Semi-analytical Analysis of Single-pass Microbunching Instability in presence of Intrabeam Scattering Effect

Cheng-Ying Tsai[1] Weilun Qin[2]

[1] Huazhong University of Science and Technology

[2] Deutsches Elektronen-Synchrotron DESY

[1] Email: jcytsai@hust.edu.cn.

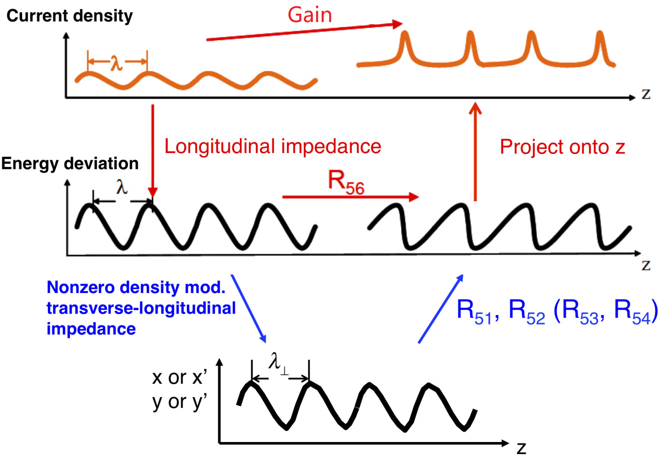
[2] Email: weilun.qin@desy.de.

Outline

- Introduction
- Motivation
- Theoretical formulation
  - Model assumptions
  - 0th- and 1st-order dynamics
- Semi-analytical Vlasov-Fokker-Planck (VFP) solver
  - Tool capabilities
  - Possible extensions
- Examples
  1. Mid-energy recirculation-IBS ring
  2. FODO-BC-FODO-BC transport line
- Summary and Discussion

Introduction: Micro-Bunching

Microbunching involves **phase space modulation** and/or **pure optics transport** and/or **high-frequency impedances**



Intrabeam scattering (IBS)

Classic models

1. Theoretical models: A. Piwinski, J.D. Bjorken, S.K. Mtingwa (2018 Wilson Prize), M. Martini, K. Bane, CIMP, etc
2. Numerical simulation: direct solution of Fokker-Planck equation, Monte Carlo method, etc

Physical processes

- small-angle, multiple particle-particle scattering (different from space charge and Touschek scattering)
- diffusion in particle momentum
- friction in particle momentum
- growth of energy spread and beam emittances

★ Our analysis employs CIMP (Completely Integrable Modified Piwinski) formula to evaluate IBS effects.

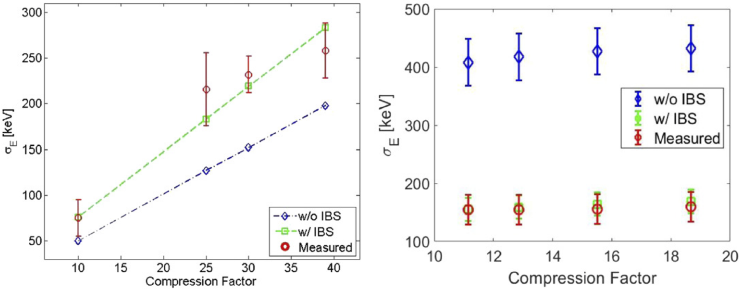
Intrabeam scattering (IBS)

According to Piwinski, calculation of IBS growth rate involves

1. Lorentz transformation from Lab frame to beam rest frame
2. Calculate momentum change due to elastic Coulomb scattering
3. Lorentz transformation back to Lab frame
4. Change of longitudinal momentum $\overset{R_{16,36}}{\Rightarrow}$ change of transverse coordinates $\Rightarrow \Delta\epsilon_{\perp}^{\mathsf{IBS}}$ (similar to $\Delta\delta^{\mathsf{IBS}}$)
5. Apply cross section formula, average over the scattering angle
6. Average over position and momentum coordinates
7. Obtain IBS growth rates

Motivation

Recent experiment at FERMI linac[3] indicates that IBS may have significant effect on FEL performance in terms of incoherent energy spread



★ Physical mechanism: Both MBI and IBS heat the beam, with different mechanisms, but are not fully independent. Existing MBI theory does not properly take IBS into account.

[3]S. DiMitri et al., New J. Phys. **22**, 083053 (2020)

Motivations

Microbunching has been one of the research focuses in accelerator physics and is expected to remain so in the years to come, as evidenced by the advent of free-electron lasers (FELs).

