# Problems with Poisson Approach

tracking example: European XFEL

collective uniform motion (CUM) approach

time dependent shape $\rightarrow$ problems

dirty trick

some conclusions

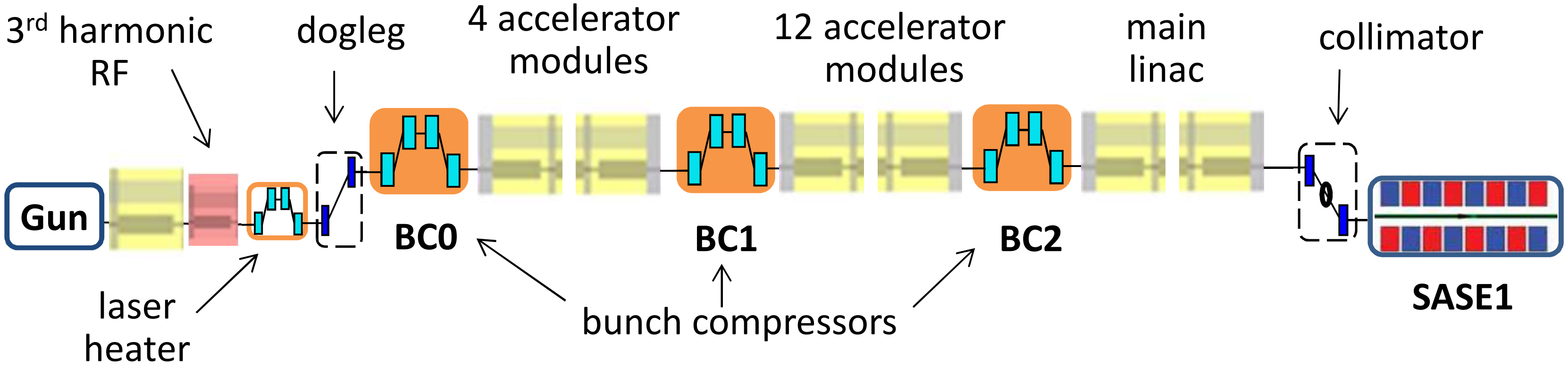
P1 and P2 approach

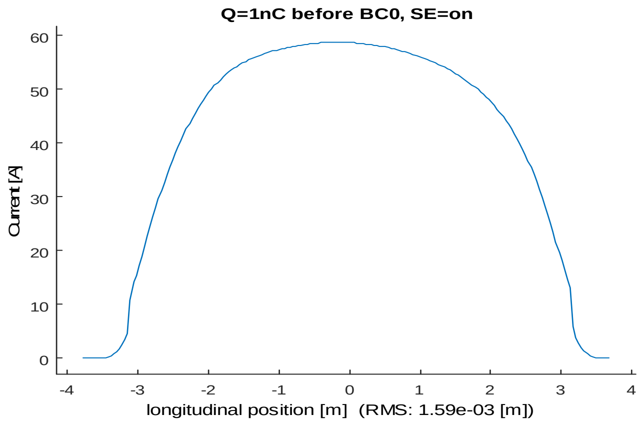
simple example

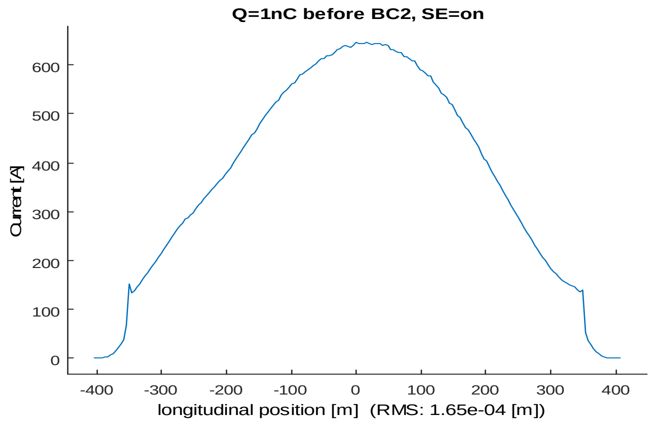
point particle / gaussian bunch / discrete quadrupole

