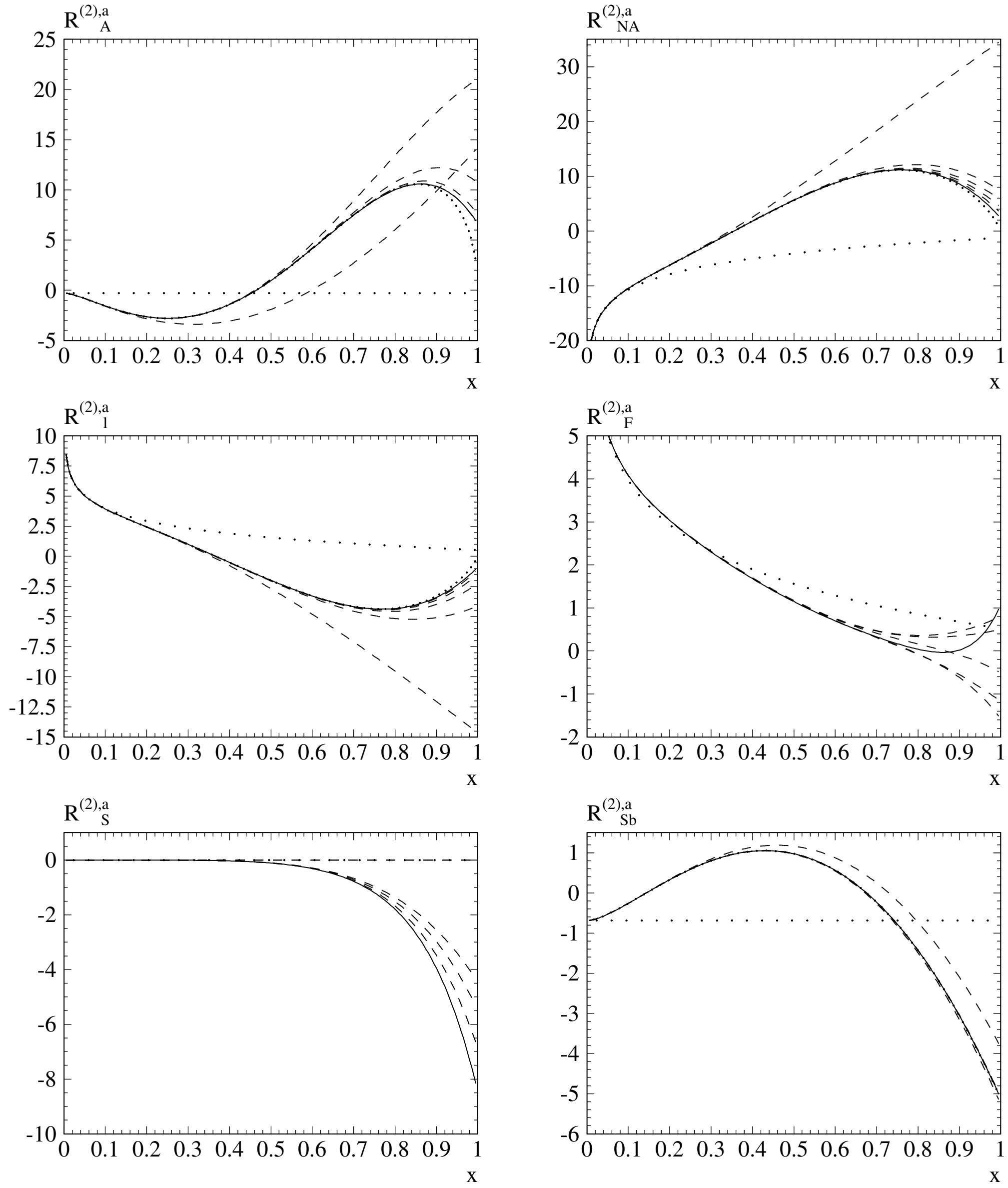


Figure 1: $R_i^{(2),a}$, $i = A, NA, l, F, S, Sb$ as functions of $x = 2M_t/\sqrt{s}$ at $\mu^2 = M_t^2$. Successively higher order terms in $(M_t^2/s)^n$: Dotted: $n = 0$; dashed: $n = 1, \ldots, 5$; solid: $n = 6$. Narrow dots: exact result ($R_l^{(2),a}$) or semi-analytical results ($R_A^{(2),a}$, $R_{NA}^{(2),a}$).