Pros and cons for particle tracking simulation vs. kinetic analysis:

- Particle tracking: time domain, can be sensitive to numerical noise ⇒ time-consuming (huge number of macroparticles, sufficient number of bins), easy to implement different physical effects[4], many available simulation packages
- Kinetic analysis: frequency domain, direct solution of microbunched phase space can be avoided ⇒ efficient and free from numerical noise, suitable for systematic studies and/or design optimization, not always straightforward to add various physical effects[5], simulation packages usually not avaiable

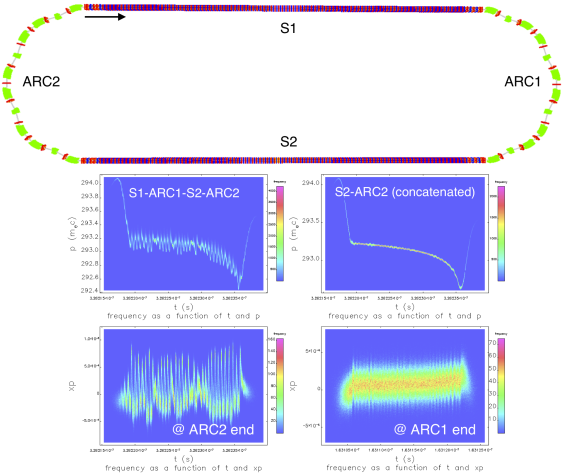
★ Goal: Develop an efficient, accurate semi-analytical analysis to clarify the interplay between MBI and IBS.

[4] For example, nonlinear single-particle effect. But, there can be one exception: it is difficult and time-consuming to simulate CSR **and** LSC relevant beam dynamics in particle tracking simulations.

[5] There can be one exception: it is straightforward to add CSR **and** LSC to the analysis.

A caveat

Better to perform 6-D start-to-end calculation for accurate analysis. Either lower-dimensional or concatenated analysis would likely underestimate MB[6].



[6] C.-Y. Tsai, NIMA **940**, 462-474 (2019), beamline lattice from S. DiMitri, PRST-AB **17**, 074401 (2014).

Kinetic analysis: Vlasov-Fokker-Planck equation

$$\frac{\mathrm{d}f}{\mathrm{d}s} = -\sum_{i=x,y,z} \frac{\partial}{\partial p_i}(D_i f) + \frac{1}{2}\sum_{i,j=x,y,z} \frac{\partial^2}{\partial p_i \partial p_j}(D_{ij} f)$$

If the friction D_i and diffusion D_{ij} can be neglected, VFP equation reduces to Vlasov equation (or collisionless Boltzmann equation). In usual situations, the time scale for the collective dynamics is shorter than that of the diffusion dynamics. For long-term dynamics and/or high-peak current, one may need to include RHS to base the analysis on VFP equation.

Direct, 6-D solution can be very complicated. One may Taylor expand $f = f_0 + f_1$ with $|f_1| \ll f_0$

- 0th order solution $\Rightarrow$ pure optics transport and/or incoherent effects (e.g., IBS, ISR), PWD (for storage ring)
- 1st order solution $\Rightarrow$ the collective dynamics

Phase space microbunching involves the dynamical evolution of **the characteristic functions of** f_1, e.g., density modulation $b(k_z; s) = \frac{1}{N}\int f_1(\mathbf{X}; s) e^{-ik_z z_s} \mathrm{d}\mathbf{X}$. Denote $\mathbf{b}_{k_z}$ as $b(k_z; s), \forall s$.

Model assumptions

1. $|f_1| \ll f_0 \Rightarrow$ this assumption would fail when phase space modulation saturates (become distorted, filamented)
2. Modulation wavelength $\ll \sigma_z$ or coasting beam approximation $\Rightarrow$ may fail when an electron bunch is critically compressed
3. Single-frequency assumption $\Rightarrow$ relevant to coasting beam approximation
   $\Rightarrow$ can be extended to quasi-multi-frequency for the case of large longitudinal phase space shearing
4. and so on

Vlasov-Fokker-Planck equation

$$\frac{df}{ds} = -\sum_{i=x,y,z} \frac{\partial}{\partial p_i}(D_i f) + \frac{1}{2}\sum_{i,j=x,y,z} \frac{\partial^2}{\partial p_i \partial p_j}(D_{ij} f)$$

