A high order FEM wakefield solver in the frequency domain

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Contents

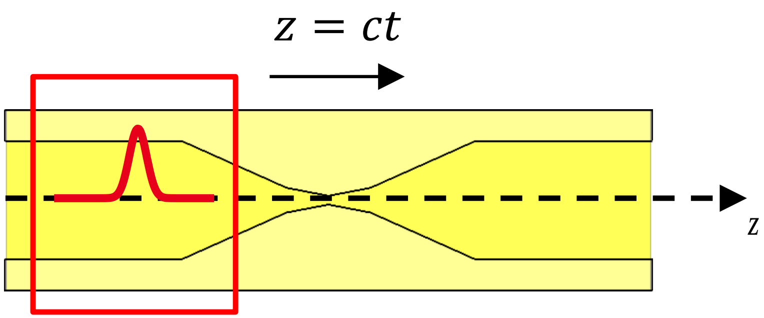
- Overview of the method
- New developments since 2018
  - Surface impedance boundary conditions / lossy walls
  - S-Parameter concatenation for impedances
  - Simulation of corrugated plate dechirper

Motivation

- Wakefields → wake potentials → impedances

$$\boxed{W_{\parallel}(r,s) = \frac{1}{Q}\int dz\, E_z\big(r,z,t(s,z)\big)} \qquad t(s,z) = \frac{s+z}{c}$$

- Solve Maxwell's equations in the time domain
  - FIT/Cartesian grids/ Dispersion-free methods
  - Co-moving computational window
  - Indirect integration
  - …



- Impedance by Fourier transform:

$$\boxed{Z_{\parallel}(r,\omega) = -\frac{1}{c\tilde{\lambda}(\omega)}\int ds W_{\parallel}(r,s)e^{-\frac{i\omega s}{c}} = -\frac{1}{Q\tilde{\lambda}(\omega)}\int dz\tilde{E}_z(r,z,\omega)e^{\frac{i\omega z}{c}}}$$

Motivation

- Long range wakefields
  - Low frequency, long bunches, bunch trains, long wake transients
- Approximation of geometry
  - Curved geometry, small details, smooth tapers
- Dispersive problems
  - Surface impedance, dielectrics
  - Free-space and waveguide boundary conditions
- Radiation fields
  - Curved beam trajectories (CSR)
  - Wakefields in β-graded cavities
- Periodic / quasi-periodic structures

Frequency Domain Formulation

- The frequency domain problem

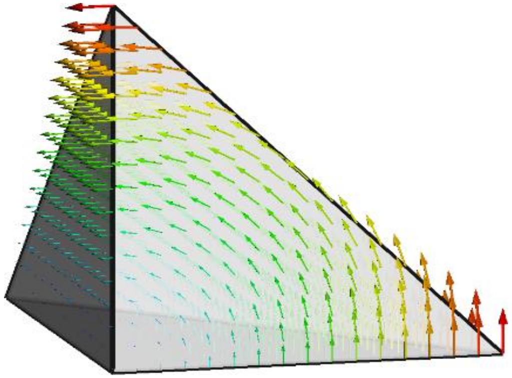
$$\boxed{\nabla \times \mu^{-1} \nabla \times E - k_0^2 \varepsilon E = -jk_0 Z_0 J_s \qquad J_s(x,y,z,\omega) = \delta(x-x_0)\delta(y-y_0)e^{-i\frac{\omega}{v}z}}$$

- Weak FEM formulation: find $E \in H(curl)$ such that:

$$\int dV \mu^{-1}\, \nabla \times E \cdot \nabla \times v_h - k_0^2 \int dV \varepsilon\, E \cdot v_h = -jk_0 Z_0 \int dV\, J_s \cdot v_h$$

$$\underbrace{+\oint_S \; dS\;\, n \cdot [v_h \times \mu^{-1}\, \nabla \times E]}_{\textbf{boundary term}}$$

$$\forall\, \boldsymbol{v_h} \in \boldsymbol{H(curl)}$$



Frequency Domain Formulation

- Treatment of boundary surfaces

$$\int dV \mu^{-1} \nabla \times E \cdot \nabla \times v_h - k_0^2 \int dV \varepsilon \, E \cdot v_h =$$

$$-jk_0 Z_0 \int dV \, J_s \cdot v_h + \underbrace{\int_{S_{SIBC}} \boldsymbol{dS \, n \cdot [v_h \times \mu^{-1} \nabla \times E]}}_{\textbf{resistive wall}} + \underbrace{\int_{S_{WG}} dS n \cdot [v_h \times \mu^{-1} \nabla \times E]}_{\textbf{in \& outgoing pipes}}$$

- Resistive wall boundary

$$\oint_{S_{SIBC}} dS \;\, n \cdot [v_h \times \mu^{-1} \nabla \times E] = \cdots = j\omega \boldsymbol{Y_S(\omega)} \oint_{S_{SIBC}} dS \;\, v_h \cdot [n \times n \times E]$$

Simple modification of the system matrix on SIBC surfaces

No fitting of the surface impedance function or ADE/convolution is needed

Frequency Domain Formulation

- Treatment of boundary surfaces

$$\int dV \mu^{-1}\, \nabla \times E \cdot \nabla \times v_h - k_0^2 \int dV \varepsilon\, E \cdot v_h =$$

$$-jk_0 Z_0 \int dV\, J_s \cdot v_h + \underbrace{\int_{S_{SIBC}} dS\, n \cdot \left[v_h \times \mu^{-1}\, \nabla \times E\right]}_{\textbf{resistive wall}} + \underbrace{\int_{S_{WG}} \boldsymbol{dSn} \cdot \left[\boldsymbol{v_h} \times \boldsymbol{\mu^{-1}}\, \boldsymbol{\nabla} \times \boldsymbol{E}\right]}_{\textbf{in \& outgoing pipes}}$$

- Beam pipe boundaries

$$n \times \nabla \times E = n \times \nabla \times E^{inc} + \sum_m a_m^{TE} \gamma_m^{TE} e_m^{TE} + \sum_m a_m^{TM} \frac{-k_0^2}{\gamma_m^{TM}} e_m^{TM}$$

$$a_m^{TE} = \int_{S_{WG}} dS e_m^{TE} \cdot \left[E - E^{inc}\right]$$

Reflection coefficients for each mode

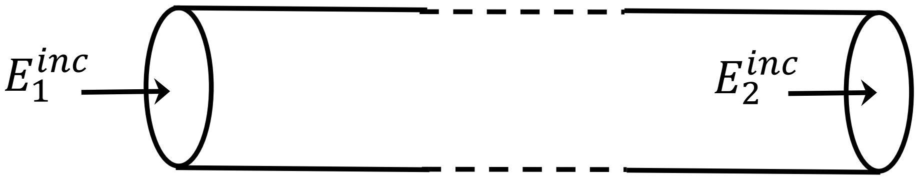
$$a_m^{TM} = \int_{S_{WG}} dS e_m^{TM} \cdot \left[E - E^{inc}\right]$$