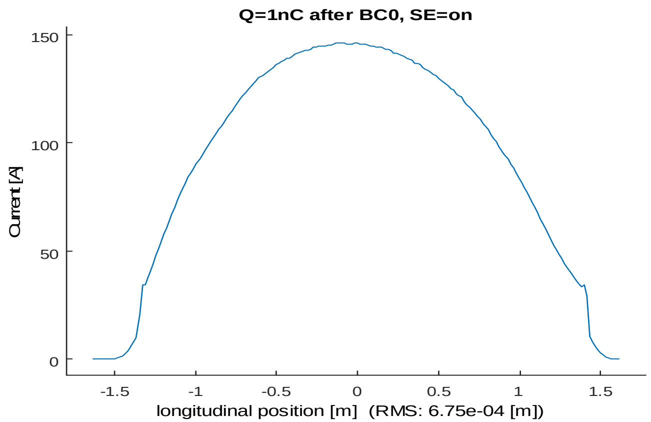
more conclusions

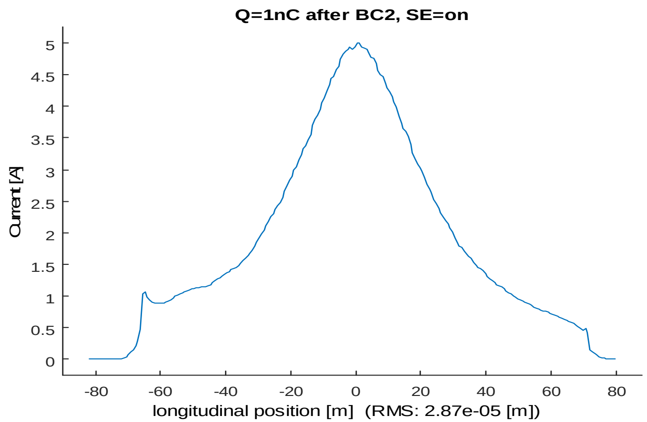
# European XFEL



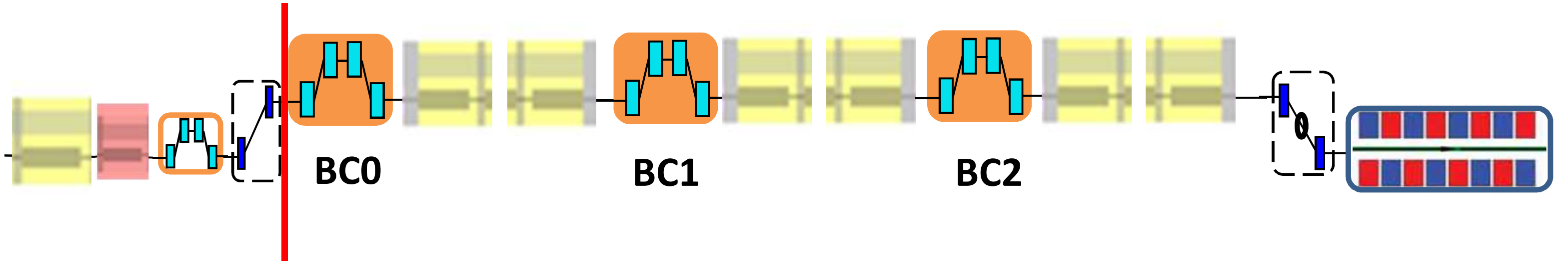