Direct, 6-D numerical solution is too complicated. Here we

1. Decompose into the 0th and 1st order terms
   - 0th order (pure optics, IBS) $\Rightarrow$ existing IBS formula[7]
     $\Rightarrow \frac{df_0}{ds} = -\sum_{i=x,y,z} \frac{\partial}{\partial p_i}(D_i f_0) + \frac{1}{2}\sum_{i,j=x,y,z} \frac{\partial^2}{\partial p_i \partial p_j}(D_{ij} f_0)$
   - 1st order (collective effect)
     $\Rightarrow \frac{df_1}{ds} \approx -\frac{\partial f_0}{\partial \delta}\left(\frac{d\delta}{ds}\right)_1 - \frac{\partial}{\partial \delta}(D_{z,0}(s) f_1) - \frac{\partial}{\partial \delta}(D_{z,1}(s) f_0) +$
     $D_{zz,0}(s)\frac{\partial^2 f_1}{\partial \delta^2} + D_{zz,1}(s)\frac{\partial^2 f_0}{\partial \delta^2} \Rightarrow$ require further simplification
2. Instead of solving $f(\mathbf{X}; s)$, we derive the evolution equations for
   - density modulation $\Rightarrow b(k_z; s) = \frac{1}{N}\int f_1(\mathbf{X}; s) e^{-ik_z z_s} d\mathbf{X}$
   - energy modulation[8]
     $\Rightarrow p(k_z; s) = \frac{1}{N}\int (\delta_s - h z_s) f_1(\mathbf{X}; s) e^{-ik_z z_s} d\mathbf{X}$
3. $\sigma_\delta^{(0)}(s), \epsilon_\perp^{(0)}(s)$ will be substituted into 1st-order equations

[13] For example, Piwinski, Bjorken-Mtingwa, K. Bane, K. Kubo, V. Lebedev, etc.

[14] The energy modulation refers to (z, δ), different from that of EEHG-like energy band structure.

Linearized integral equations[9]

From definition of the diffusion and friction coefficients in VFP equation, for IBS, they can be derived

$$D_z(s) = -\left(\frac{r_e[\text{Log}]}{\gamma^2\epsilon_{\perp,N}^2}\frac{I_b}{I_A}\right)\text{erf}\left(\frac{\delta}{\sqrt{2}\sigma_\delta}\right)$$

$$D_{zz}(s) = \frac{\sqrt{\pi}}{2}\left(\frac{r_e[\text{Log}]}{\gamma^2\epsilon_{\perp,N}\sigma_\perp}\frac{I_b}{I_A}\right)$$

Substituting $f = f_0 + f_1$ into VFP and neglecting higher order terms of f_1, we would obtain the linearized VFP equation. Expressed in terms of the density and energy modulations,

$$b(k_z;s) = \frac{1}{N}\int f_1(\mathbf{X};s)e^{-ik_z z_s}\text{d}\mathbf{X}$$

$$p(k_z;s) = \frac{1}{N}\int \delta_s f_1(\mathbf{X};s)e^{-ik_z z_s}\text{d}\mathbf{X}$$

we would obtain a set of linear coupled integral equations.

[9] C.-Y. Tsai, W. Qin et al., Phys. Rev. Accel. Beams **23**, 124401 (2020)

Vlasov-Fokker-Planck equation$^{\text{0th order}}$ (CIMP[10])

$$\frac{1}{\sigma_\delta}\frac{\mathrm{d}\sigma_\delta}{\mathrm{d}s} = \tau^{-1}_{\mathrm{IBS},\delta} + \frac{1}{C}\frac{\mathrm{d}C}{\mathrm{d}s}$$

$$\tau^{-1}_{\mathrm{IBS},\delta} = 2\times 2\pi^{3/2}A\left[\frac{\sigma_H^2}{\sigma_\delta^2}\left([\mathrm{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathrm{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$\frac{1}{\epsilon_x^G}\frac{\mathrm{d}\epsilon_x^G}{\mathrm{d}s} = \tau^{-1}_{x,\mathrm{IBS}} = 4\pi^{3/2}A\left[-a[\mathrm{Log}]_x g\left(\frac{b}{a}\right) + \frac{\mathcal{H}_x\sigma_H^2}{\epsilon_x^G}\left([\mathrm{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathrm{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$\frac{1}{\epsilon_y^G}\frac{\mathrm{d}\epsilon_y^G}{\mathrm{d}s} = \tau^{-1}_{y,\mathrm{IBS}} = 4\pi^{3/2}A\left[-b[\mathrm{Log}]_y g\left(\frac{a}{b}\right) + \frac{\mathcal{H}_y\sigma_H^2}{\epsilon_y^G}\left([\mathrm{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathrm{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$A = \frac{I_b}{\left(2\sqrt{2\ln 2}\right)64\pi^2\gamma^2\epsilon_x^N\epsilon_y^N\sigma_\delta}\frac{r_e^2}{ce}, [\mathrm{Log}]_x = \ln\left(\frac{q^2}{a^2}\right), [\mathrm{Log}]_y = \ln\left(\frac{q^2}{b^2}\right)$$

$$\mathcal{H}_{x,y} = \frac{R_{16,36}^2 + \left(\beta_{x,y}R_{26,46} + \alpha_{x,y}R_{16,36}\right)^2}{\beta_{x,y}}, q = \sigma_H\beta\sqrt{2d/r_e}, d = \min\left\{\sigma_x,\sigma_y,\lambda_D\right\}$$

$$g(w) = \sqrt{\frac{\pi}{w}}\left[P^0_{-1/2}\left(\frac{w^2+1}{2w}\right) \pm P^{-1}_{-1/2}\left(\frac{w^2+1}{2w}\right)\right]$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_x}{\epsilon_x^G}}, b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_y}{\epsilon_y^G}}, \frac{1}{\sigma_H^2} = \frac{1}{\sigma_\delta^2} + \frac{\mathcal{H}_x}{\epsilon_x^G} + \frac{\mathcal{H}_y}{\epsilon_y^G}$$