that result by taking the imaginary part according to Eq.(4). In Fig. 1 the terms for the five different color contributions are plotted against $x = 2M_t/\sqrt{s}$ including successively higher orders in $M_t^2/s$. For $R_A^{(2),a}$ and $R_{NA}^{(2),a}$ a comparison with a recently evaluated semi-analytical result (narrow dots) [9] is possible and agreement up to $x \approx 0.8$ is found. The light fermion contribution, $R_l^{(2),a}$, may be compared with exact results [10] (narrow dots)

| $\sqrt{s}$ (GeV) | $x$ | $\alpha_s^{(6)}(s)$ | $R_t^{(0)}$ | $C_F R_t^{(1)}$ | $R_t^{(2)}$ | $R_t(s)$ |
|---|---|---|---|---|---|---|
| 500 | 0.70 | 0.095 | 1.419 | 6.021 | 29.902 | 1.629 |
| 1000 | 0.35 | 0.088 | 1.732 | 2.842 | 6.016 | 1.816 |
| 1500 | 0.23 | 0.085 | 1.771 | 2.291 | 3.709 | 1.836 |
| 2000 | 0.18 | 0.083 | 1.784 | 2.091 | 3.044 | 1.841 |

Table 1: Numerical values for the contributions of $\mathcal{O}(\alpha_s^i)$, $R_t^{(i)}$, to the normalized cross section $R_t$. The values for $\alpha_s^{(6)}(s)$ are based on $\alpha_s^{(5)}(M_Z^2) = 0.118$. The scale $\mu^2 = s$ has been adopted. Also the values of $x = 2M_t/\sqrt{s}$ are shown.

and also shows agreement up to $x \approx 0.8$. Concerning $R_F^{(2),a}$ and $R_S^{(2),a}$ the situation is less satisfactory. It seems that there is reasonable convergence up to $x \approx 0.7$ which is also motivated by the behaviour of the vector case where analytical results for $x > 0.5$ are available (see [4]). However, for $x > 0.7$ the behaviour of the curve including all known power correction terms (solid line) indicates that close to $x = 1$ the convergence fails to work. The reason presumably is connected to the four particle cut starting at $x = 0.5$. Although $R_A^{(2),a}$ and $R_{NA}^{(2),a}$ also exhibit a four particle cut it seems to be somehow less dominant for these contributions. In Fig. 1 also the contribution $R_{Sb}^{(2),a}$ is shown. Here, already the curve including power corrections up to order $(M_t^2/s)^2$ is practically indistinguishable from the exact result. Note that only the difference of the two plots in the bottom line of Fig. 1 enters $R_t(s)$.

Recalling that the first seven terms for $R^v(s)$ approximate the exact result up to $x \approx 0.7$ [4] we are now prepared to present predictions for $R_t$ valid up to $\mathcal{O}(\alpha_s^2)$ for $\sqrt{s} \gtrsim 500$ GeV which corresponds to $x \lesssim 0.7$. Therefore we insert the isospin and charge quantum numbers into Eq. (5) and choose $n_l = 5$, $s_\Theta^2 = 0.2315$, $\alpha_s^{(5)}(M_Z^2) = 0.118$, $M_Z = 91.187$ GeV and $M_t = 175$ GeV. Then the expansion of $R_t(s)$ looks as follows:

$$R_t(s) \;=\; R_t^{(0)}(s) + \frac{\alpha_s^{(6)}(s)}{\pi} C_F\, R_t^{(1)}(s) + \left(\frac{\alpha_s^{(6)}(s)}{\pi}\right)^2 R_t^{(2)}(s). \qquad (6)$$

In Tab. 1 the coefficients $R_t^{(i)}$ are listed for different values of the center of mass energy $\sqrt{s}$. One observes that for $\sqrt{s} = 500$ GeV, which is a proposed option for the NLC, the $\mathcal{O}(\alpha_s^2)$ QCD corrections amount to $\approx 2\%$. For higher values of the center of mass energy these terms get less important. In Fig. 2 the normalized cross section $R_t$ is plotted against $\sqrt{s}$.