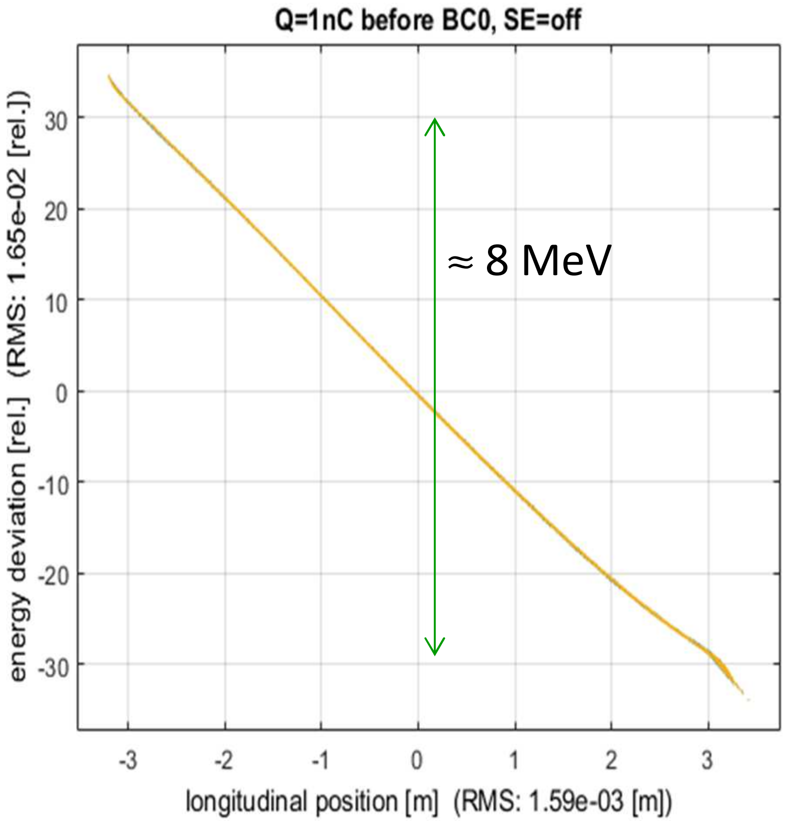



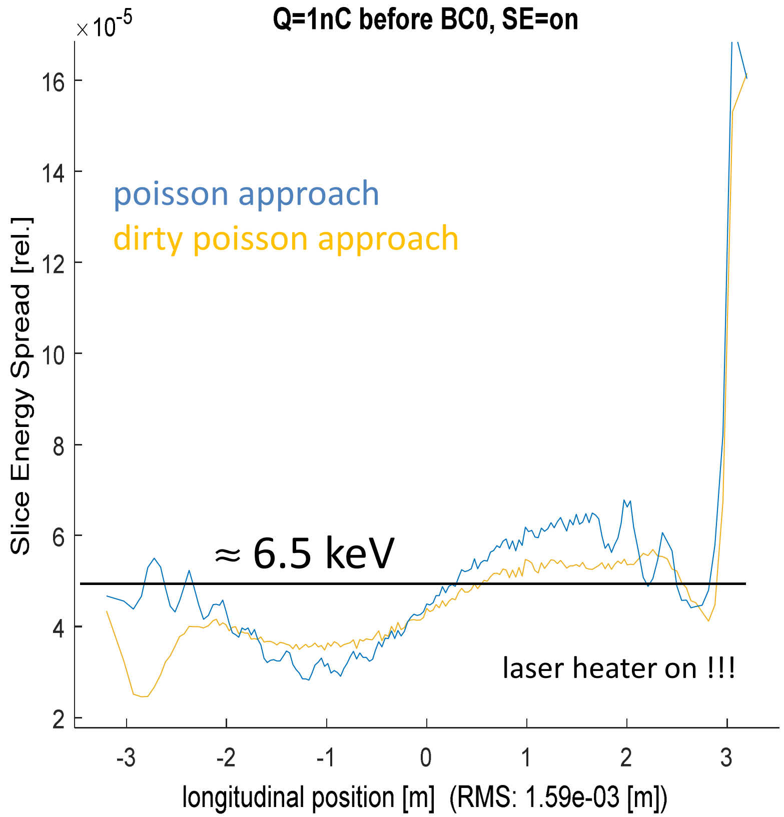
before BC0, Z=73m, 130 MeV, ~60A