[10] K. Kubo et al., PRST-AB **8**, 081001 (2005)

Linearized matrix equations

Skipping the lengthy derivation, the set of linear integral equations can be expressed in the matrix equation in a compact way

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{R} & \mathcal{S} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{k_z} \\ \mathbf{p}_{k_z} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{k_z}^{(0)} \\ \mathbf{p}_{k_z}^{(0)} \end{bmatrix}$$

where

$$\mathcal{P} = \mathcal{I} - i\mathcal{K}_{Z_\parallel}^{(1)} - \mathcal{K}_{\mathsf{IBS},z}^{(1)} + 2\mathcal{K}_{\mathsf{IBS},zz}^{(2)}$$

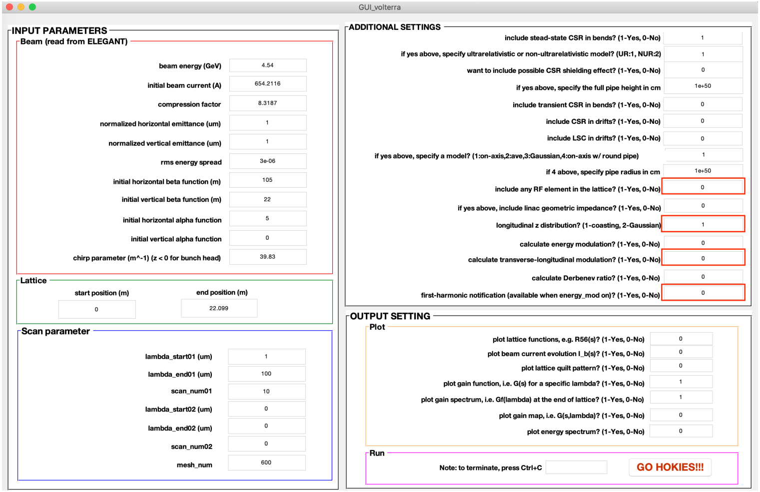
$$\mathcal{Q} = -i\mathcal{K}_{\mathsf{IBS},z}^{\perp(0)} - i\mathcal{K}_{\mathsf{IBS},zz}^{(3)}$$

$$\mathcal{R} = \mathcal{K}_{Z_\parallel}^{(0)} - \mathcal{K}_{Z_\parallel}^{(2)}\sigma_{\delta\tau}^2 - i\mathcal{K}_{\mathsf{IBS},z}^{(0)} - 2i\mathcal{K}_{\mathsf{IBS},z}^{(1)} + 4i\mathcal{K}_{\mathsf{IBS},zz}^{(1)} - 2i\mathcal{K}_{\mathsf{IBS},zz}^{(3)}\sigma_{\delta\tau}^2$$

$$\mathcal{S} = \mathcal{I} + \mathcal{K}_{\mathsf{IBS},z}^{\perp(0)} - \mathcal{K}_{\mathsf{IBS},z}^{\perp(2)} + 3\mathcal{K}_{\mathsf{IBS},zz}^{(2)} - \mathcal{K}_{\mathsf{IBS},zz}^{(4)}\sigma_{\delta\tau}^2$$

The kernel functions $\mathcal{K}_{Z_\parallel}$ involve collective effects and $\mathcal{K}_{\mathsf{IBS}}$ reflect the IBS effect.

An efficient, accurate tool for microbunching analysis



Input files: `elegant *.ele & *.lte`
Available on Github: `https://github.com/jcytsai/volterra_mat`, version 4.2

More refined, friendly GUI is under development

Tool capabilities

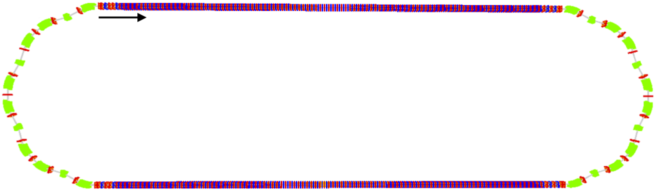
	Our Vlasov solver	Heifets *et al.*	Huang and Kim
Vlasov model	linear, semi-analytical		linear, analytical
transverse emittance effect	yes	yes	yes
bending plane	horizontal & vertical	horizontal	horizontal
beam acceleration	yes	no	no
energy modulation	yes	no	yes, approximate expression
transverse-longitudinal modulation (x,z) or (x',z) (y,z) or (y',z)	yes	no	no
IBS	yes	no	no