Frequency Domain Formulation

- Beam pipe boundary excitation
  - For an ultra-relativistic bunch (same idea for β < 1):

$$\nabla_t \cdot E^{inc} = \frac{1}{\varepsilon_0}\rho(x,y)e^{-ik_0 z_0}$$

$$\nabla \times E^{inc} = 0$$



2D-electrostatic problem at both ends of the pipe

  - Modal contribution to the RHS

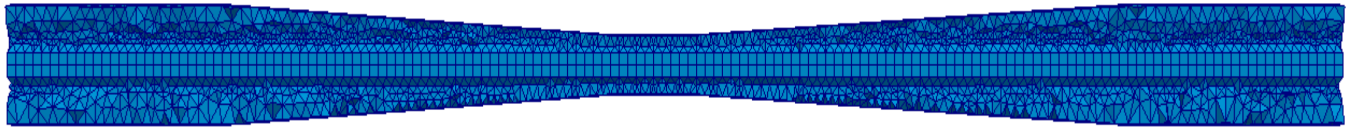
$$U_m^{TE}\left(E^{inc}\right) = -\gamma_m^{TE}\left(\int_{S_{WG}} dS\, v_h \cdot e_m^{TE}\right)\left(\int_{S_{WG}} dS\, e_m^{TE} \cdot E^{inc}\right)$$

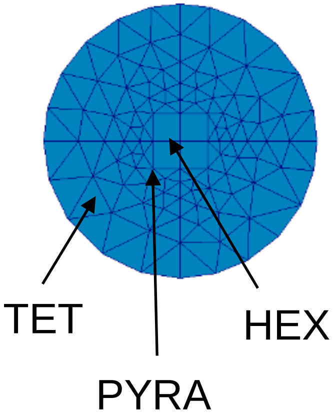
$$\boxed{U_m^{TE}\left(E^{inc}\right) \;\; \rightarrow\cdot\; \mathbf{U}_m^{TE}\cdot \mathbf{e}^{inc} = -\gamma_0^{TE}\mathbf{R}^{T} \cdot \mathbf{M}_m^{TE}\cdot \mathbf{R}^{2D} \cdot \mathbf{e}^{inc}}$$

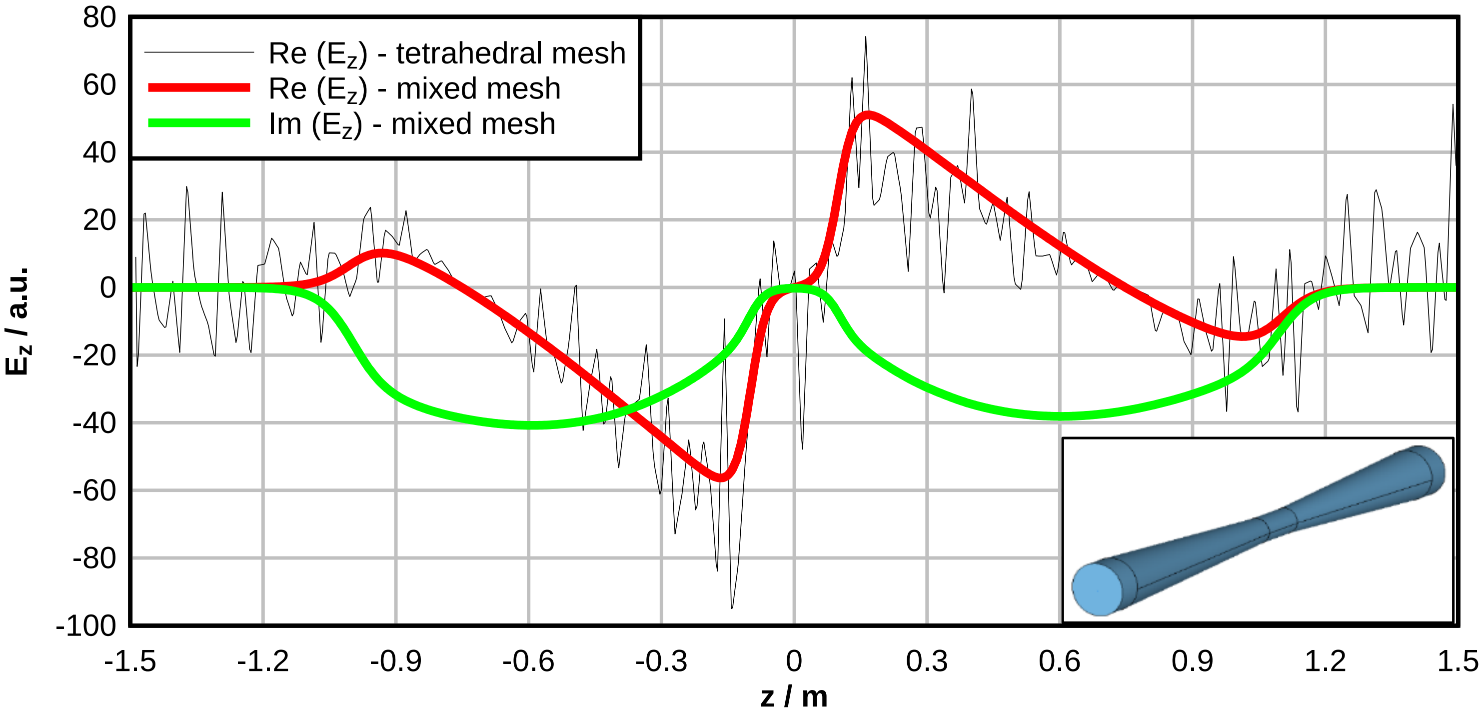
…do this for all waveguide modes supported in the pipe

Frequency Domain Formulation

- Collimator – hybrid meshes



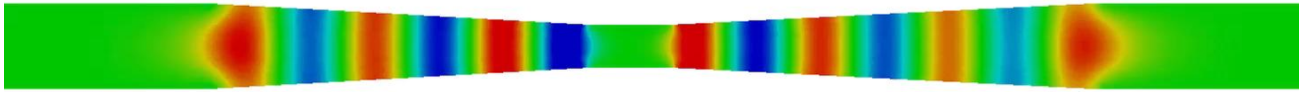




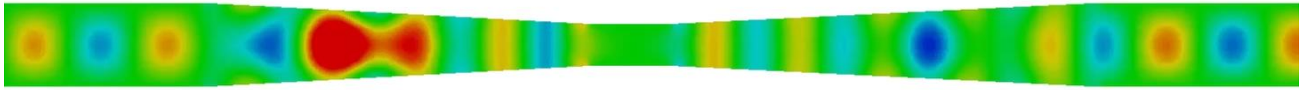
Longitudinal wakefield on axis at 100MHz