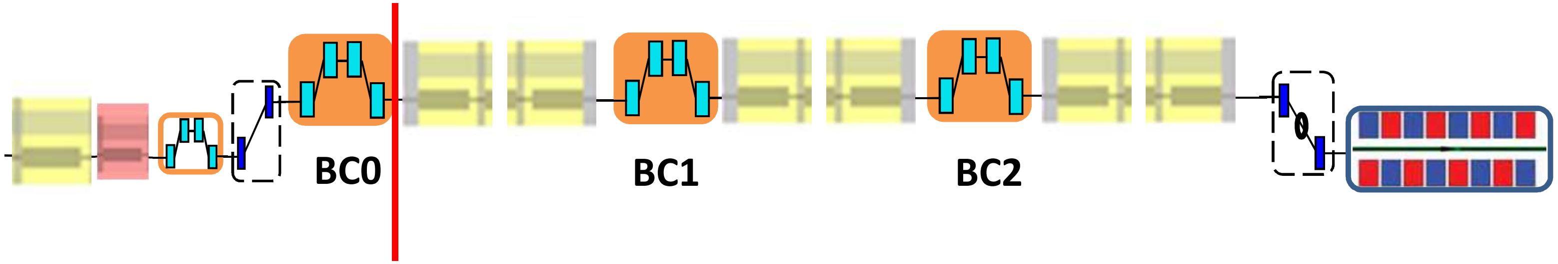

longitudinal phase space



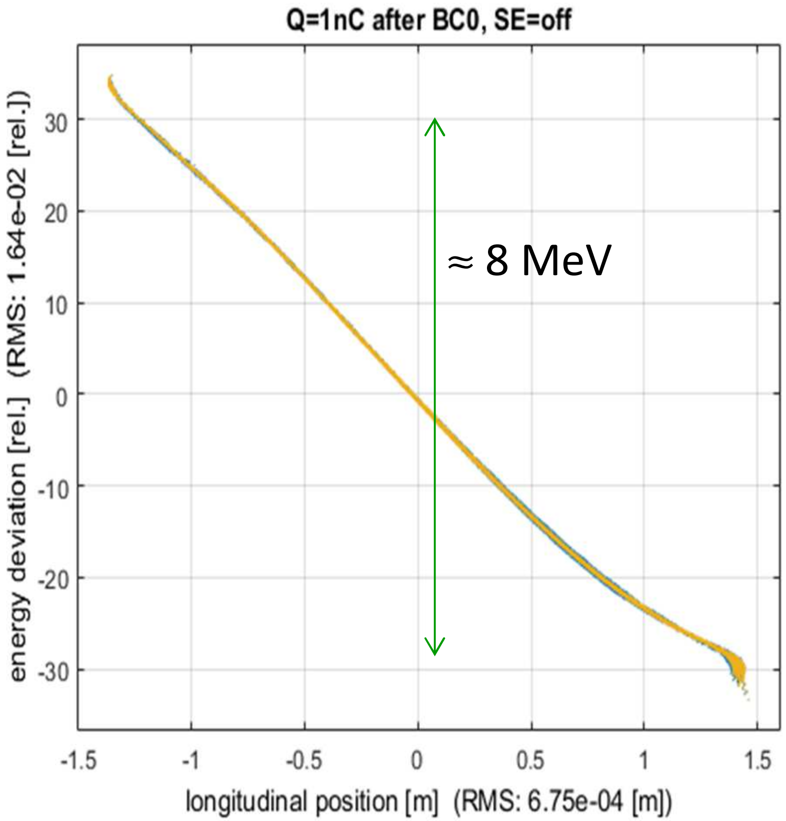

slice energy spread