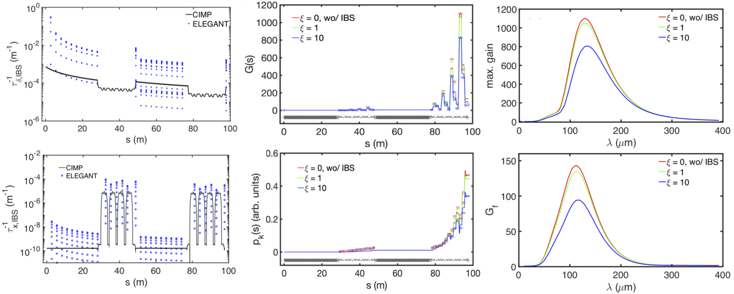
		Our Vlasov solver	Heifets *et al* Huang and Kim
1-D CSR	steady-state free-space	yes NUR & UR	yes only UR
	entrance transient free-space	yes UR	no
	exit transient free-space	yes NUR & UR	no
	steady-state with shielding	yes	no
LSC		yes	no
linac geometric effect		yes	no

Note: NUR: Non-UltraRelativistic, UR: UltraRelativistic

Example 1: 150-MeV quasi-isochronous ring[11]

IBS may play a negligible effect on MB for one turn





[11] C.-Y. Tsai et al., Phys. Rev. Accel. Beams **23**, 124401 (2020), beamline lattice from S. DiMitri.

Order of magnitude estimate

	Storage ring light source	Middle-energy single-pass accelerator
Beam energy	$\sim$GeV	$\sim$100 MeV
Particles per bunch	10^{10} or more	$10^8 \sim 10^9$
Peak current	50$\sim$100 A	100 $\sim$ a few kA
Normalized emittances	$\sim \mu$m	1 μm or lower
Fractional energy spread	$10^{-3} \sim 10^{-4}$	10^{-4} or smaller
Effective distance	∞	100 m$\sim$a few km

$$\text{IBS growth} \quad \tau_{\text{IBS}}^{-1} \left(\equiv \frac{1}{\left(\epsilon_\perp^N, \sigma_\delta\right)} \frac{\mathrm{d}\left(\epsilon_\perp^N, \sigma_\delta\right)}{\mathrm{d}s} \right) \propto \frac{N_b}{\gamma^2 \epsilon_x^N \epsilon_y^N \sigma_z \sigma_\delta}$$

$$\Rightarrow \tau_{\text{IBS,single-pass}}^{-1} \approx 10^{2\sim3} \tau_{\text{IBS,storage-ring}}^{-1}$$

Energy chirp & bunch compression $\Rightarrow$ another factor of $10 \sim 10^2$ enhancement

Slice energy spread (SES)

In addition to MBI gain, one may care more about SES. Short wavelength energy modulation ≈ SES, which may be attributed to

1. pure optics ⇒ $\sigma_\delta^{\text{pure optics}} \approx C_{\text{tot}}\sigma_{\delta 0}$. Bunch compression increases SES.
2. IBS ⇒ $\sigma_{\delta,\text{IBS}}$ obtained from CIMP formula. Bunch compression will locally increase IBS growth rate.
3. collective effect ⇒ $\sigma_{\delta,\text{coll}}$ evaluated from energy modulation. Bunch compression increases peak current, thus enhancing collective effect

$$\sigma_{\delta,\text{coll}}^2 = \frac{8}{n_b} C_{\text{tot}} \int_0^{\lambda^*} \frac{d\lambda}{\lambda^2} \left| \int_0^{s_f} d\tau \frac{I_b(\tau)}{\gamma I_A} Z_0^{\parallel}(\lambda;\tau)\tilde{G}(\lambda;\tau) \right|^2$$

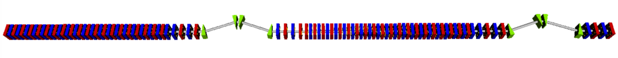
$$\sigma_{\delta,\text{tot}} \approx \begin{cases} \sqrt{C_{\text{tot}}^2\sigma_{\delta 0}^2 + C_{\text{tot}}^2\sigma_{\delta,\text{coll}}^2}, & \text{without IBS} \\ \sqrt{\sigma_{\delta,\text{IBS}}^2 + C_{\text{tot}}^2\sigma_{\delta,\text{coll}}^2}, & \text{with IBS} \end{cases}$$