To conclude, the large momentum procedure has been applied to the axial-vector polarization function and terms up to order $(M_t^2/q^2)^6$ have been determined. The imaginary part in combination with the result recently obtained for the vector case was used to predict the production of top quarks at future $e^+e^-$ colliders up to $\mathcal{O}(\alpha_s^2)$.

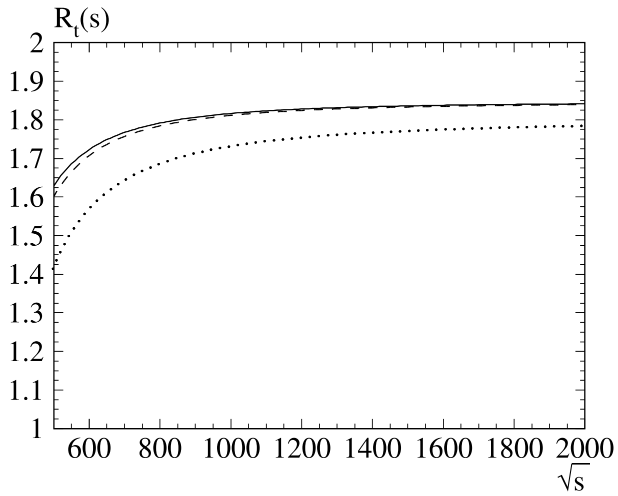


Figure 2: Normalized cross section $R_t$ as a function of the center of mass energy $\sqrt{s}$. Dotted: Born approximation; dashed: $\mathcal{O}(\alpha_s)$, solid: $\mathcal{O}(\alpha_s^2)$. The scale $\mu^2 = s$ has been adopted.

**Acknowledgments**

We would like to thank K.G. Chetyrkin and J.H. Kühn for suggesting our collaboration in this project and for carefully reading the manuscript. M.S. thanks B.A. Kniehl for useful discussions. The work of R.H. was supported by the "Landesgraduiertenförderung" at the University of Karlsruhe.

# References

[1] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, *Phys. Rept.* **277** (1996) 189.

[2] K.G. Chetyrkin and J.H. Kühn, Report Nos. MPI/PhT/96-84, TTP97-14 and hep-ph/9609202.

[3] K.G. Chetyrkin, A.H. Hoang, J.H. Kühn, M. Steinhauser and T. Teubner, in proceedings of $e^+e^-$ *collisions at TeV energies*; Annecy, Gran Sasso, Hamburg; Feb. 1995 to Sept. 1995; edited by P.M. Zerwas; Report Nos. TTP96-11, MPI/PhT/96-24, DPT/96/34 and hep-ph/9605311.

[4] K.G. Chetyrkin, R. Harlander, J.H. Kühn and M. Steinhauser, Report Nos. MPI/PhT/97-012, TTP97-11 and hep-ph/9704222 (to be published in *Nucl. Phys.* **B**).

[5] J. Jersák, E. Laermann and P. Zerwas, *Phys. Rev.* **D 25** (1982) 1218; (E) ibid. **D 36** (1987) 310.

[6] R. Harlander and M. Steinhauser, Report Nos. MPI/PhT/97-013, TTP97-12 and hep-ph/9704436.

[7] B.A. Kniehl and J.H. Kühn, *Phys. Lett.* **B 224** (1989) 229; *Nucl. Phys.* **B 329** (1990) 547.

[8] A.H. Hoang, M. Ježabek, J.H. Kühn and T. Teubner, *Phys. Lett.* **B 338** (1994) 330.

[9] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, Report Nos. MPI/PhT/97-029, TTP97-18 and hep-ph/9705254.

[10] A.H. Hoang and T. Teubner, *private communication.*