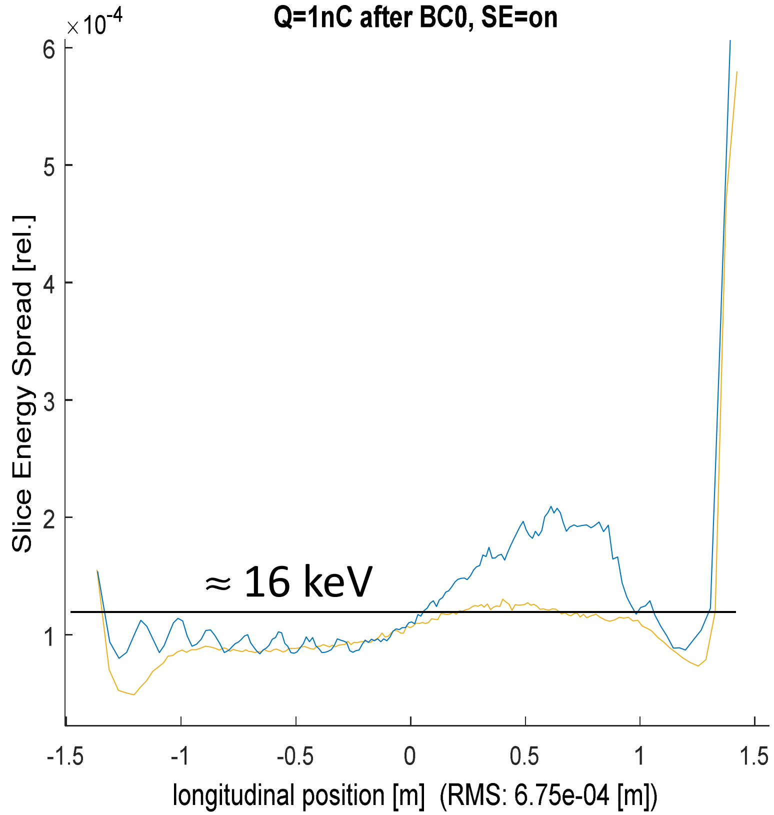
after BC0, Z=80m, 130 MeV, ~150A



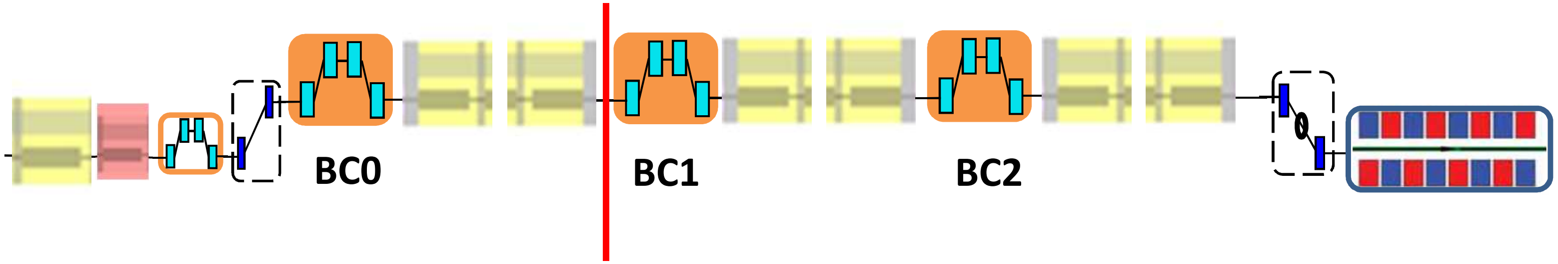

longitudinal phase space