Frequency Domain Formulation

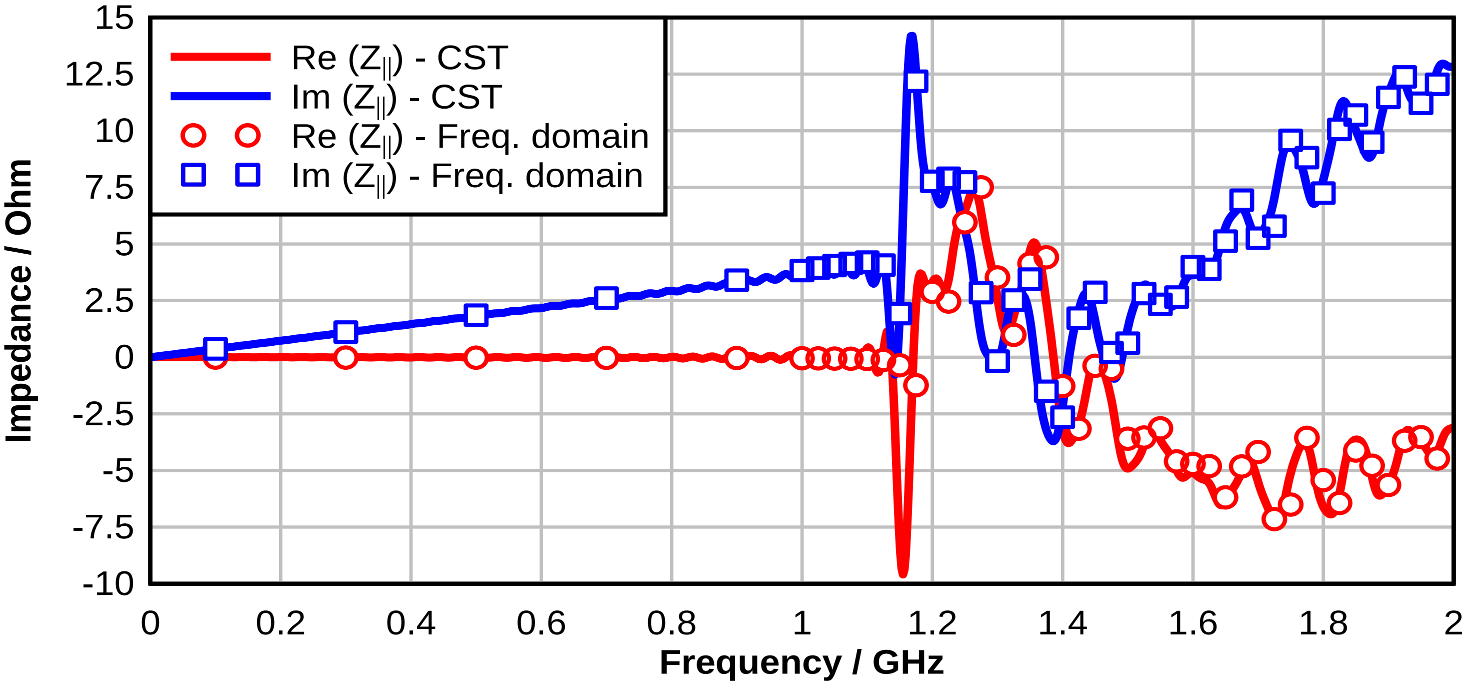
- Collimator – impedance



E_z – 1GHz



E_z – 1.5GHz

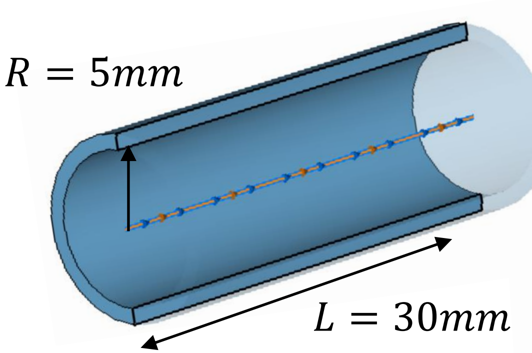


Comparison with CST PS

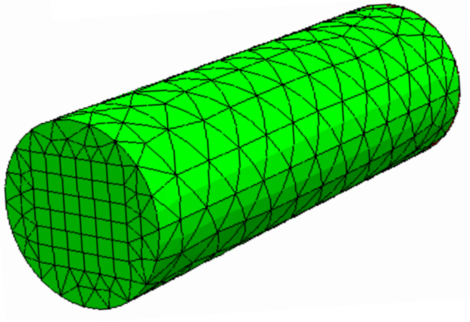
Resistive Wall Impedance

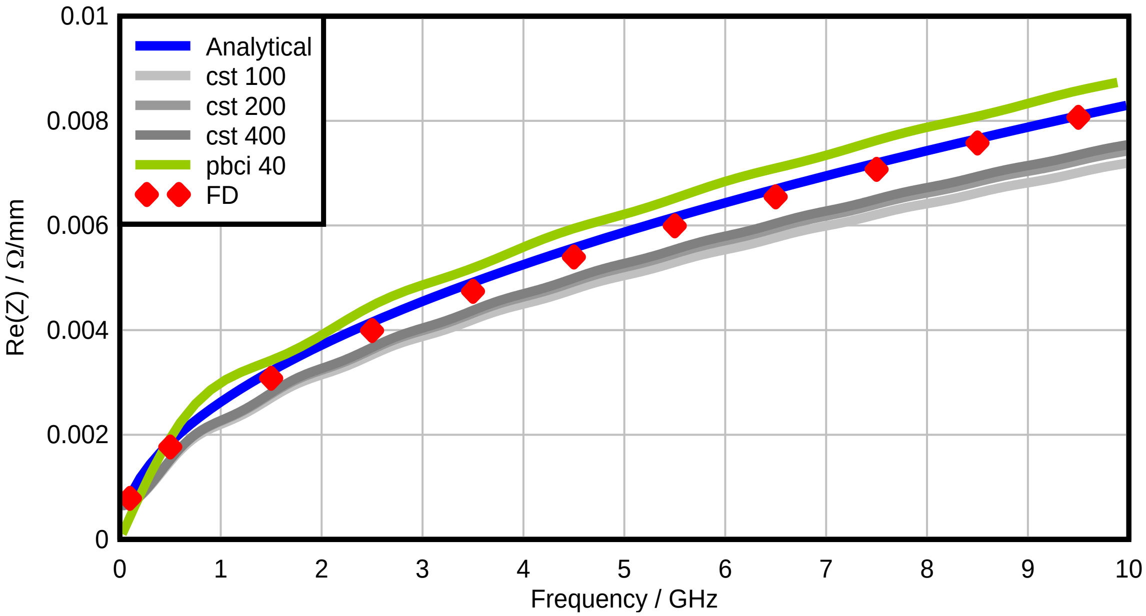
- Resistive wall round pipe
  - Analytical solution