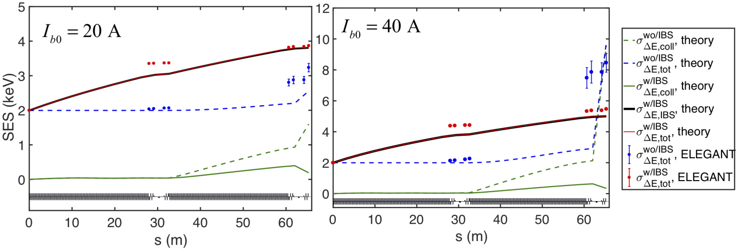
★ When is IBS beneficial to mitigate MBI? ⇒ $\sigma_{\delta,\text{tot}}^{\text{wo/IBS}} \gtrsim \sigma_{\delta,\text{tot}}^{\text{w/IBS}}$

Example 2: FODO-BC-FODO-BC transport line[12]

Both MBI and IBS heat the beam. However IBS-induced slice energy spread (SES) may further mitigate MBI.

Name	Value	Unit
Beam energy	150	MeV
Peak current	5∼40	A
Initial energy spread	1.33×10^{-5}	
Normalized emittances	0.4	μm
Momentum compaction	24.45	cm





[12] C.-Y. Tsai and W. Qin, Phys. Plasmas **28**, 013112 (2020)

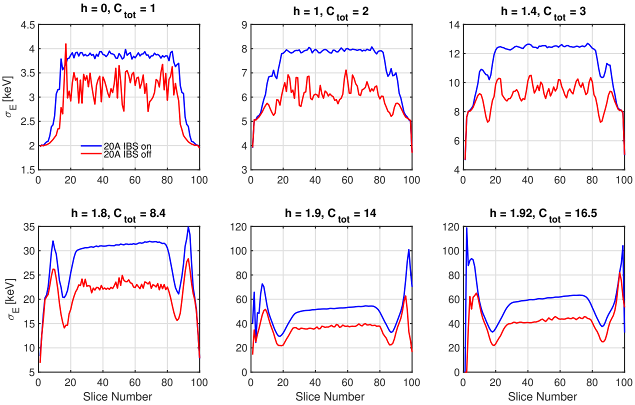


Figure: Slice energy spread for $I_{b0} = 20$ A for different energy chirps.

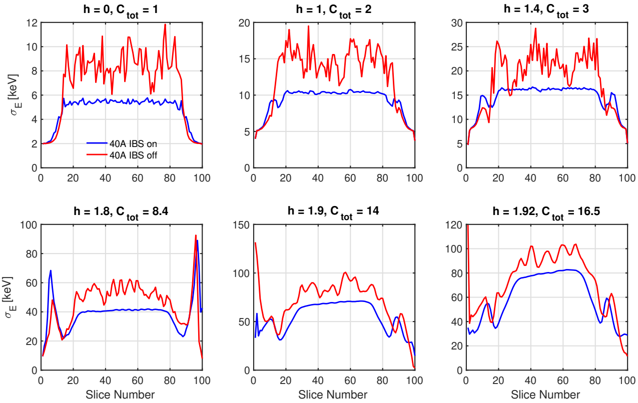


Figure: Slice energy spread for $I_{b0} = 40$ A for different energy chirps.

Threshold condition

Below the contour plot draws $\sigma_{\Delta E,\text{tot}}^{\text{wo/IBS}} - \sigma_{\Delta E,\text{tot}}^{\text{w/IBS}}$

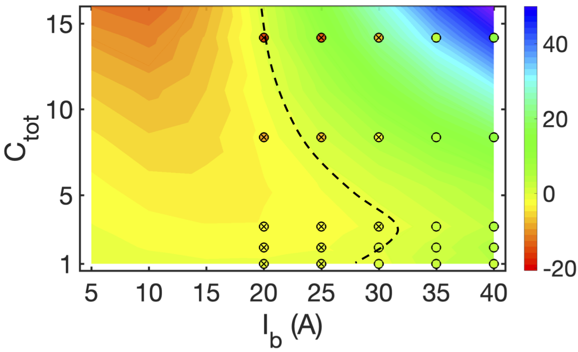


Figure: ○ and ⊗ are elegant tracking results. Background are results from VFP calculation. Dashed line refers to the case $\sigma_{\Delta E,\text{tot}}^{\text{wo/IBS}} = \sigma_{\Delta E,\text{tot}}^{\text{w/IBS}}$.

Using multi-stage coefficient[13], a semi-analytical expression of the threshold current can be found.

[13] C.-Y. Tsai, NIMA **943**, 162499 (2019).

