Discrete Resonator Model

Maxwell approach:

DRM from eigenmode expansion
example: Tesla cavity ⟶ cavity signals

empiric approach:

network models for (quasi) periodic cavities
field flatness and cavity spectrum
field flatness and loss-parameter
transient detuning

summary/conclusions

DRM from eigenmode expansion

$$\nabla\times\nabla\times\mathbf{E}+\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E}=-\frac{\partial}{\partial t}\mu\mathbf{J}$$

$$\nabla\times\nabla\times\mathbf{E}_\nu=\mu\varepsilon\omega_\nu^2\mathbf{E}_\nu$$

$$\mathbf{E}(\mathbf{r},t)=\sum\alpha_\nu(t)\mathbf{E}_\nu(\mathbf{r})$$

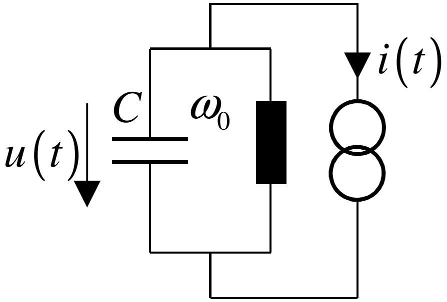
$$\mu\varepsilon\sum\left(\omega_\nu^2+\frac{\partial^2}{\partial t^2}\right)\alpha_\nu(t)\mathbf{E}_\nu(\mathbf{r})=-\frac{\partial}{\partial t}\mu\mathbf{J}$$

$$\tfrac{1}{2}\int\varepsilon\mathbf{E}_\nu\mathbf{E}_\mu dV=W_\nu\delta_{\nu\mu}$$

$$\boxed{\rightarrow\left(\omega_\nu^2+\frac{\partial^2}{\partial t^2}\right)\alpha_\nu(t)=-\frac{\partial}{\partial t}\underbrace{\tfrac{1}{2}\int\mathbf{E}_\nu\mathbf{J}dV}_{\boxed{g_\nu(t)}}}$$

beam

$$\mathbf{J}=q\dot{\mathbf{r}}_p(t)\delta\left(\mathbf{r}-\mathbf{r}_p(t)\right)\rightarrow g_\nu(t)=\tfrac{1}{2}q\dot{\mathbf{r}}_p(t)\cdot\mathbf{E}_\nu\left(\mathbf{r}_p(t)\right)$$



$$\boxed{\rightarrow\left(\omega_0^2+\frac{d^2}{dt^2}\right)u(t)=-\frac{1}{C}\frac{d}{dt}i(t)}$$

$$u(t)=-\frac{1}{C}\int_0^t i(\tau)\cos\left(\omega_0(t-\tau)\right)d\tau$$

EoM + eigenmode approach

$$\frac{d}{dt}\mathbf{r}_\mu = \mathbf{v}\left(\mathbf{p}_\mu\right)$$

$$\frac{d}{dt}\mathbf{p}_\mu = q\left\{\mathbf{E}\left(\mathbf{r}_\mu,t\right)+\mathbf{v}\left(\mathbf{p}_\mu\right)\times\mathbf{B}\left(\mathbf{r}_\mu,t\right)\right\}$$

$$\frac{d}{dt}\begin{pmatrix}\alpha_\nu\\ \beta_\nu\end{pmatrix}=\begin{pmatrix}0 & \omega_\nu\\ -\omega_\nu & 0\end{pmatrix}\begin{pmatrix}\alpha_\nu\\ \beta_\nu\end{pmatrix}-\begin{pmatrix}\overset{\text{beam}}{g_\nu}+\overset{\text{port}}{h_\nu}\\ 0\end{pmatrix}$$

dynamic equations for
state quantities

with

all particles μ and
all modes ν

$$\mathbf{E}(t)=\sum\alpha_\nu(t)\mathbf{E}_\nu(\mathbf{r})$$

$$\mathbf{B}(t)=\sum\beta_\nu(t)\mathbf{B}_\nu(\mathbf{r})$$

$$\mathbf{B}_\nu=\frac{1}{\omega_\nu}\nabla\times\mathbf{E}_\nu \quad \text{for } \omega_\nu\neq 0$$

$$\boxed{g_\nu(t)=\tfrac{1}{2W_\nu^{(m)}}\int\mathbf{E}_\nu\mathbf{J}dV \qquad \mathbf{J}=q\sum\mathbf{v}_\mu\delta\left(\mathbf{r}-\mathbf{r}_\mu\right)}$$

beam

$$\boxed{h_\nu(t)\sim(a-b)\int\mathbf{E}_\nu\mathbf{E}_{\perp,\text{port}}dA}$$

port

$$a+b\sim\sum_\nu\alpha_\nu\int\mathbf{E}_\nu\mathbf{E}_{\perp,\text{port}}dA$$

port coupling

“instantaneous” equations

port modes in matrix formalism

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{D}_\omega\\ -\mathbf{D}_\omega & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}-\begin{pmatrix}\boxed{\mathbf{g}}+\mathbf{V}\boxed{(\mathbf{a}-\mathbf{b})}\\ \mathbf{0}\end{pmatrix}$$

$$\boxed{\mathbf{a}+\mathbf{b}}=-2\mathbf{V}^t\mathbf{D}_{Wm}\boldsymbol{\alpha}$$

beam	$\mathbf{g}$
port	$\mathbf{a}$ = all forward waves
	$\mathbf{b}$ = all backward waves
modes	$\boldsymbol{\alpha}$ = all $\mathbf{E}$ mode-amplitudes
	$\boldsymbol{\beta}$ = all $\mathbf{B}$ mode-amplitudes

all coefficients can be calculated from EM eigenmode results
(but MWS does not support it)

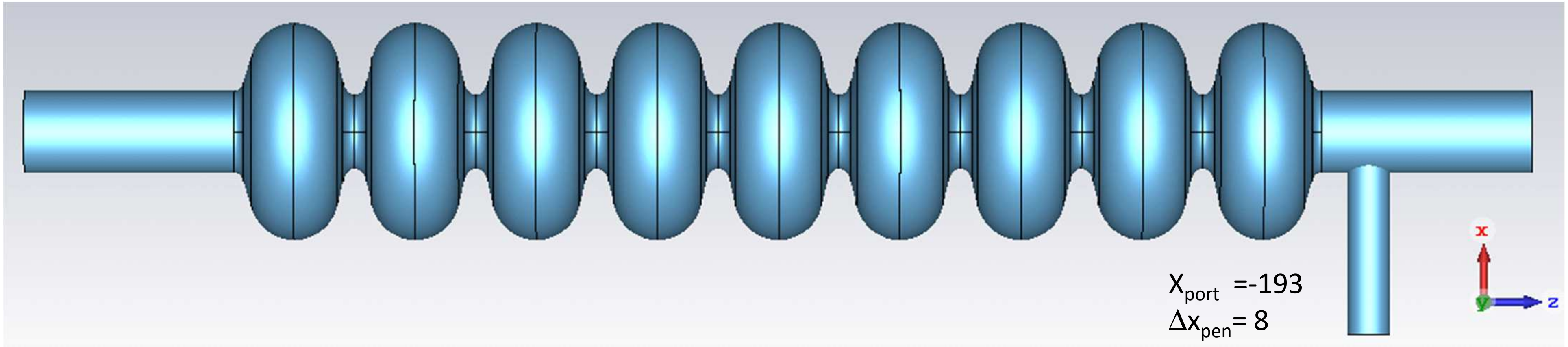
the lossy eigenmode problem: $\mathbf{a}=\mathbf{0},\ \mathbf{g}=\mathbf{0}$

$$\downarrow$$

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\begin{pmatrix}-2\mathbf{V}\mathbf{V}^t\mathbf{D}_{Wm} & \mathbf{D}_\omega\\ -\mathbf{D}_\omega & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$$

but it does not work too well if one considers not enough modes
example: TESLA cavity with modes below 2 GHz

example: TESLA cavity



direct calculation (modes below 2GHz):

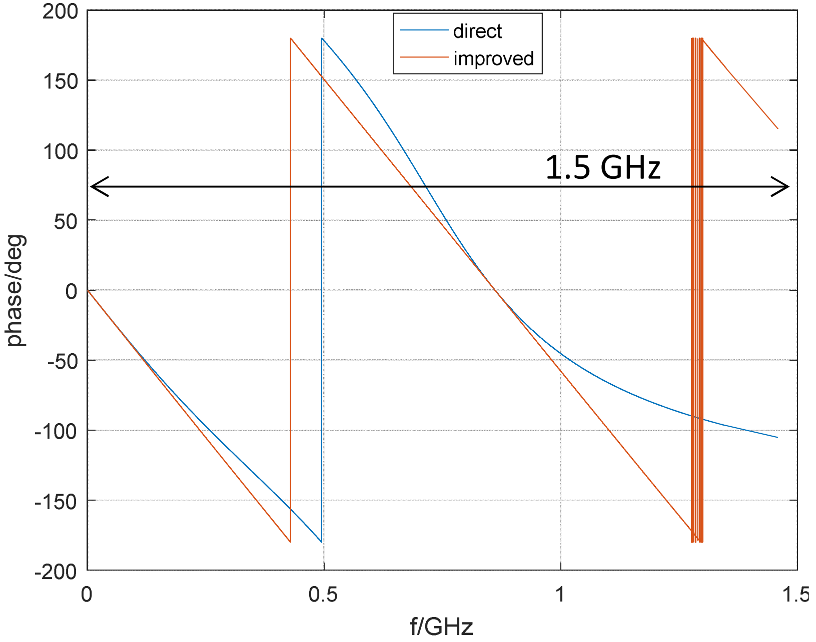
f/GHz	Q/1E6
0	0
0.721865	0.000001
1.2763	88.2703
1.27838	22.6809
1.28157	10.5396
1.28551	6.32979
1.28973	4.41263
1.29373	3.41628
1.29701	2.87407
1.29917	2.57969
1.29991	**4.97314**

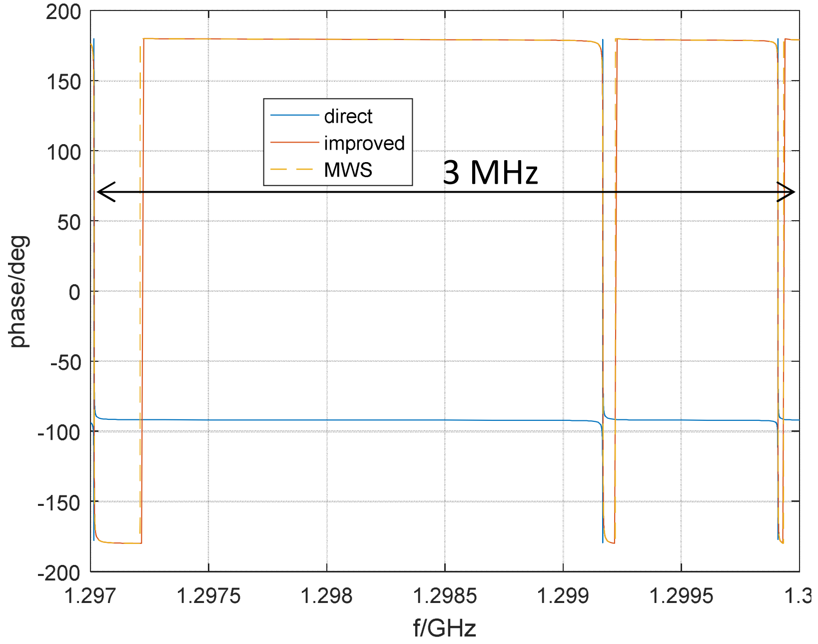
improved calculation*:

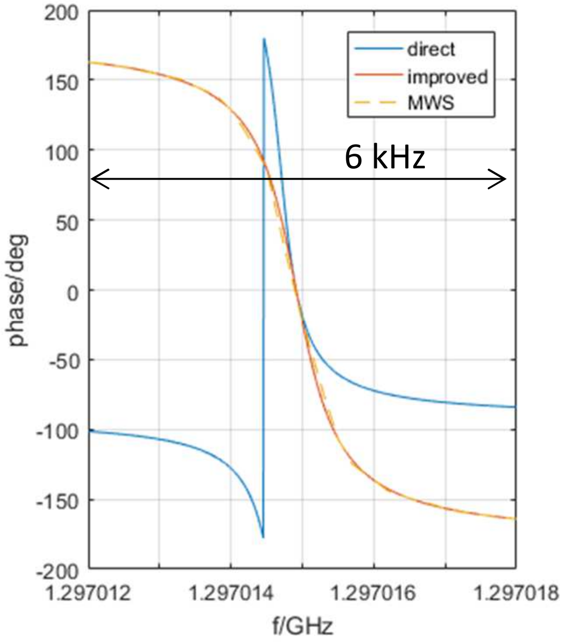
f/GHz	Q/1E6
0	0
0.833513	0.000001
1.2763	44.2134
1.27838	11.3918
1.28157	5.31692
1.28551	3.21128
1.28973	2.25295
1.29373	1.7553
1.29701	1.48468
1.29917	1.33733
1.29991	**2.57894**

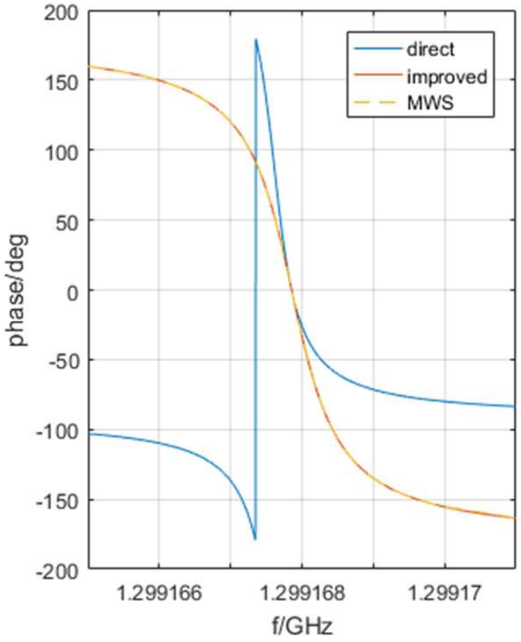
* compare:
Dohlus, Schuhmann, Weiland: Calculation of Frequency Domain Parameters Using 3D Eigensolutions. Special Issue of International Journal of Numerical Modelling: Electronic Networks, Devices and Fields 12 (1999) 41–68

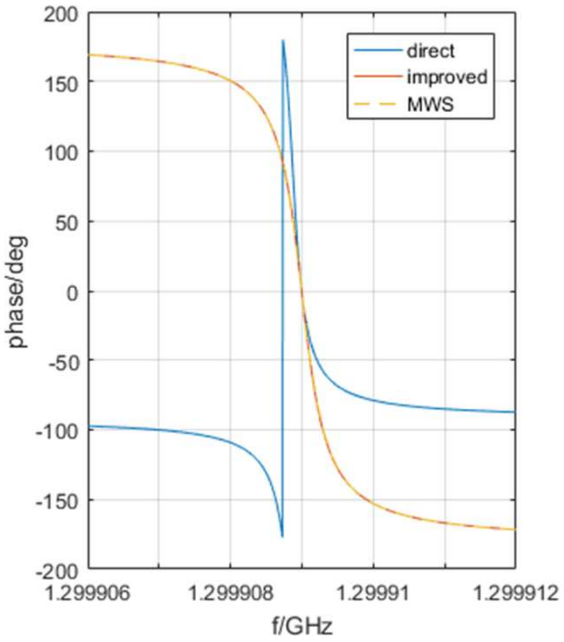
phase of reflection coefficient:











the improved method

frequency domain (quantities with tilde)

$$j\omega\begin{pmatrix}\tilde{\boldsymbol{\alpha}}\\ \tilde{\boldsymbol{\beta}}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{D}_{\omega}\\ -\mathbf{D}_{\omega} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\tilde{\boldsymbol{\alpha}}\\ \tilde{\boldsymbol{\beta}}\end{pmatrix}-\begin{pmatrix}\mathbf{V}\left(\tilde{\mathbf{a}}-\tilde{\mathbf{b}}\right)\\ \mathbf{0}\end{pmatrix}$$

$$\tilde{\mathbf{a}}+\tilde{\mathbf{b}}=-2\mathbf{V}^{t}\mathbf{D}_{Wm}\tilde{\boldsymbol{\alpha}}$$

$$\downarrow$$

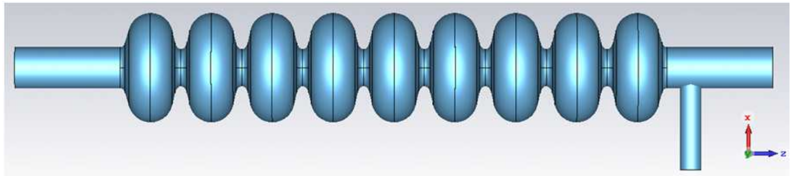
$$\tilde{\mathbf{a}}+\tilde{\mathbf{b}}=\tilde{\mathbf{Z}}(\omega)\left(\tilde{\mathbf{a}}-\tilde{\mathbf{b}}\right)$$ with $$\tilde{\mathbf{Z}}(\omega)=\sum\frac{2j\omega}{\omega_{\nu}^{2}-\omega^{2}}W_{\nu}\mathbf{v}_{\nu}^{t}\mathbf{v}_{\nu}$$

known part unknown part

$$\tilde{\mathbf{Z}}(j\omega)=\boxed{\underbrace{\sum_{\nu=1}^{N}\frac{2j\omega}{\omega_{\nu}^{2}-\omega^{2}}W_{\nu}\mathbf{v}_{\nu}^{t}\mathbf{v}_{\nu}}_{\tilde{\mathbf{Z}}_{k}(j\omega)}}+\boxed{\underbrace{\sum_{\nu>N}\frac{2j\omega}{\omega_{\nu}^{2}-\omega^{2}}W_{\nu}\mathbf{v}_{\nu}^{t}\mathbf{v}_{\nu}}_{\tilde{\mathbf{Z}}_{u}(j\omega)}}$$

this part is smooth in the frequency range of the known part;
with some additional information one can find a good approximation

example: TESLA cavity

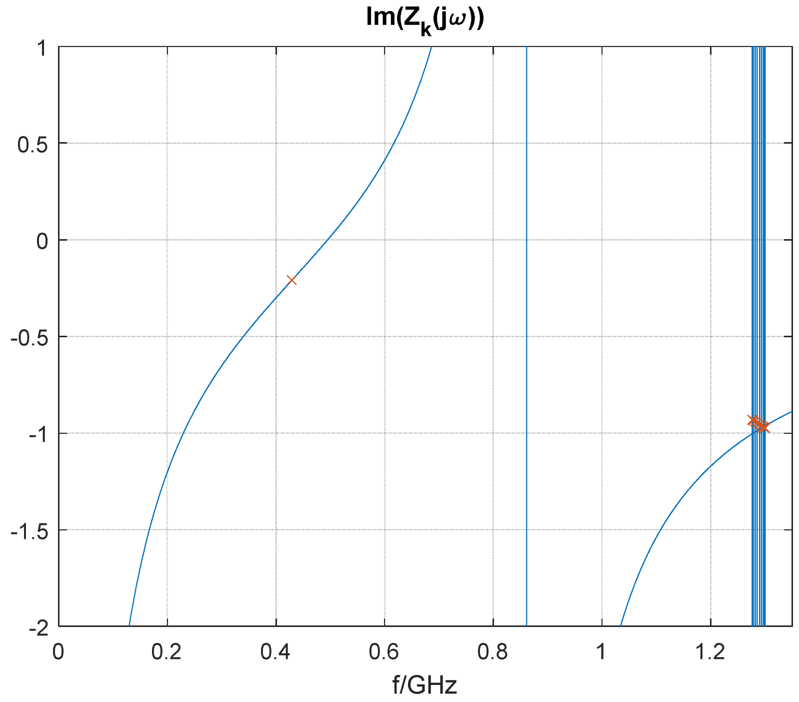


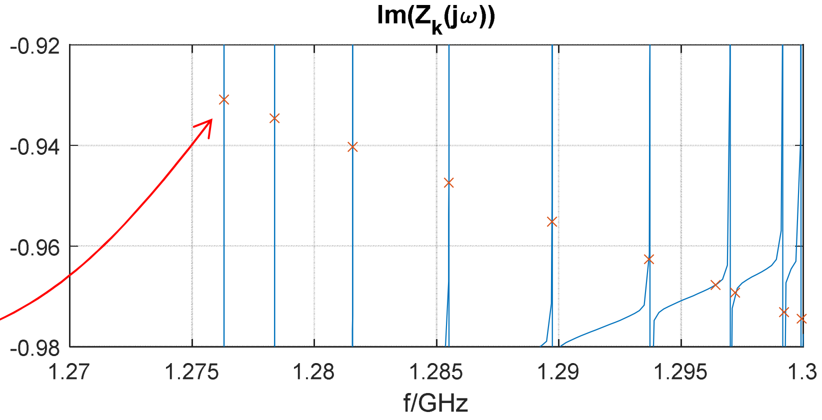
f(PMC)/GHz	f(PEC)/GHz
0	0.429000747
0.8612027	1.276303633
1.276303833	1.278376664
1.278377530	1.281567291
1.281569543	1.285503885
1.285508949	1.289722730
1.289734390	1.293696870
1.293732580	1.296413920
1.297014913	1.297209500
1.299167855	1.299220100
1.299908996	1.299932200

$$\tilde{Z}_k\left(j\omega_{PMC}\right)=\infty$$

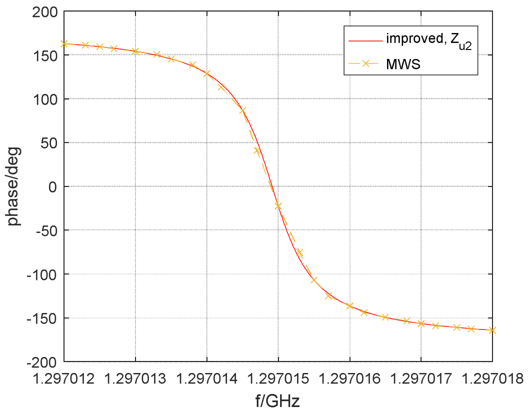
$$\tilde{Z}_k\left(j\omega_{PEC}\right)+\tilde{Z}_u\left(\omega_{PEC}\right)=0$$