$$Z(\omega) = L\frac{1+j}{2\pi R}\sqrt{\frac{\omega Z_0}{2c\sigma(\omega)}}$$



$\mathbf{TiAl} - \boldsymbol{\sigma} = \mathbf{0.58MS/m}$

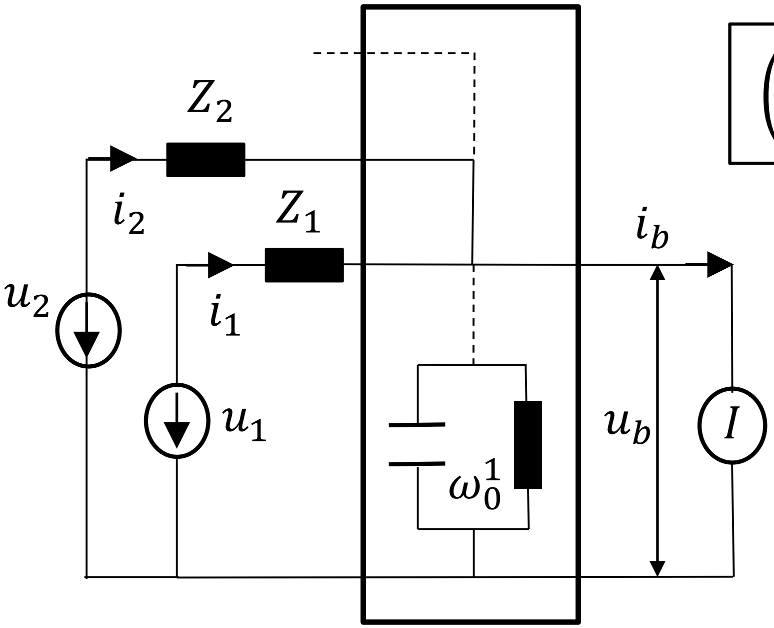




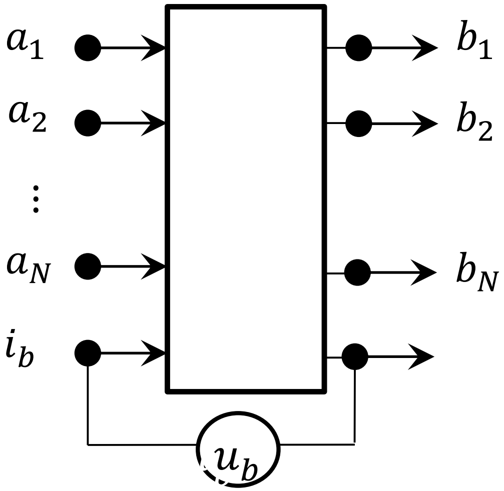
- Large mesh errors in time domain codes
- Sparse unstructured mesh
- Few evaluation points in the frequency domain

Generalized S-Matrix Formulation

- Equivalent circuit representation



$$\boxed{\begin{pmatrix} S & k \\ h & Z_b \end{pmatrix}\begin{pmatrix} a_m \\ i_b \end{pmatrix} = \begin{pmatrix} b_m \\ u_b \end{pmatrix}}$$



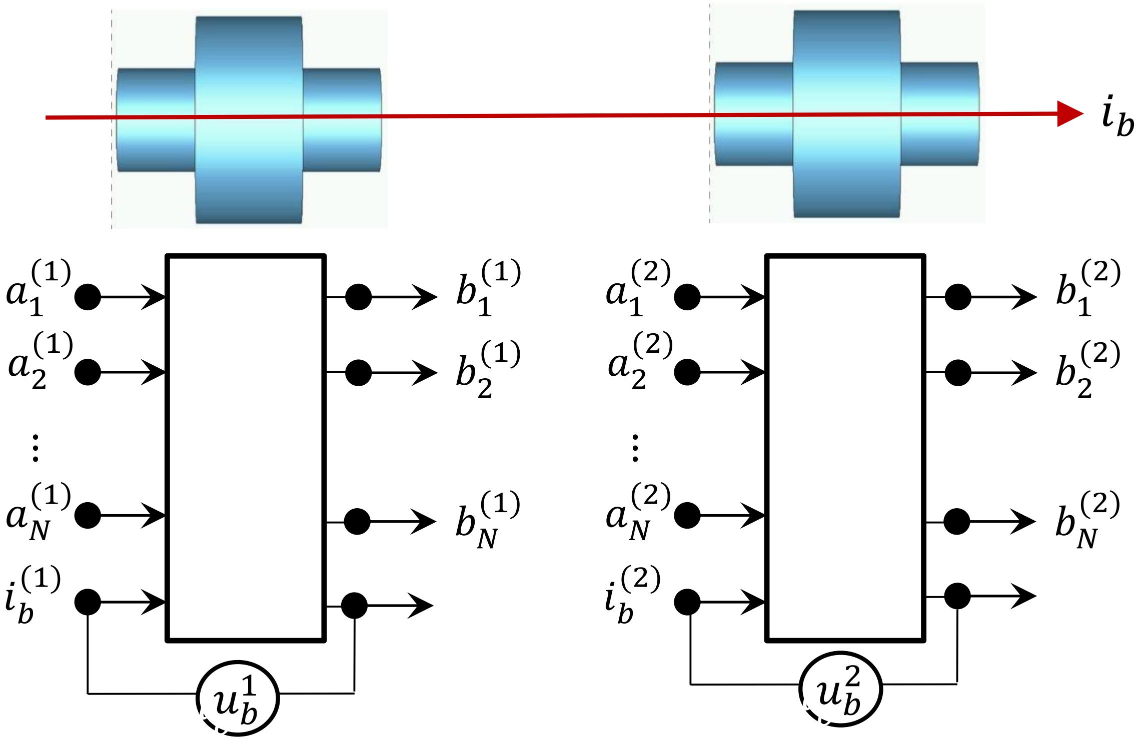
$$k_m(\omega) = \frac{b_m(\omega)}{i_b(\omega)}: \; a_m(\omega) = 0$$

$$a_i = \frac{u_i + Z_i i_i}{2\sqrt{Z_i}}, \qquad b_i = \frac{u_i - Z_i i_i}{2\sqrt{Z_i}}$$

$$h^m(\omega) = \int dz \, E_z^m(r, z, \omega) e^{-i\frac{\omega}{v}z}$$

Generalized S-Matrix Formulation

- Coupled S-Parameter Calculation with Beam (CSC-Beam)



Matching conditions:

$$b_i^{(n)} = a_i^{(n+1)}$$

$$i_b^{(n)} = i_b^{(n-1)} e^{ik_0 L_{n-1}}$$

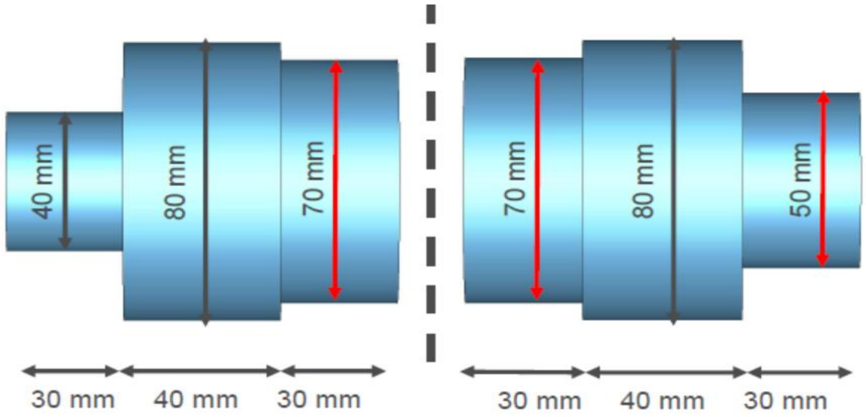
$$\sum_n u_b^{(n)} = u_b^{tot}$$

- Total coupling matrix:

$$\boxed{\begin{pmatrix} S^{tot} & k^{tot} \\ h^{tot} & Z_b^{tot} \end{pmatrix} \begin{pmatrix} a_m \\ i_b \end{pmatrix} = \begin{pmatrix} b_m \\ u_b^{tot} \end{pmatrix}}$$