Summary and Discussion

- To more accurately evaluate microbunching performance, it is better to perform 6-D start-to-end analysis. Either lower-dimensional or concatenated analysis would likely underestimate microbunching performance
- Detailed optics balance is key to control microbunching
  - Variation of lattice functions would matter for microbunched beam $\Rightarrow$ has been taken into account in our VFP solver
- A convenient semi-analytical VFP solver is developed and benchmarked with particle tracking simulations. Many extensions are ongoing
- Tool capabilities of the existing solver are summarized, including beam and field dynamics
- We expect that after possible extension this analysis may be applicable to
  - improved performance estimate of advanced FEL schemes
  - SSMB beam dynamics analysis

Acknowledgements

- Thank Dr. Torsten Limberg and Dr. Najmeh Mirian for the invitation
- Development of the semi-analytical VFP solver is a long, ongoing process. During these years, I benefit much from Steve Benson, Slava Derbenev, David Douglas, Rui Li, Chris Tennant (JLab), Irwan Setija (ASML), Weilun Qin (DESY)
- Stimulating discussions from colleagues/supervisors/friends: Simone DiMitri, Giovanni Perosa (FERMI-Elettra), Yi Jiao (IHEP), He Zhang, Yuhong Zhang (JLab), Juhao Wu, Guanqun Zhou (SLAC), Xiujie Deng (Tsinghua University)

Acknowledgements

Thank you for your attention

Vlasov-Fokker-Planck equation

$$\frac{\mathrm{d}f}{\mathrm{d}s} = -\sum_{i=x,y,z}\frac{\partial}{\partial p_i}\left(D_i f\right) + \frac{1}{2}\sum_{i,j=x,y,z}\frac{\partial^2}{\partial p_i \partial p_j}\left(D_{ij} f\right)$$

Direct, 6-D numerical solution is too complicated. Here we

1. Decompose into the 0th and 1st order terms
   - 0th order (pure optics, IBS) $\Rightarrow$ existing IBS formula[14]
     $\Rightarrow \frac{\mathrm{d}f_0}{\mathrm{d}s} = -\sum_{i=x,y,z}\frac{\partial}{\partial p_i}\left(D_i f_0\right) + \frac{1}{2}\sum_{i,j=x,y,z}\frac{\partial^2}{\partial p_i \partial p_j}\left(D_{ij} f_0\right)$
   - 1st order (collective effect)
     $\Rightarrow \frac{\mathrm{d}f_1}{\mathrm{d}s} \approx -\frac{\partial f_0}{\partial \delta}\left(\frac{\mathrm{d}\delta}{\mathrm{d}s}\right)_1 - \frac{\partial}{\partial \delta}\left(D_{z,0}(s) f_1\right) - \frac{\partial}{\partial \delta}\left(D_{z,1}(s) f_0\right) + D_{zz,0}(s)\frac{\partial^2 f_1}{\partial \delta^2} + D_{zz,1}(s)\frac{\partial^2 f_0}{\partial \delta^2}$ $\Rightarrow$ require further simplification
2. Instead of solving $f(\mathbf{X}; s)$, we derive the evolution equations for
   - density modulation $\Rightarrow b(k_z; s) = \frac{1}{N}\int f_1(\mathbf{X}; s) e^{-ik_z z_s}\mathrm{d}\mathbf{X}$
   - energy modulation[15]
     $\Rightarrow p(k_z; s) = \frac{1}{N}\int \left(\delta_s - h z_s\right) f_1(\mathbf{X}; s) e^{-ik_z z_s}\mathrm{d}\mathbf{X}$
3. $\sigma_\delta^{(0)}(s), \epsilon_\perp^{(0)}(s)$ will be substituted into 1st-order equations

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[13] For example, Piwinski, Bjorken-Mtingwa, K. Bane, K. Kubo, V. Lebedev, etc.

[14] The energy modulation refers to (z, δ), different from that of EEHG-like energy band structure.

Vlasov-Fokker-Planck equation$^{\text{0th order}}$ (CIMP[16])

$$\frac{1}{\sigma_\delta}\frac{\mathsf{d}\sigma_\delta}{\mathsf{d}s} = \tau^{-1}_{\mathsf{IBS},\delta} + \frac{1}{C}\frac{\mathsf{d}C}{\mathsf{d}s}$$

$$\tau^{-1}_{\mathsf{IBS},\delta} = 2\times 2\pi^{3/2}A\left[\frac{\sigma_H^2}{\sigma_\delta^2}\left([\mathsf{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathsf{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$\frac{1}{\epsilon_x^G}\frac{\mathsf{d}\epsilon_x^G}{\mathsf{d}s} = \tau^{-1}_{x,\mathsf{IBS}} = 4\pi^{3/2}A\left[-a[\mathsf{Log}]_x g\left(\frac{b}{a}\right) + \frac{\mathcal{H}_x\sigma_H^2}{\epsilon_x^G}\left([\mathsf{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathsf{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$\frac{1}{\epsilon_y^G}\frac{\mathsf{d}\epsilon_y^G}{\mathsf{d}s} = \tau^{-1}_{y,\mathsf{IBS}} = 4\pi^{3/2}A\left[-b[\mathsf{Log}]_y g\left(\frac{a}{b}\right) + \frac{\mathcal{H}_y\sigma_H^2}{\epsilon_y^G}\left([\mathsf{Log}]_x\frac{g\left(\frac{b}{a}\right)}{a} + [\mathsf{Log}]_y\frac{g\left(\frac{a}{b}\right)}{b}\right)\right]$$