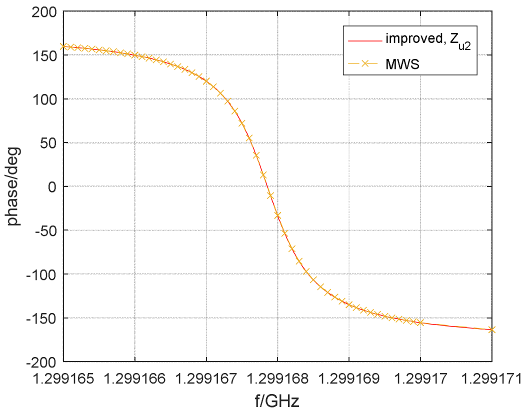
one-port system: Z_u can be calculated
for PEC resonance frequencies

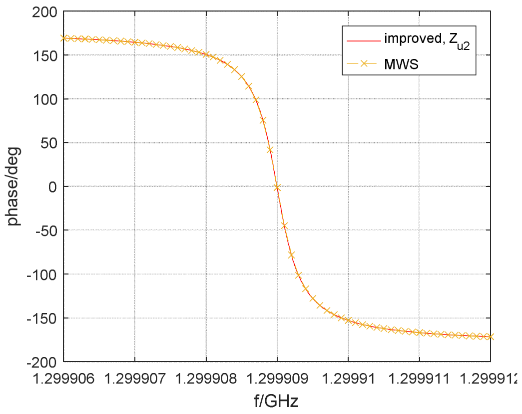




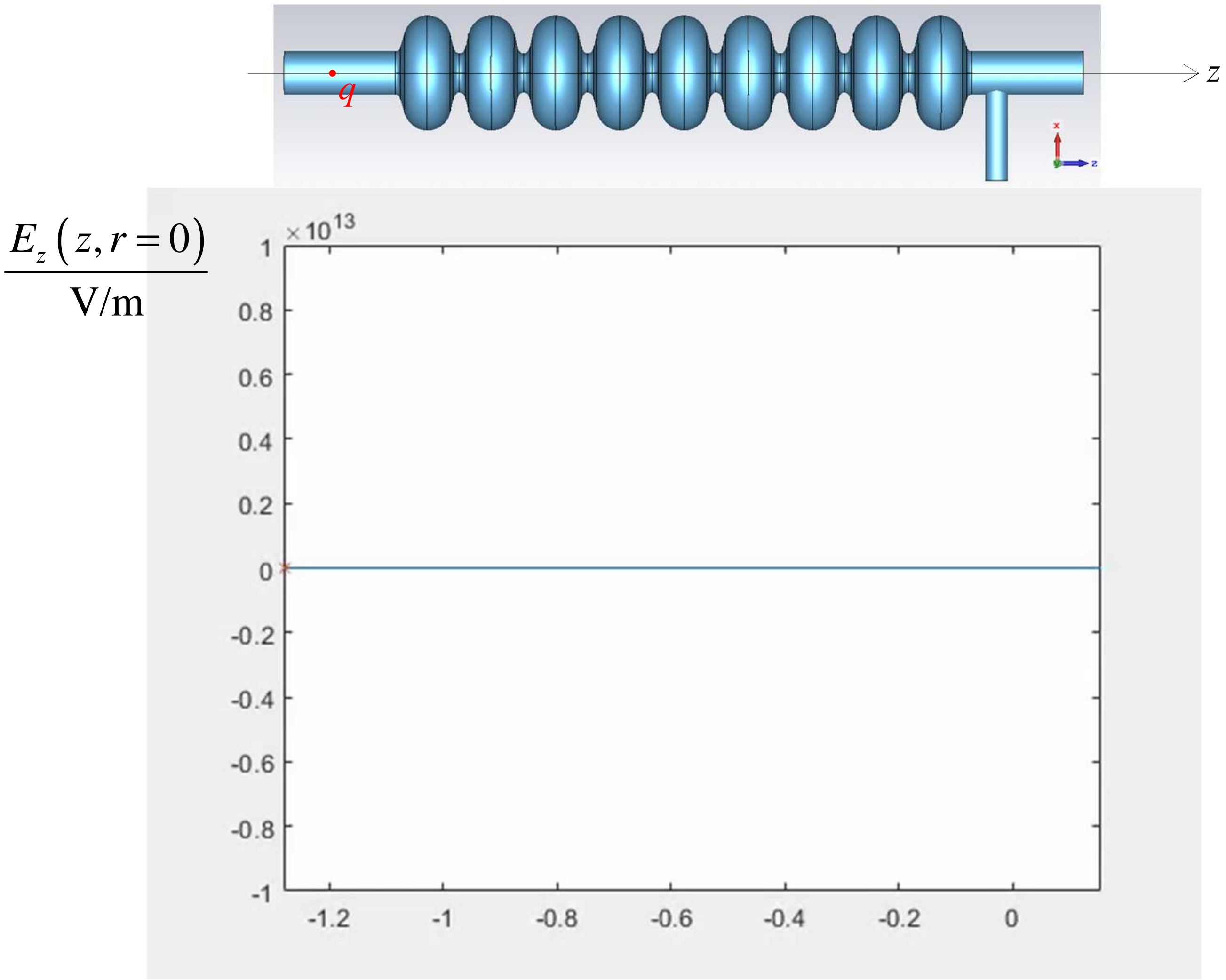
phase of reflection coefficient (again):







excitation of the modes* 1 to 11 by an ultra-relativistic point particle (q=1C)

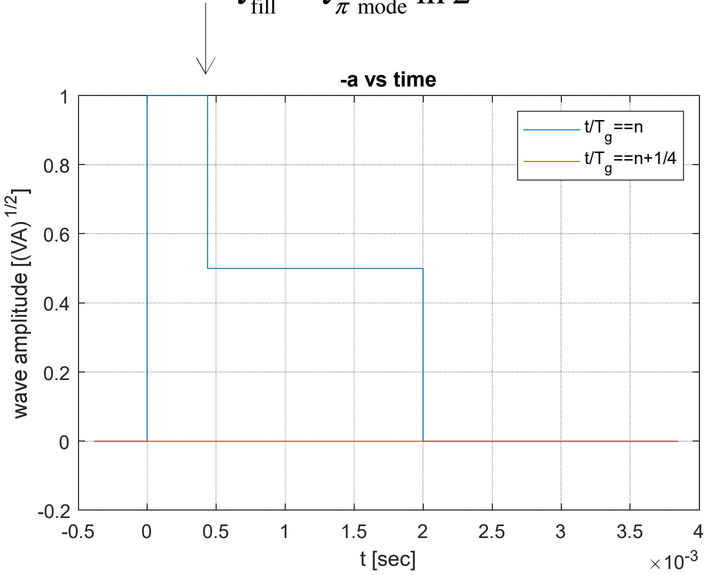


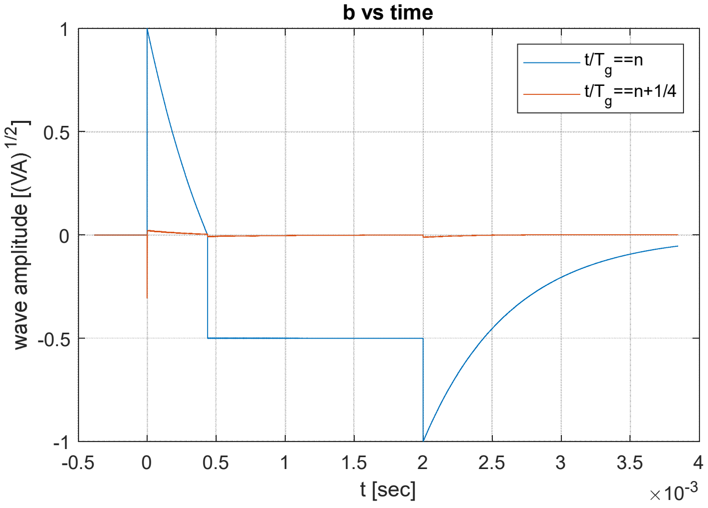
these are the first 11 modes without divergence (div E) in vacuum
modes 3-11 are related to the first band of monopole modes

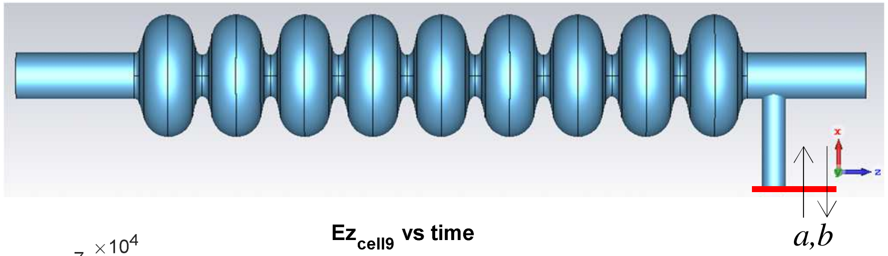
remark:
higher modes
static modes
causality