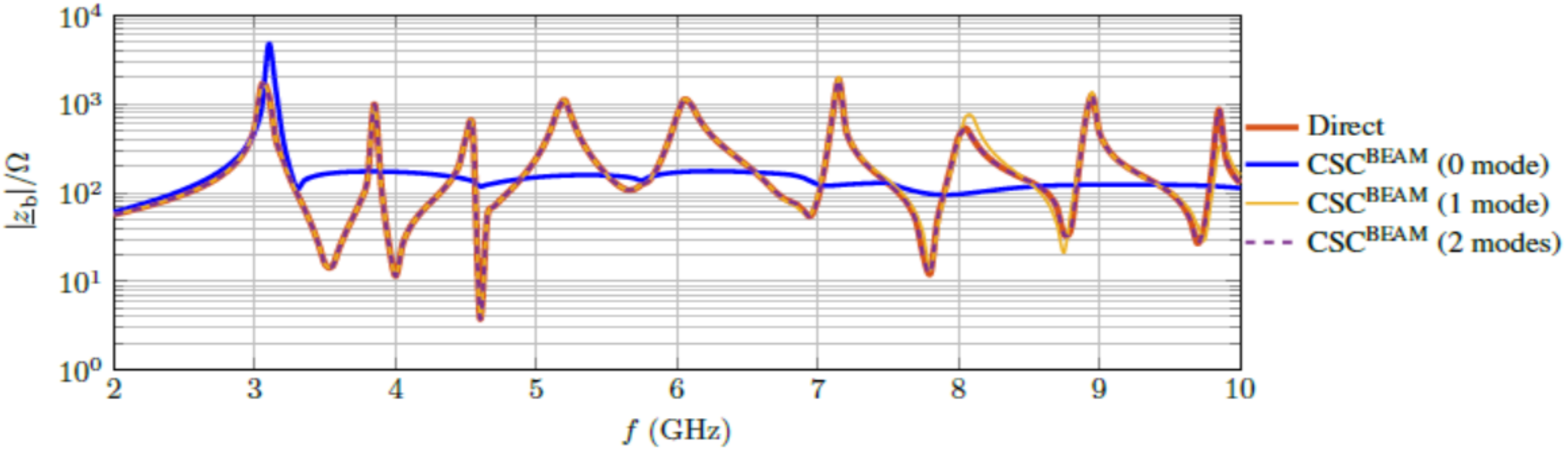
Generalized S-Matrix Formulation

- Two cavity / two mode coupling example



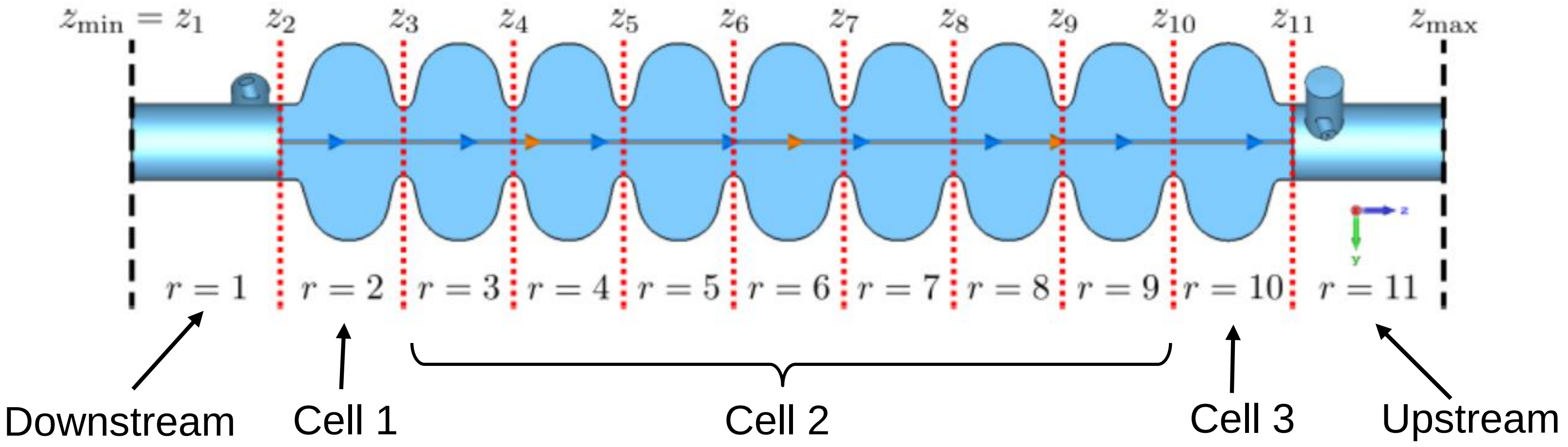
Cavity 1: $\tilde{S}\begin{pmatrix} a^{(1)}_{TM_{01}} \\ i^{(1)}_{b} \end{pmatrix} = \begin{pmatrix} b^{(1)}_{TM_{01}} \\ b^{(1)}_{TM_{02}} \\ u^{(1)}_{b} \end{pmatrix}$

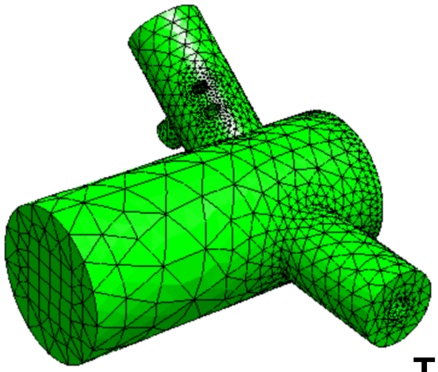
Cavity 2: $\tilde{S}\begin{pmatrix} a^{(2)}_{TM_{01}} \\ a^{(2)}_{TM_{02}} \\ i^{(2)}_{b} \end{pmatrix} = \begin{pmatrix} b^{(2)}_{TM_{01}} \\ u^{(2)}_{b} \end{pmatrix}$