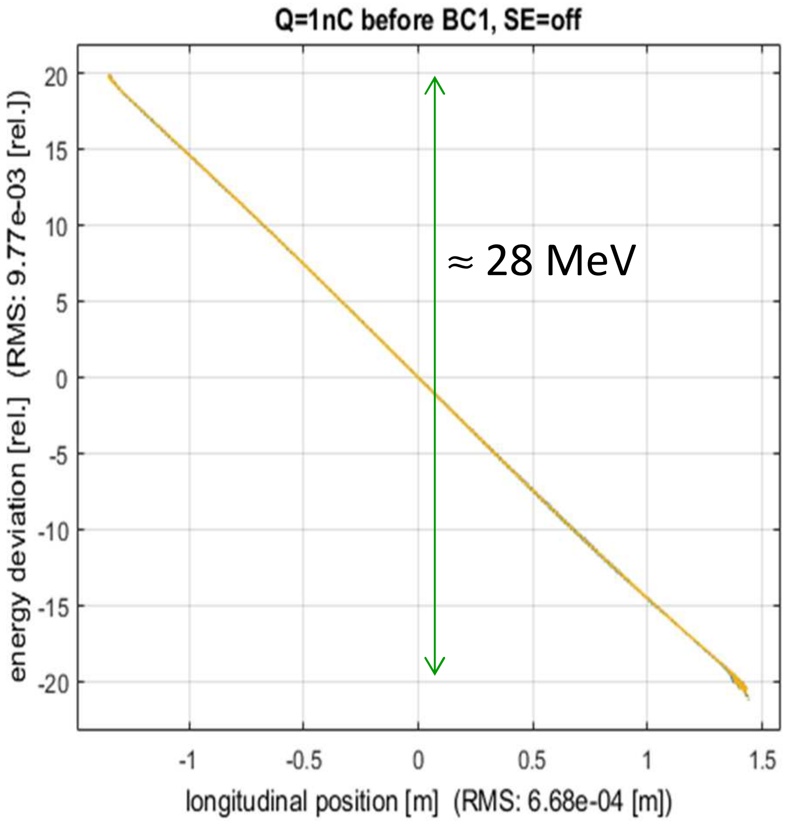

slice energy spread



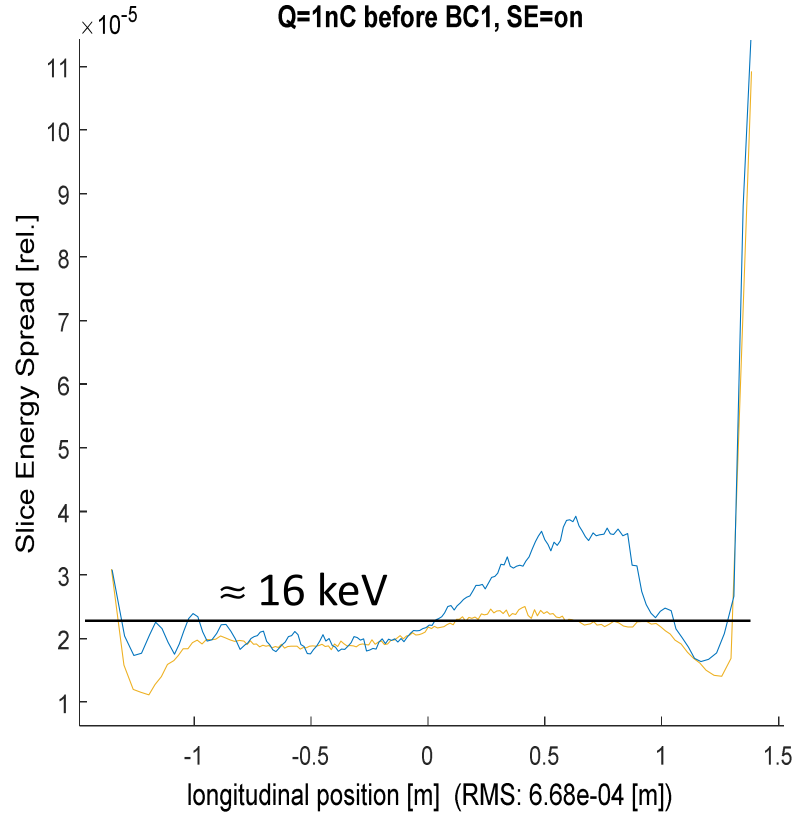
before BC1, Z=159m, 700 MeV, ~150A