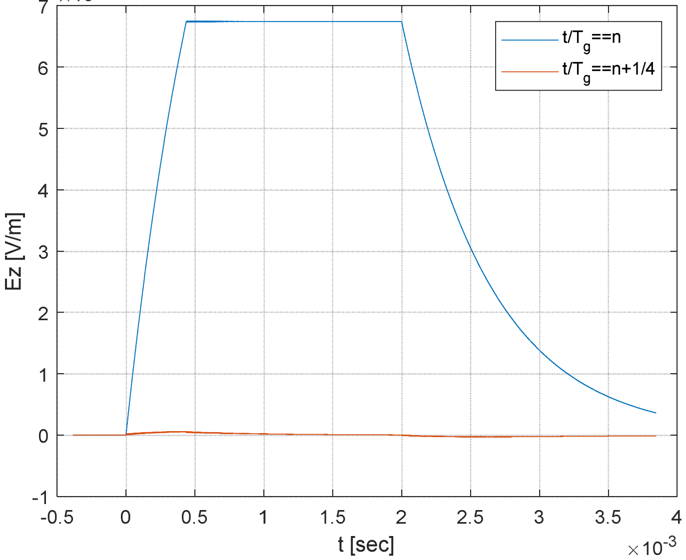
port stimulation

$$\tau_{\text{fill}} = \tau_{\pi\ \text{mode}} \ln 2$$







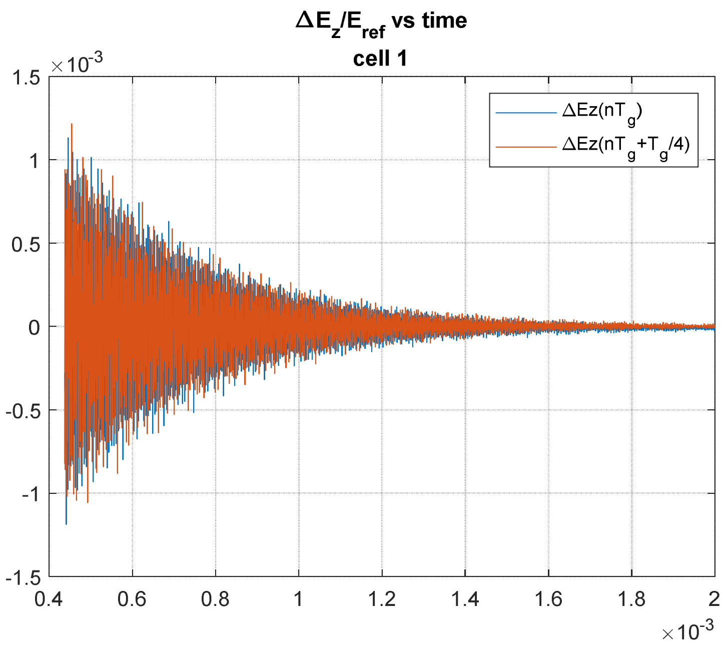


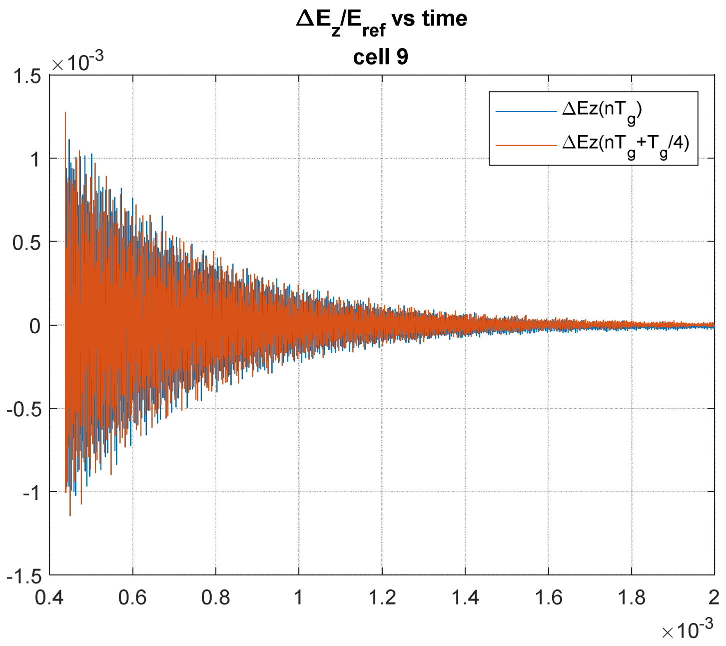
this is time domain, ~ 5E6 periods
stimulation ω_g on resonance of “pi-mode”

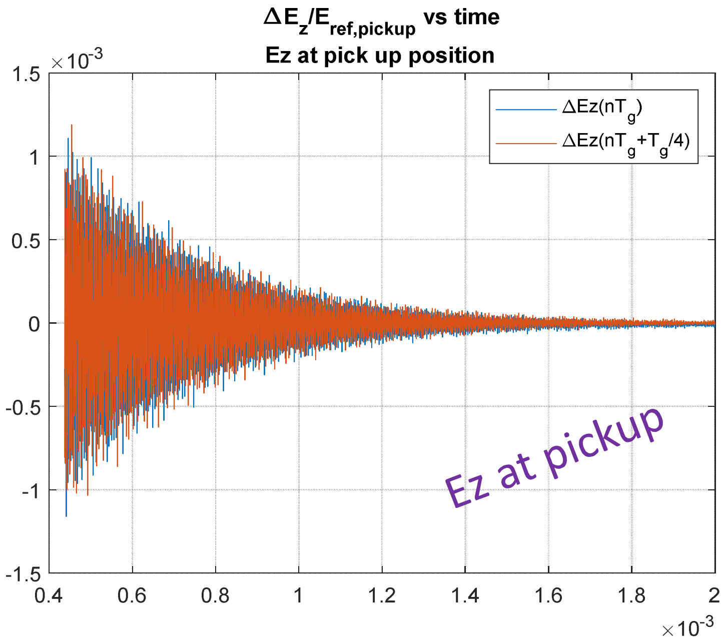
blue curves are sampled at $t = nT_g$
they can be interpreted as real part

red curves are sampled at $t = nT_g + T_g/4$
they can be interpreted as real part