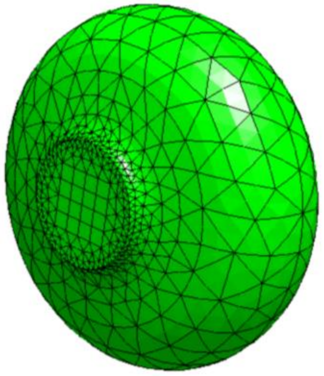
1.3 GHz TESLA cavity

- Tesla 1.3GHz cavity





15 TE-Modes
$(f_{max} = 8.2\text{GHz})$
15 TM-Modes
$(f_{max} = 10.6\text{GHz})$



15 TE-Modes
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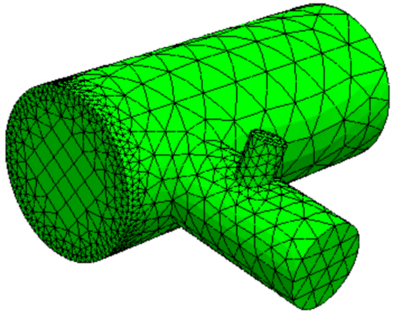
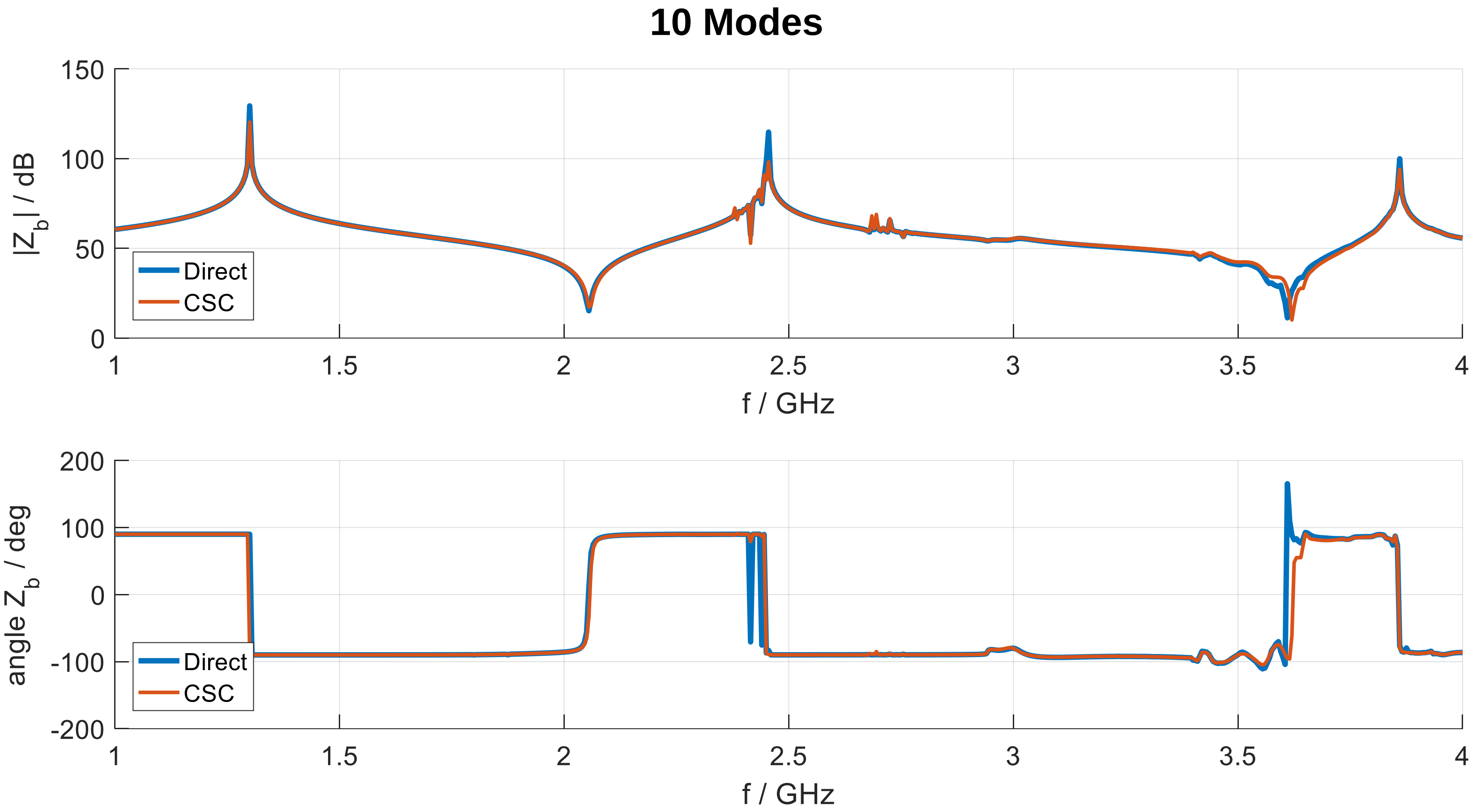