$$A = \frac{I_b}{\left(2\sqrt{2\ln 2}\right)64\pi^2\gamma^2\epsilon_x^N\epsilon_y^N\sigma_\delta}\frac{r_e^2}{ce},\ [\mathsf{Log}]_x = \ln\left(\frac{q^2}{a^2}\right),\ [\mathsf{Log}]_y = \ln\left(\frac{q^2}{b^2}\right)$$

$$\mathcal{H}_{x,y} = \frac{R_{16,36}^2 + \left(\beta_{x,y}R_{26,46} + \alpha_{x,y}R_{16,36}\right)^2}{\beta_{x,y}},\ q = \sigma_H\beta\sqrt{2d/r_e},\ d = \min\left\{\sigma_x,\sigma_y,\lambda_D\right\}$$

$$g(w) = \sqrt{\frac{\pi}{w}}\left[P^0_{-1/2}\left(\frac{w^2+1}{2w}\right) \pm P^{-1}_{-1/2}\left(\frac{w^2+1}{2w}\right)\right]$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_x}{\epsilon_x^G}},\ b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_y}{\epsilon_y^G}},\ \frac{1}{\sigma_H^2} = \frac{1}{\sigma_\delta^2} + \frac{\mathcal{H}_x}{\epsilon_x^G} + \frac{\mathcal{H}_y}{\epsilon_y^G}$$

[16] K. Kubo et al., PRST-AB **8**, 081001 (2005)

Vlasov-Fokker-Planck equation$^{\text{1st order}}$

The integral equation of Volterra type for the density modulation

$$b(k_z;s) = b_0(k_z;s) + i\int_0^s K_{Z_\parallel}^{(1)}(\tau,s)b(k_z;\tau)\mathsf{d}\tau + \int_0^s K_{\mathsf{IBS},z}^{(1)}(\tau,s)b(k_z;\tau)\mathsf{d}\tau + i\int_0^s K_{\mathsf{IBS},z}^{\perp(0)}(\tau,s)p(k_z;\tau)\mathsf{d}\tau - 2\int_0^s K_{\mathsf{IBS},zz}^{(2)}(\tau,s)b(k_z;\tau)\mathsf{d}\tau + i\int_0^s K_{\mathsf{IBS},zz}^{(3)}(\tau,s)p(k_z;\tau)\mathsf{d}\tau$$

Vlasov-Fokker-Planck equation$^{\text{1st order}}$

The integral equation of Volterra type for the energy modulation

$$
\begin{aligned}
p(k_z;s) = p_0(k_z;s) &- \int_0^s \left[K_{Z_\parallel}^{(0)}(\tau,s) - K_{Z_\parallel}^{(2)}(\tau,s)\sigma_{\delta 0}^2\right] b(k_z;\tau)\mathrm{d}\tau \\
&+ i\int_0^s K_{\mathsf{IBS},z}^{(0)}(\tau,s) b(k_z;\tau)\mathrm{d}\tau + 2i\int_0^s K_{\mathsf{IBS},z}^{\perp(1)}(\tau,s) b(k_z;\tau)\mathrm{d}\tau \\
&- \int_0^s K_{\mathsf{IBS},z}^{\perp(0)}(\tau,s) p(k_z;\tau)\mathrm{d}\tau + \int_0^s K_{\mathsf{IBS},z}^{\perp(2)}(\tau,s) p(k_z;\tau)\mathrm{d}\tau \\
&- 4i\int_0^s K_{\mathsf{IBS},zz}^{(1)}(\tau,s) b(k_z;\tau)\mathrm{d}\tau + 2i\int_0^s K_{\mathsf{IBS},zz}^{(3)}(\tau,s)\sigma_{\delta\tau}^2 b(k_z;\tau)\mathrm{d}\tau \\
&- 3\int_0^s K_{\mathsf{IBS},zz}^{(2)}(\tau,s) p(k_z;\tau)\mathrm{d}\tau + \int_0^s K_{\mathsf{IBS},zz}^{(4)}(\tau,s)\sigma_{\delta\tau}^2 p(k_z;\tau)\mathrm{d}\tau
\end{aligned}
$$