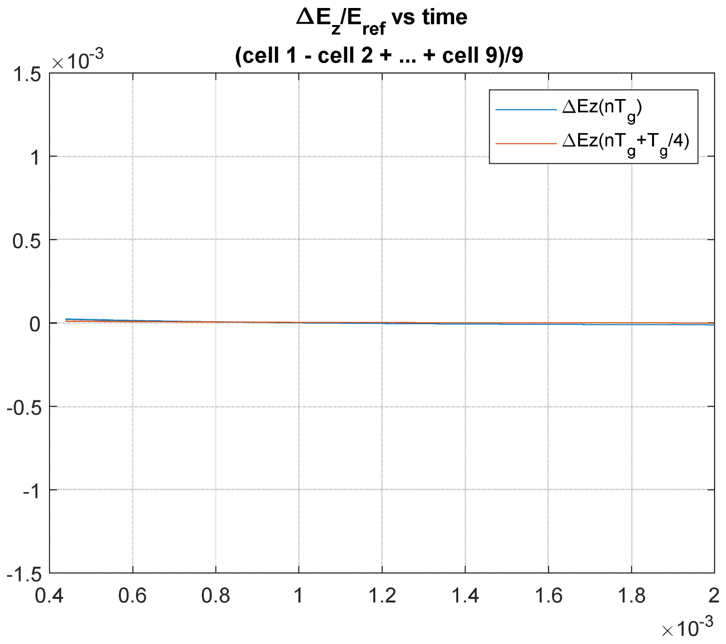
deviation from flat top

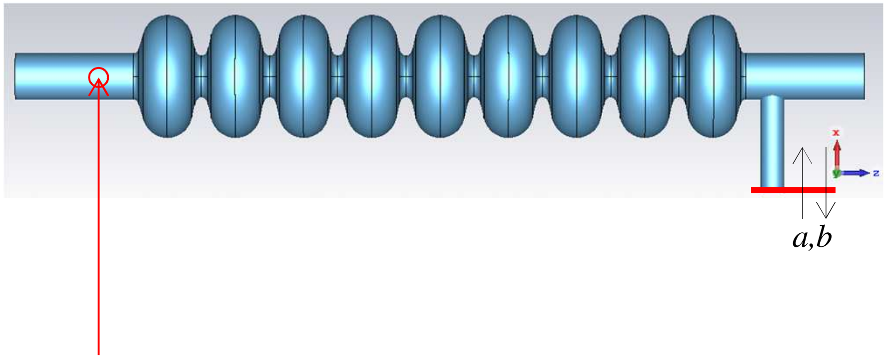

$\Delta E_z/E_{ref}$ vs time
cell 1


$\Delta E_z/E_{ref}$ vs time
cell 9


$\Delta E_z/E_{ref,pickup}$ vs time
Ez at pick up position

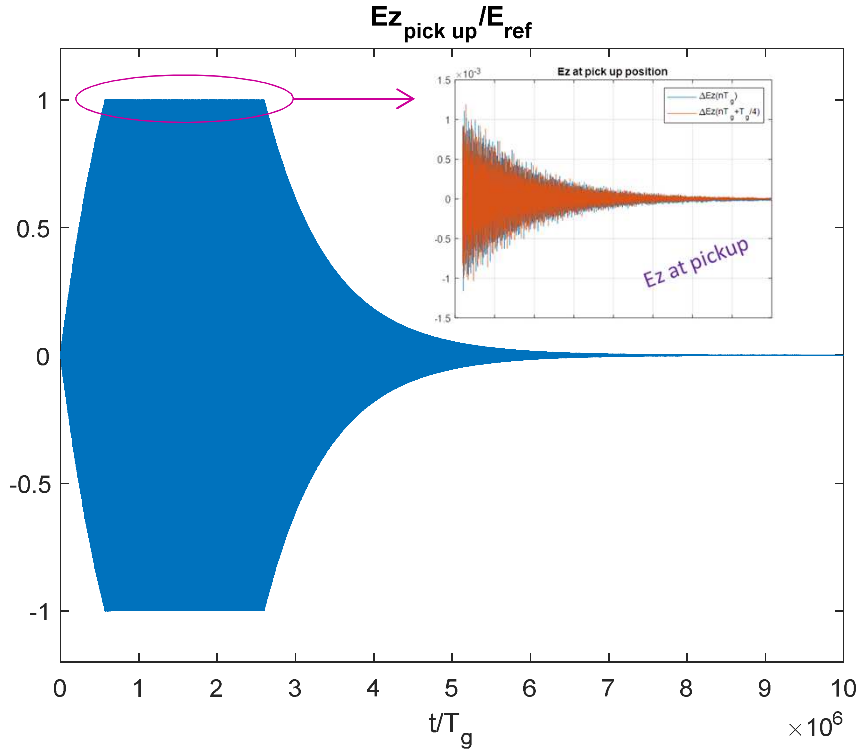
Ez at pickup

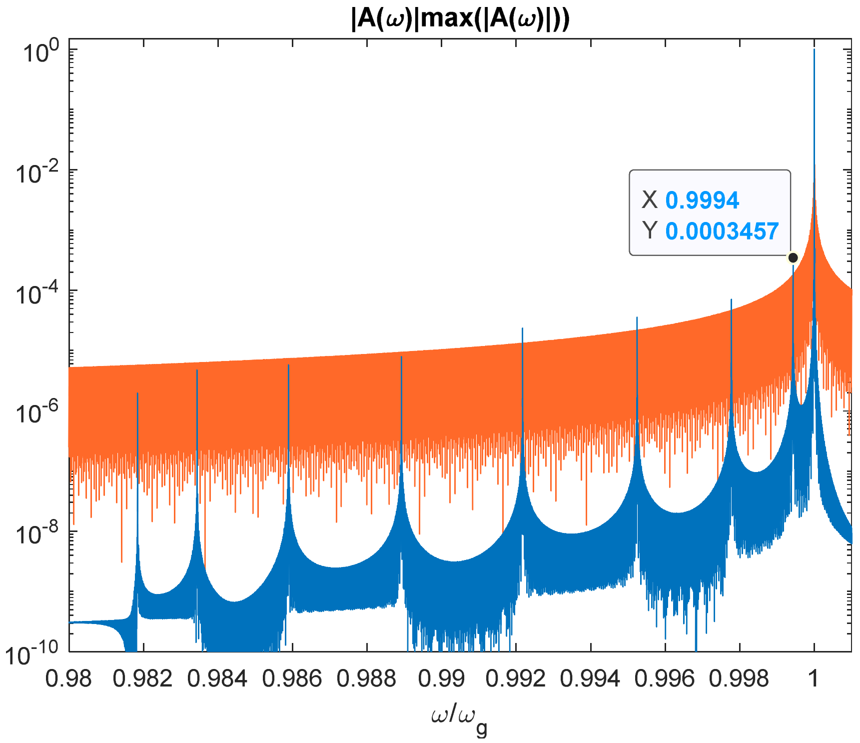

$\Delta E_z/E_{ref}$ vs time
(cell 1 - cell 2 + ... + cell 9)/9

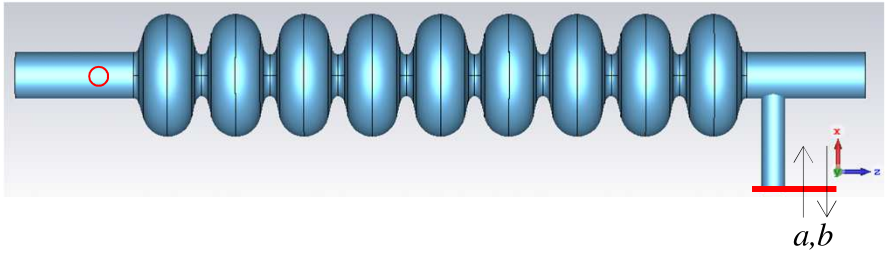


“pickup” signal

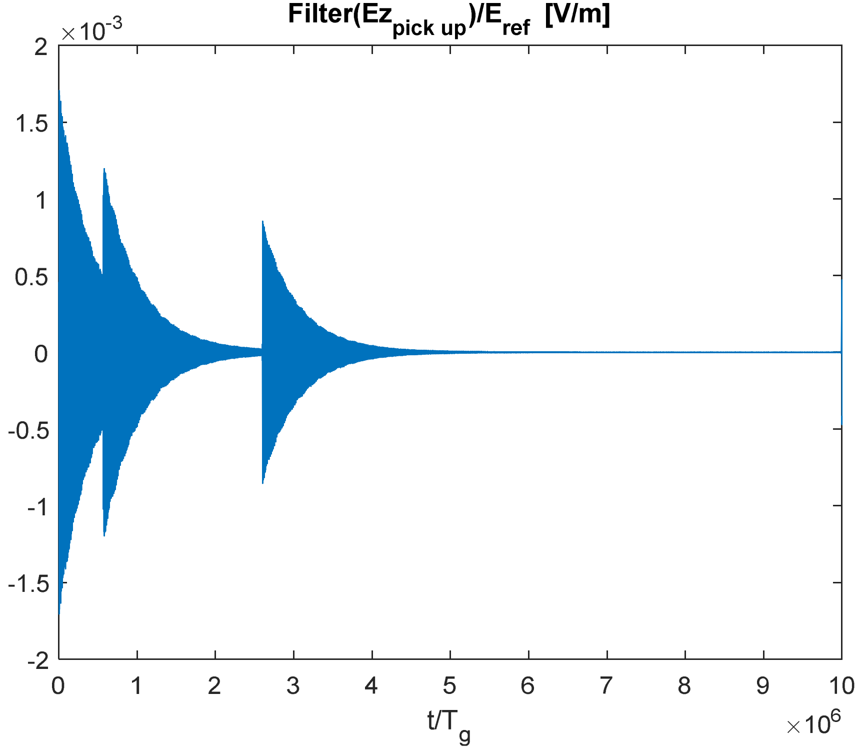
spectrum of “pickup” signal and stimulation



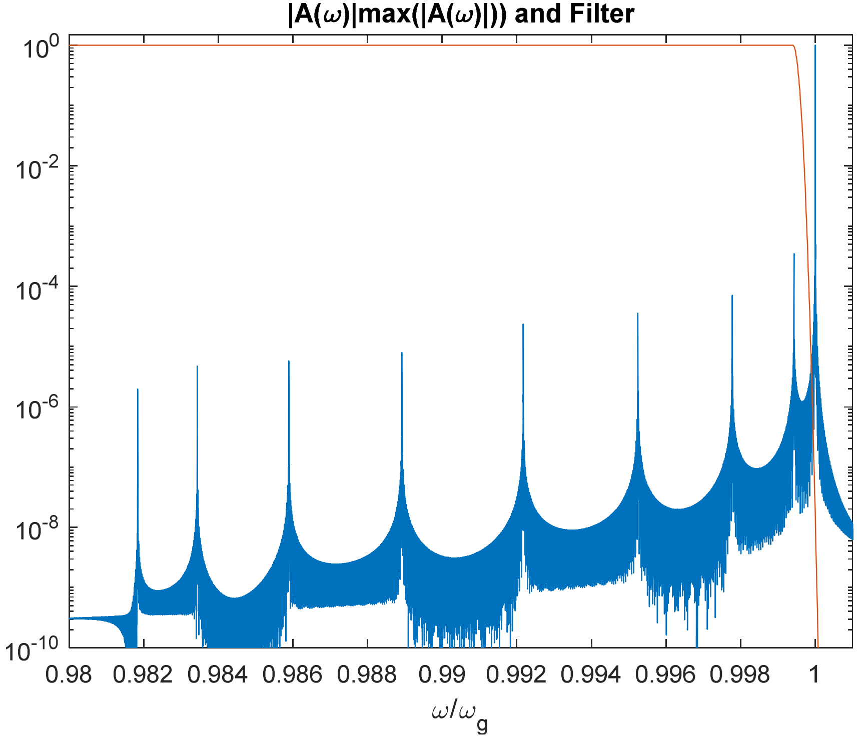


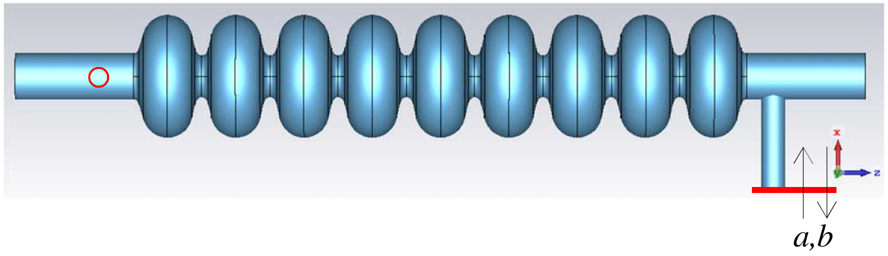


“pickup” signal after LP filter

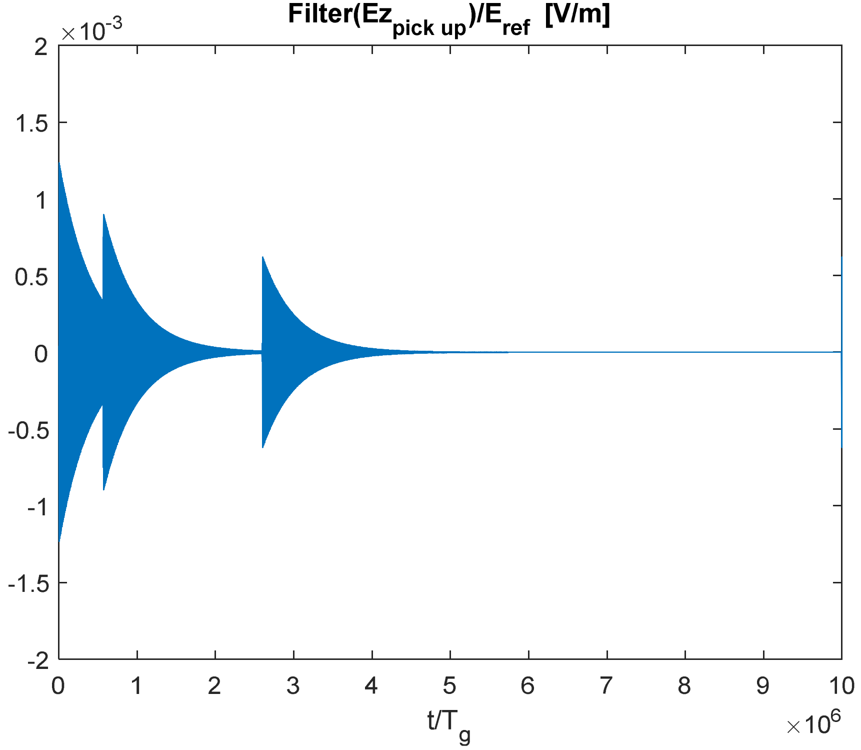

Filter($Ez_{\text{pick up}}$)/E_{ref} [V/m]