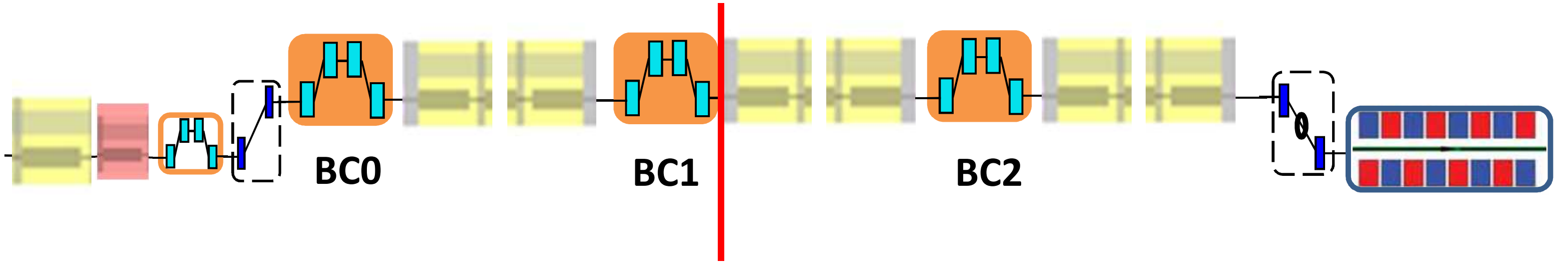

longitudinal phase space



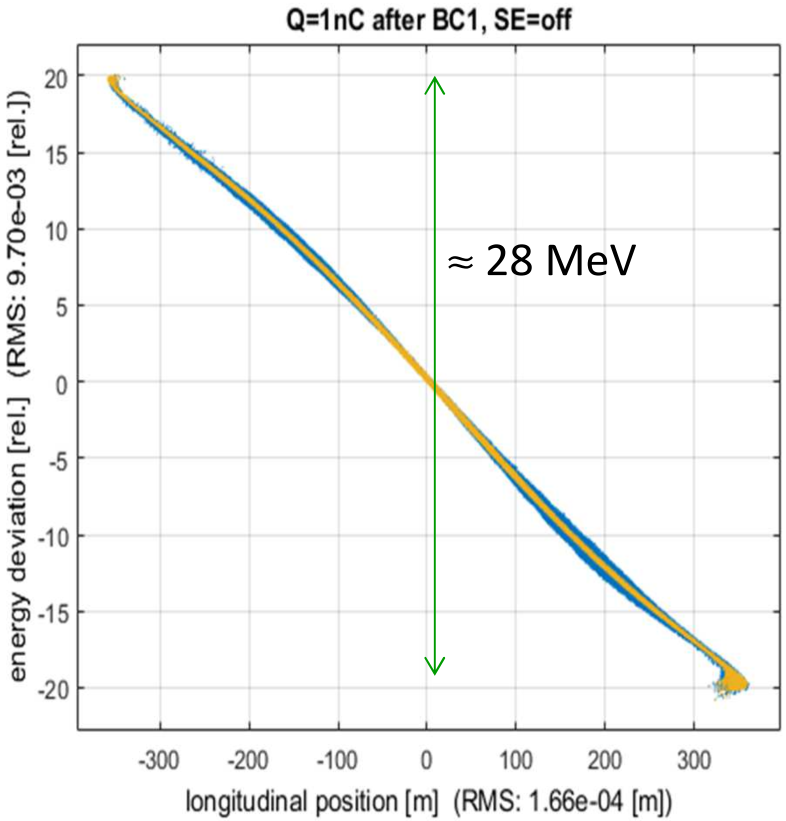

slice energy spread