TEM, …

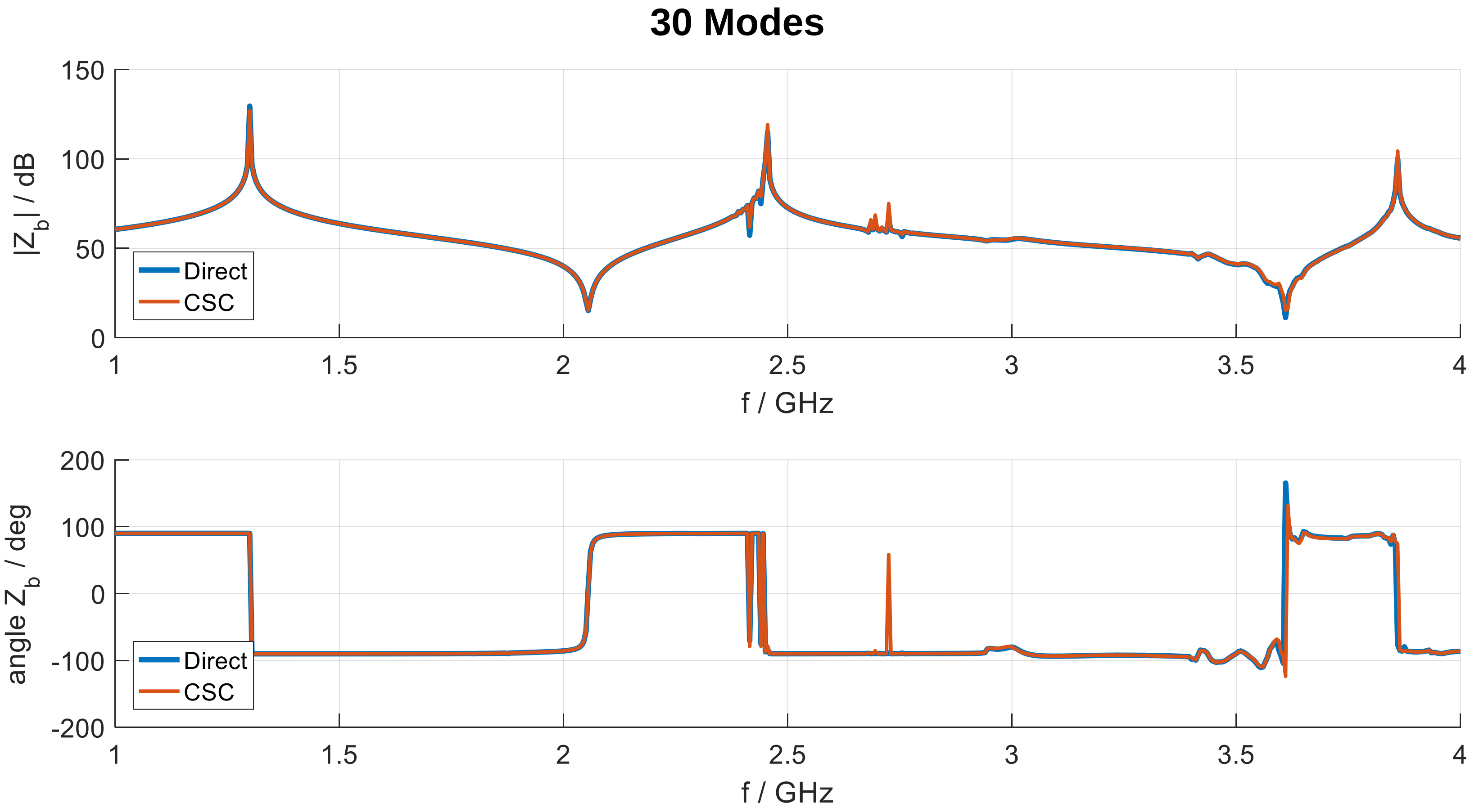
1.3 GHz TESLA cavity

- Tesla 1.3GHz cavity



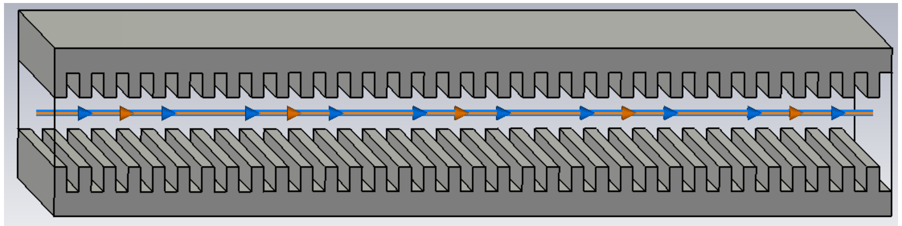
1.3 GHz TESLA cavity

- Tesla 1.3GHz cavity

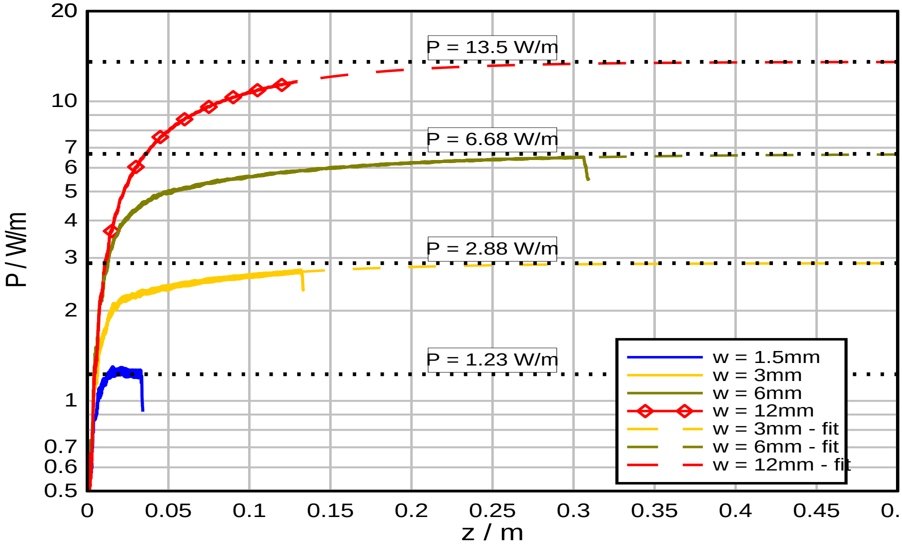


Periodic Structures

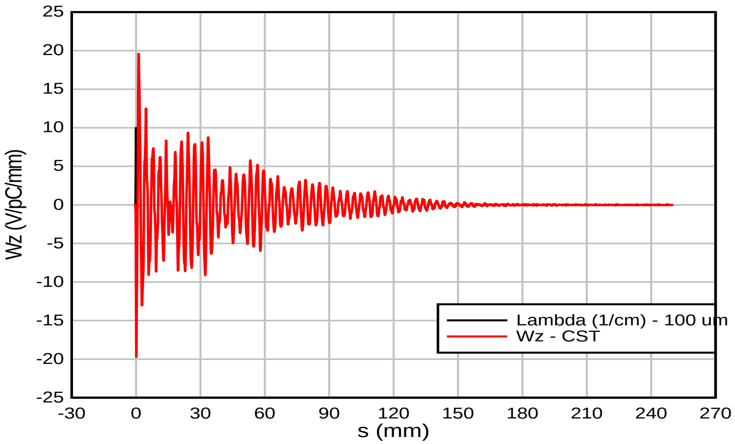
- Dechirper (LCLS, E-XFEL)

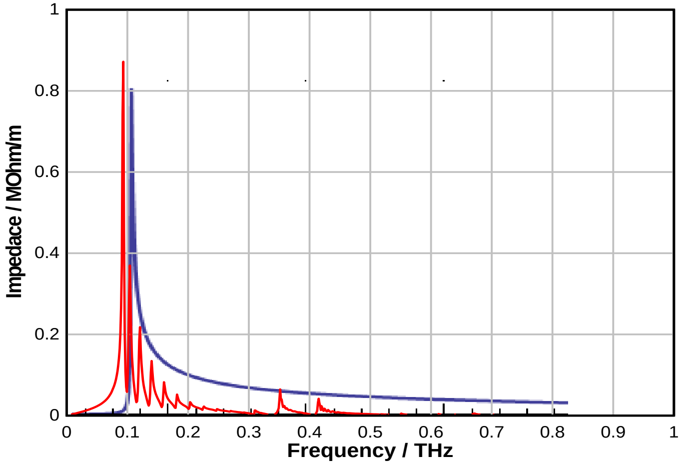


  - Steady state losses



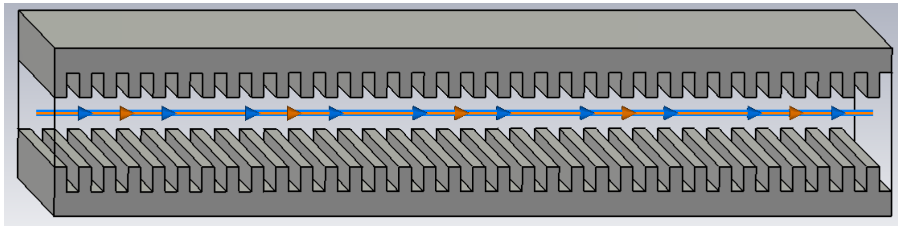
100µm bunch



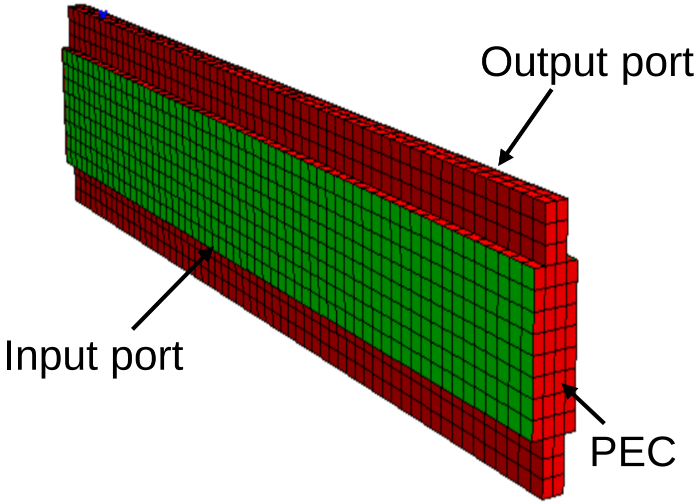


Periodic Structures

- Dechirper (LCLS, E-XFEL)