spectrum of “pickup” signal and LP filter

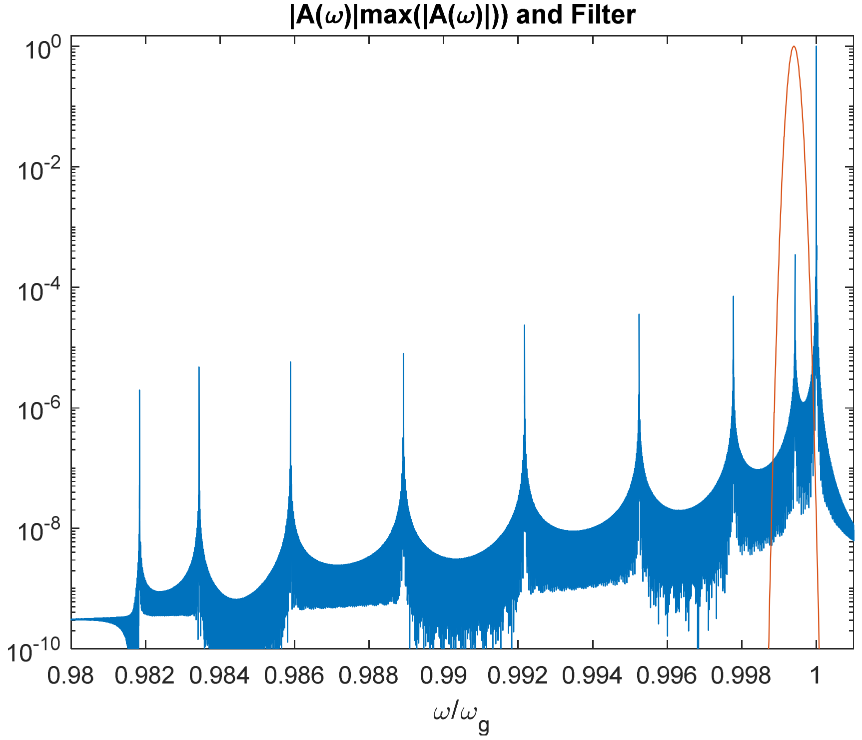

|A(ω)|max(|A(ω)|)) and Filter



“pickup” signal after BP filter

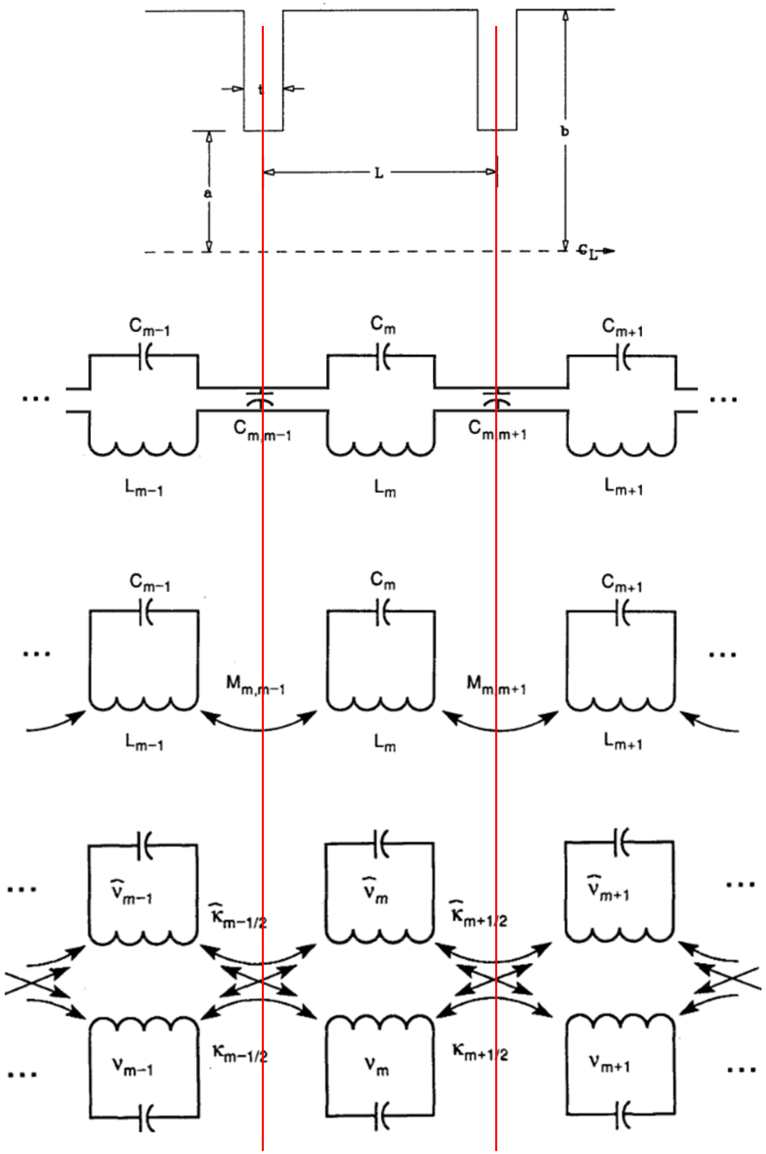


spectrum of “pickup” signal and BP filter



empiric approach:

network models for (quasi) periodic cavities



geometry

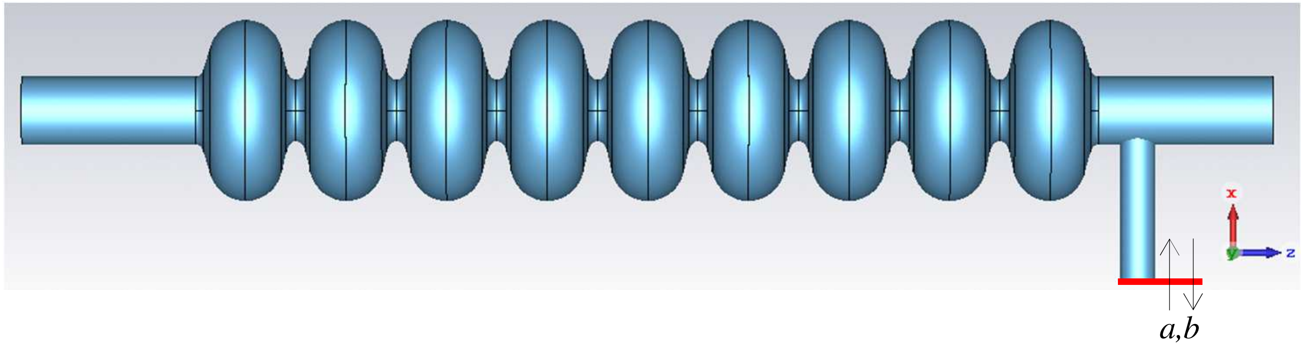
1 resonance, 2x1 ports, electric coupling

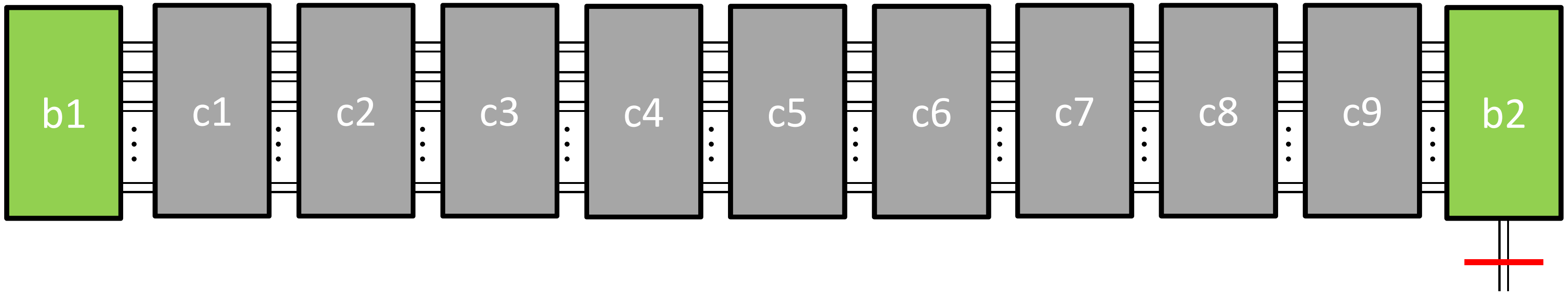
1 resonance, 2x1 ports, magnetic coupling

2 resonances, 2x2 ports, magnetic coupling

K. Bane, R. Gluckstern: The Transverse Wakefield of a Detunded X-Band Accelerator Structure, SLAC-PUB-5783, March 1992

approach from network theory

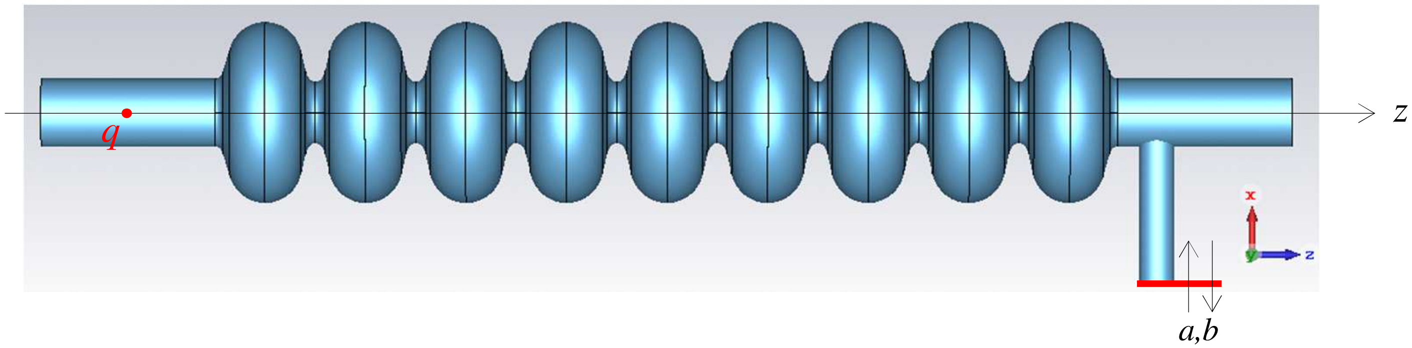


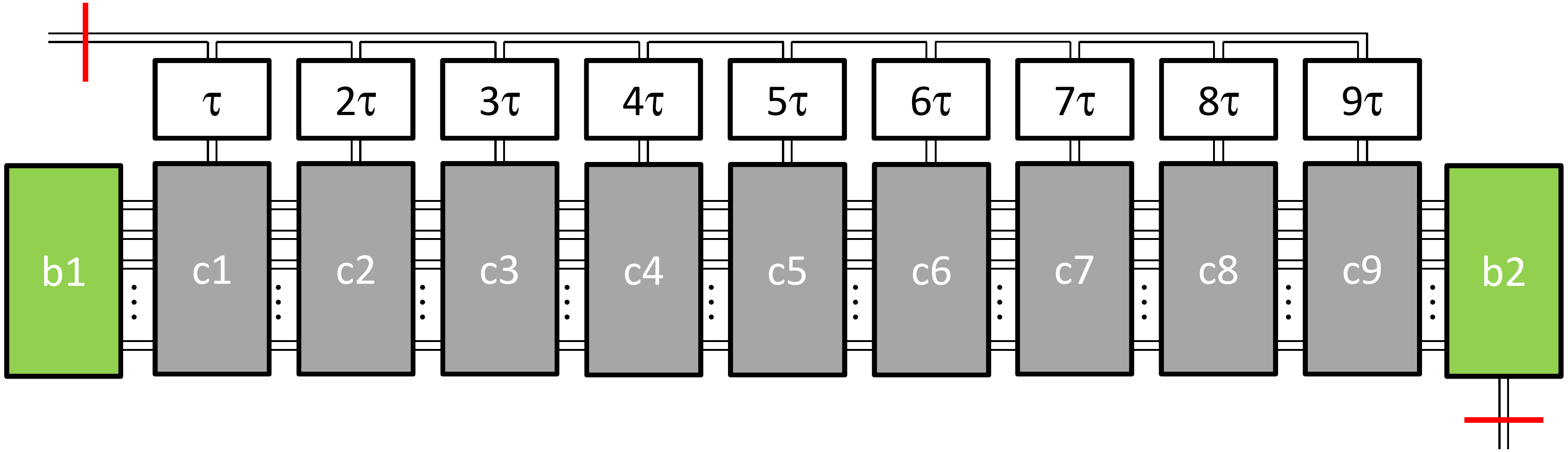


b1, b2: boundary blocks with n respectively $n+1$ ports
c1, c2, … c9: cell blocks with $2n$ ports

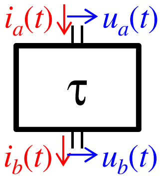
simplifications: c2, … c8 (or c1, ... c9) are identical and symmetric
use periodic solutions of 3d system to characterize cell blocks
special treatment of boundary blocks

system with beam





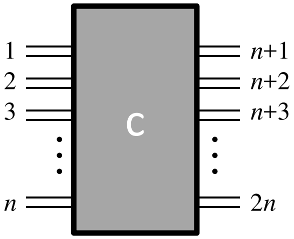
beam ports with delay are connected in series



$$i_b(t) = i_a(t-\tau)$$

$$u_b(t) = u_a(t-\tau)$$

the general symmetric 2n-port network



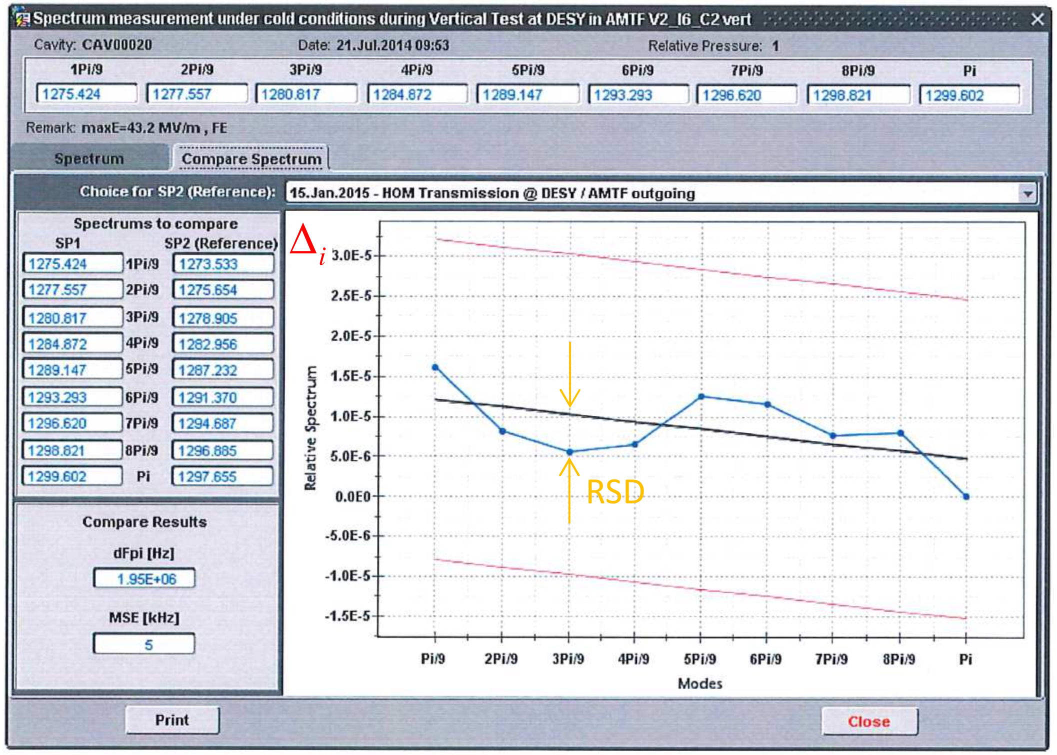
$$\begin{pmatrix} \tilde{u}_1(j\omega) \\ \tilde{u}_2(j\omega) \\ \vdots \\ \tilde{u}_{2n}(j\omega) \end{pmatrix} = \tilde{\mathbf{Z}}(j\omega) \begin{pmatrix} \tilde{i}_1(j\omega) \\ \tilde{i}_2(j\omega) \\ \vdots \\ \tilde{i}_{2n}(j\omega) \end{pmatrix}$$

with $\tilde{\mathbf{Z}}(j\omega) = \sum_{\nu=1}^{M} \frac{j\omega}{\omega_\nu^2 - \omega^2} \mathbf{A}_\nu + j\omega \mathbf{A}_\infty$

and $\mathbf{A}_\nu = \begin{pmatrix} \mathbf{r}_\nu \\ \mathbf{s}_\nu \end{pmatrix} \begin{pmatrix} \mathbf{r}_\nu^t & \mathbf{s}_\nu^t \end{pmatrix} + \begin{pmatrix} \mathbf{s}_\nu \\ \mathbf{r}_\nu \end{pmatrix} \begin{pmatrix} \mathbf{s}_\nu^t & \mathbf{r}_\nu^t \end{pmatrix}$

$\mathbf{A}_\infty = \cdots$

relative spectral deviation



Spectrum measurement under cold conditions during Vertical Test at DESY in AMTF V2_I6_C2 vert

Cavity: CAV00020

Date: 21.Jul.2014 09:53

Relative Pressure: 1

1Pi/9	2Pi/9	3Pi/9	4Pi/9	5Pi/9	6Pi/9	7Pi/9	8Pi/9	Pi
1275.424	1277.557	1280.817	1284.872	1289.147	1293.293	1296.620	1298.821	1299.602

Remark: maxE=43.2 MV/m , FE

Choice for SP2 (Reference): 15.Jan.2015 - HOM Transmission @ DESY / AMTF outgoing

Spectrums to compare

SP1		SP2 (Reference)
1275.424	1Pi/9	1273.533
1277.557	2Pi/9	1275.654
1280.817	3Pi/9	1278.905
1284.872	4Pi/9	1282.956
1289.147	5Pi/9	1287.232
1293.293	6Pi/9	1291.370
1296.620	7Pi/9	1294.687
1298.821	8Pi/9	1296.885
1299.602	Pi	1297.655

Compare Results

dFpi [Hz]: 1.95E+06

MSE [kHz]: 5

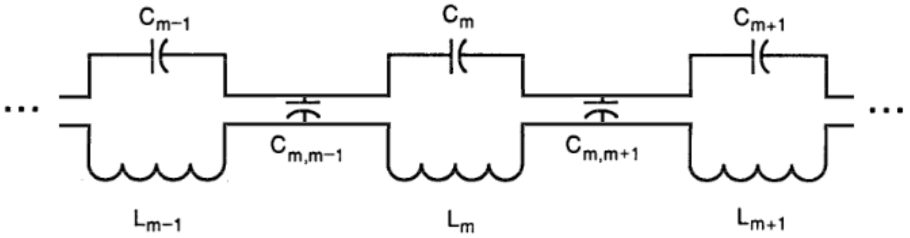
$$\Delta_i = \frac{f_i}{F_i} - \frac{f_9}{F_9}$$

RSD is a measure for the change of the mode spectrum of one cavity compared to itself

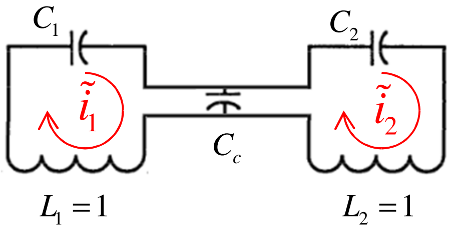
field flatness and mode spectrum

Is there a direct relation between the field flatness of the accelerating mode and the resonance frequencies of modes in the same band**?**

test it for a simpler problem:



even simpler:



$$\tilde{\mathbf{i}} = \begin{pmatrix} \tilde{i}_1 \\ \tilde{i}_2 \end{pmatrix}$$

no: both setups have the same eigen-frequencies, but different flatness/eigenvectors

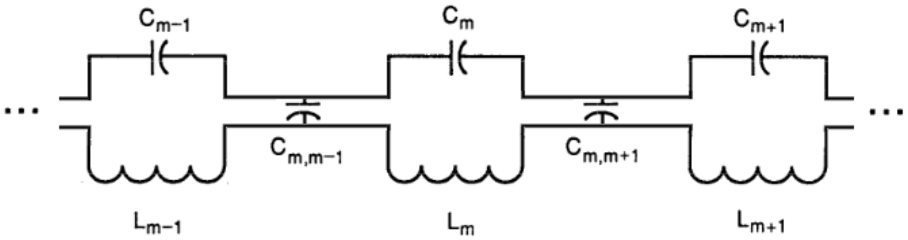
$$\left.\begin{array}{l} C_1 = C_2 = 1.25 \\ C_c = 5 \end{array}\right\} \to \tilde{\mathbf{i}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \tilde{\mathbf{i}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\left. C_1 = \frac{10}{11-\sqrt{3}},\ C_2 = \frac{10}{9-\sqrt{3}},\ C_c = 10/\sqrt{3} \right\} \to \tilde{\mathbf{i}}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},\ \tilde{\mathbf{i}}_2 = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$

$$\omega_1 = \sqrt{4/5}$$
$$\omega_2 = \sqrt{6/5}$$

field flatness and loss parameter

again the discrete network:



pi-mode

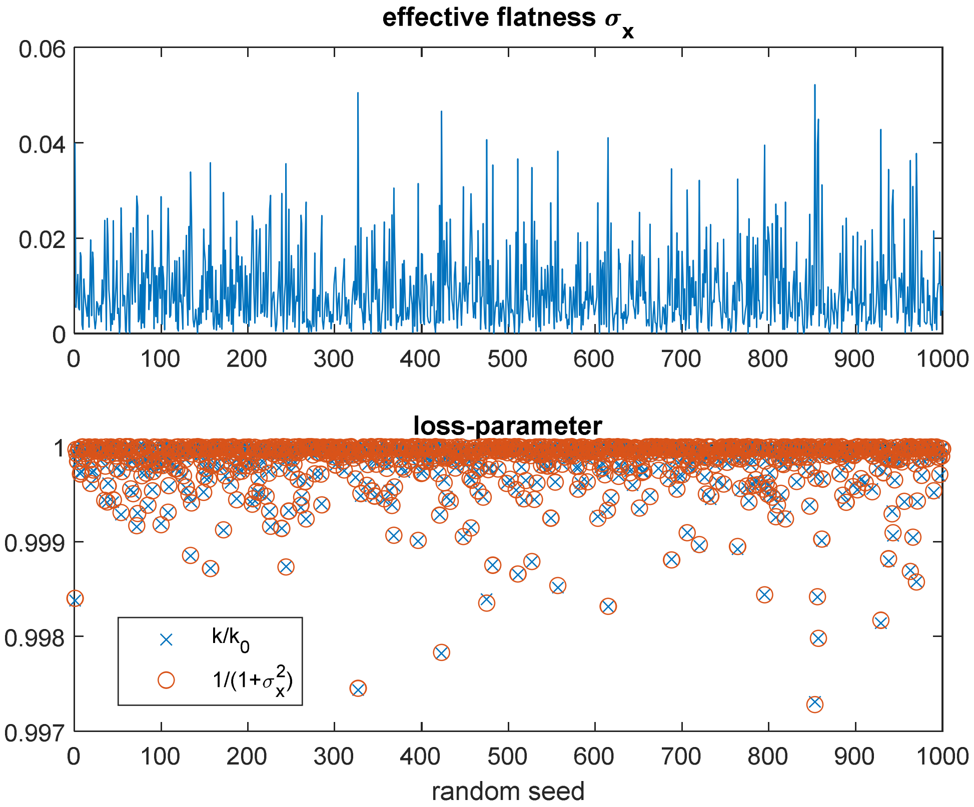
loss-parameter: $$k = \frac{|V|^2}{4W_{tot}} \approx \frac{1}{2\omega_0^2} \frac{\left|\sum \boxed{(-1)^\nu} C_\nu^{-1} \tilde{i}_\nu \right|^2}{\sum L_\nu \tilde{i}_\nu^2} \approx \frac{1}{2C} \frac{\omega_\pi^2}{\omega_0^2} \frac{\left|\sum \boxed{(-1)^\nu} \tilde{i}_\nu \right|^2}{\sum \tilde{i}_\nu^2} \qquad \nu = 1 \cdots 9$$

$$\tilde{i}_\nu \sim \boxed{(-1)^\nu} (1 + \boxed{x_\nu})$$ deviation from flat field

$$k \sim \frac{\left|\sum (1 + x_\nu)\right|^2}{9 \sum (1 + x_\nu)^2} \approx \frac{1}{1 + \langle x_\nu^2 \rangle} \qquad \text{if} \qquad \langle x_\nu \rangle \approx 0$$