  - Single period computation layout

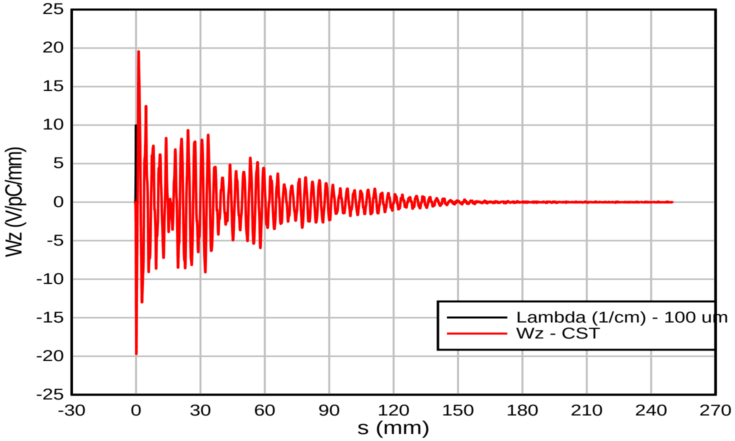


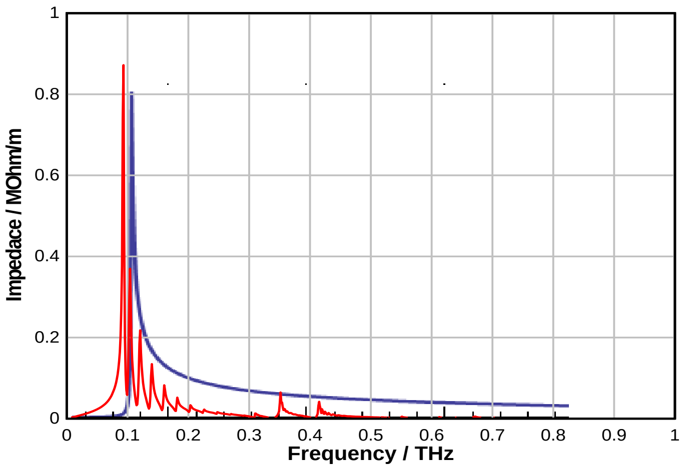
No. elements: 2640

On each port:

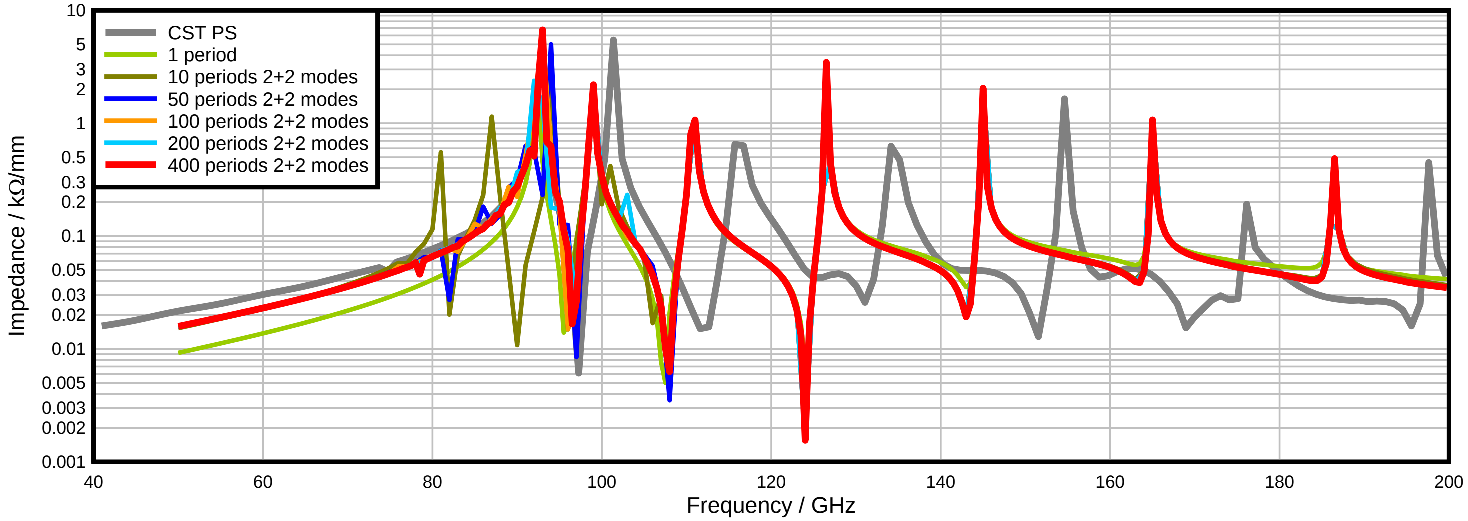
100 TE-Modes
$(f_{max} = 375\text{Ghz})$

100 TM-Modes
$(f_{max} = 448\text{Ghz})$

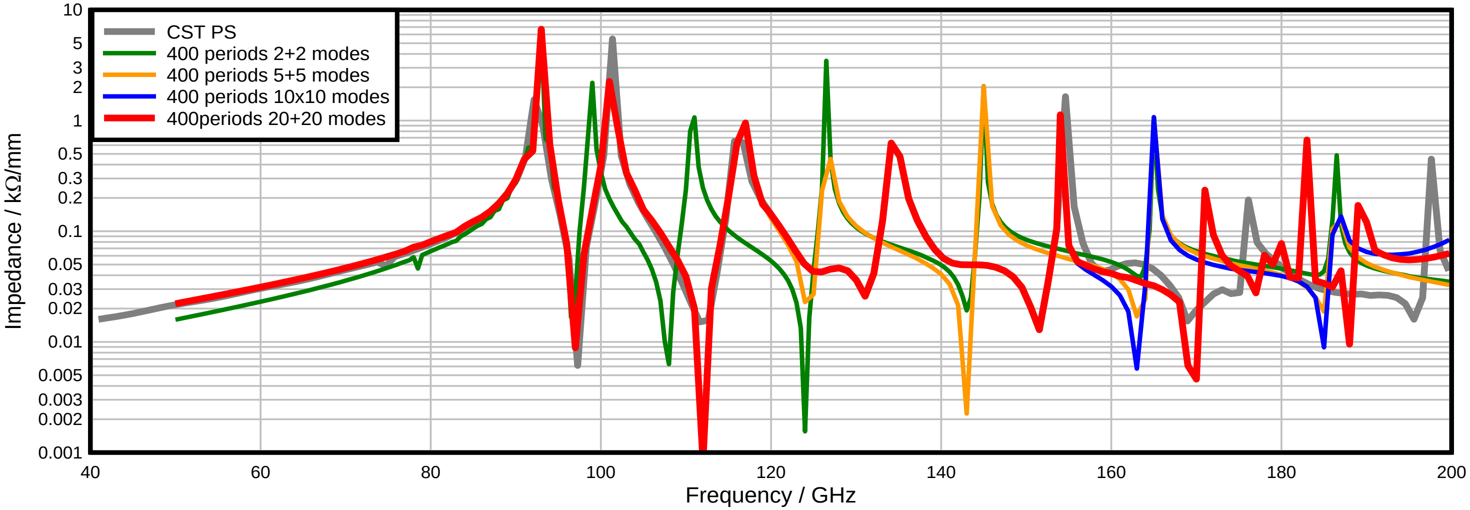

100µm bunch



Periodic Structures



Impedance vs. no. periods



Impedance vs. coupling modes

Summary & Conclusions

- The frequency domain approach
  - **Fills the gap for some important wakefield/impedance problems**
    - Complicated chamber geometry, long range / low frequency fields, resistive, rough surfaces, dispersive materials, beam signals on waveguide openings
  - **FEM Frequency domain formulation**
    - Beam port boundary conditions
    - Mixed mesh discretization
  - **Concatenation using generalized S-Matrix formulation**
    - Efficient /accurate impedance computation of large cavity chains
    - Periodic structures (dechirper)
  - **Accurate lossy wall impedances**
  - **Limitations: huge size of discrete problem for ultra-high frequencies**
    - ToDo: Domain decomposition
    - ToDo: Parallel multigrid solvers
    - ToDo: Fast frequency sweeps and spectral evaluation by MOR,…

Thank You for your attention