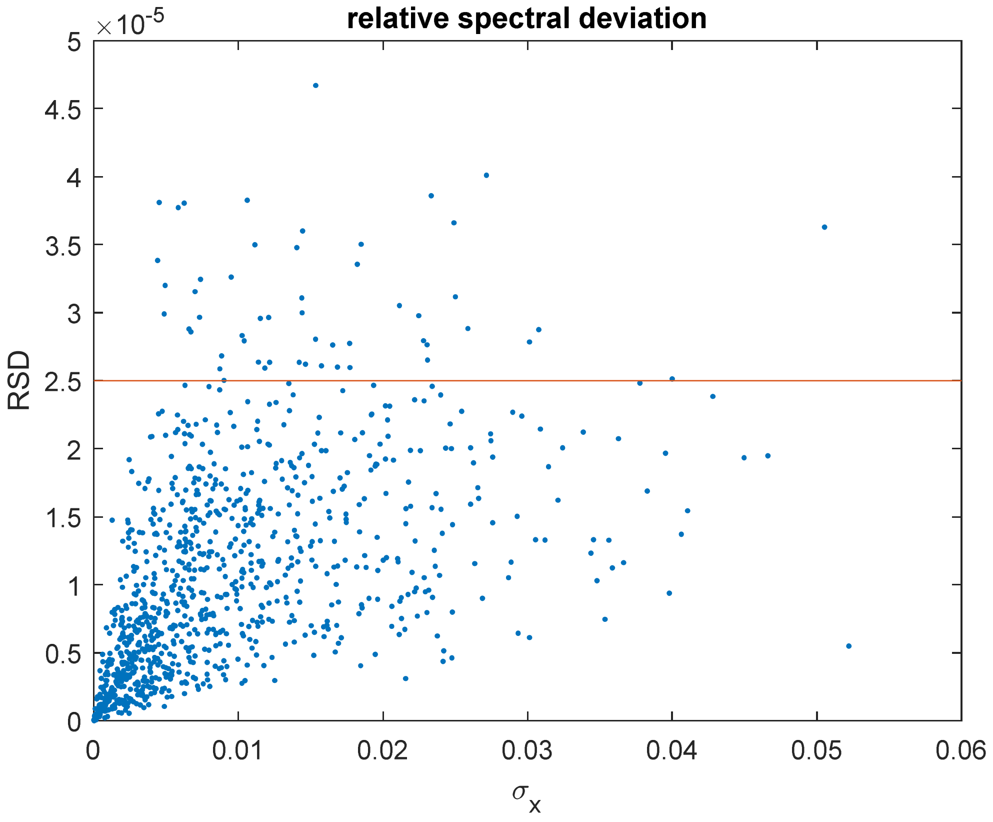
for $\sigma_x = 0.1$ the loss parameter is reduced by only 1%!

simulation for network with tolerances



assumption: error tolerances of network parameters, uncorrelated …

correlation between field flatness and relative spectral deviation



assumption: error tolerances of network parameters, uncorrelated …

time dependent detuning

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{D}_\omega\\ -\mathbf{D}_\omega & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}-\begin{pmatrix}\mathbf{V}\left(\mathbf{a}-\mathbf{b}\right)\\ \mathbf{0}\end{pmatrix}$$

$$\mathbf{a}+\mathbf{b}=-2\mathbf{V}^t\boldsymbol{\alpha}$$

$$\Big\downarrow \; ?$$

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{D}_\omega(t)\\ -\mathbf{D}_\omega(t) & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}-\begin{pmatrix}\mathbf{V}(t)\left(\mathbf{a}-\mathbf{b}\right)\\ \mathbf{0}\end{pmatrix}$$

$$\mathbf{a}+\mathbf{b}=-2\mathbf{V}^t(t)\boldsymbol{\alpha}$$

for instance no excitation (**a** = **b**) $\overset{?}{\rightarrow}$ $\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \int_0^t \mathbf{D}_\omega(\tau)\,d\tau\\ -\int_0^t \mathbf{D}_\omega(\tau)\,d\tau & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}_0\\ \boldsymbol{\beta}_0\end{pmatrix}$

This would mean all the modes are independently ringing, there is no mode conversion.

example:

$$\frac{d}{dt}\mathbf{z} = \begin{pmatrix} \mathbf{0} & \mathbf{R}(t) \\ \mathbf{R}(t) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{D}_\omega \\ -\mathbf{D}_\omega & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{R}(t) \\ \mathbf{R}(t) & \mathbf{0} \end{pmatrix}^t + \mathbf{q} \quad \text{with} \quad \mathbf{D}_\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

$$\mathbf{R}(t) = \begin{pmatrix} \cos\varphi(t) & \sin\varphi(t) \\ -\sin\varphi(t) & \cos\varphi(t) \end{pmatrix}$$

system matrix is anti symmetric → energy conservation with $W \sim \mathbf{z}^t\mathbf{z}$

eigenvalues are time invariant $\lambda = \{\pm j\omega_1, \pm j\omega_2\}$

but eigenvectors are time dependent

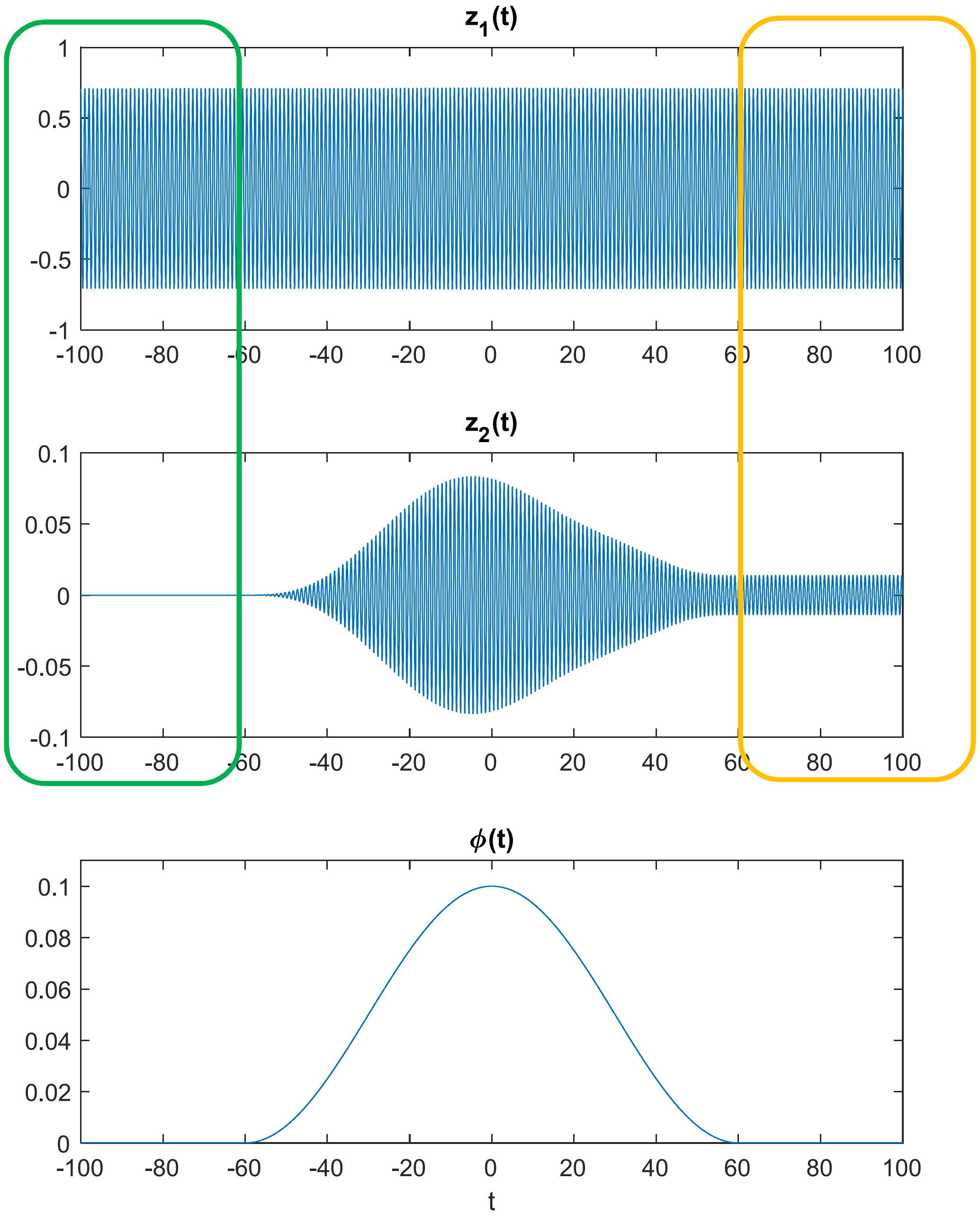
next slide:

numerical calculation for $\omega_1 = 1.98\pi,\ \omega_2 = 2.02\pi$

the effect of mode conversion is weak if the time scale of $\varphi(t)$ is long compared to the resonances no excitation

$\mathbf{q} = 0$

initial state:
ω_1 resonance
is ringing



final state:
both resonances
are ringing

summary/conclusion

Maxwell approach: field eigenmode expansion

- eigenmode expansion is effective to analyse cavity signals in the frequency range of the first monopole band → signals seen by couplers and pickups
- pickups are more sensitive to non-accelerating monopole modes than the beam
- eigenmode expansion is standard for long range effects

empiric approach: discrete network

- discrete network models allow qualitative insight
- it is easy to analyze discrete models and to consider random effects
- it is difficult to relate network parameters to geometric properties and imperfections; it is in principle possible
- loss-parameter is very insensitive to field flatness; but the peak field is sensitive!
- no sharp correlation between relative spectral deviation and flatness
- it is possible to calculate time dependent resonance, but modeling requires (some) caution