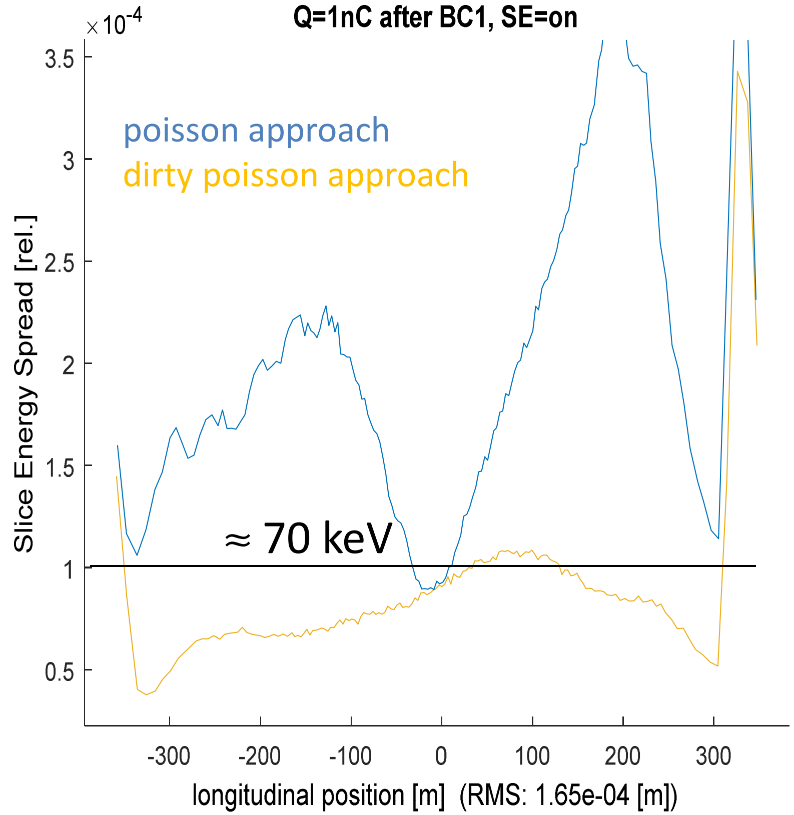
after BC1, Z=181m, 700 MeV, ~650A



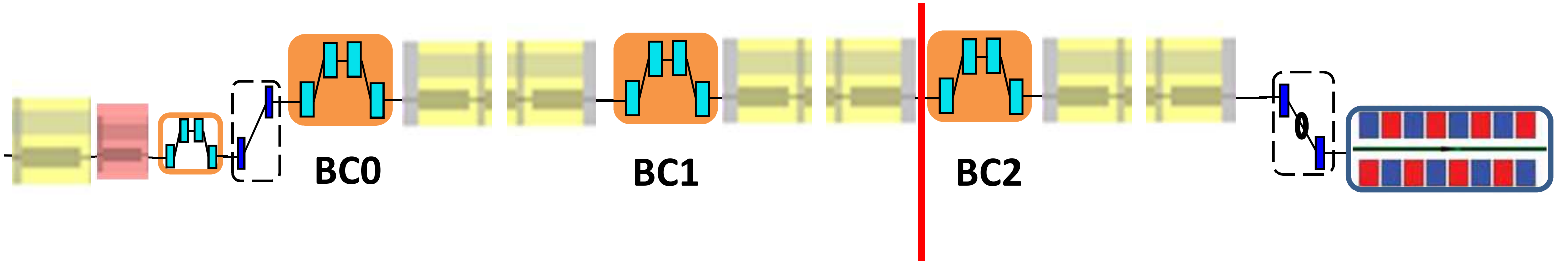

longitudinal phase space



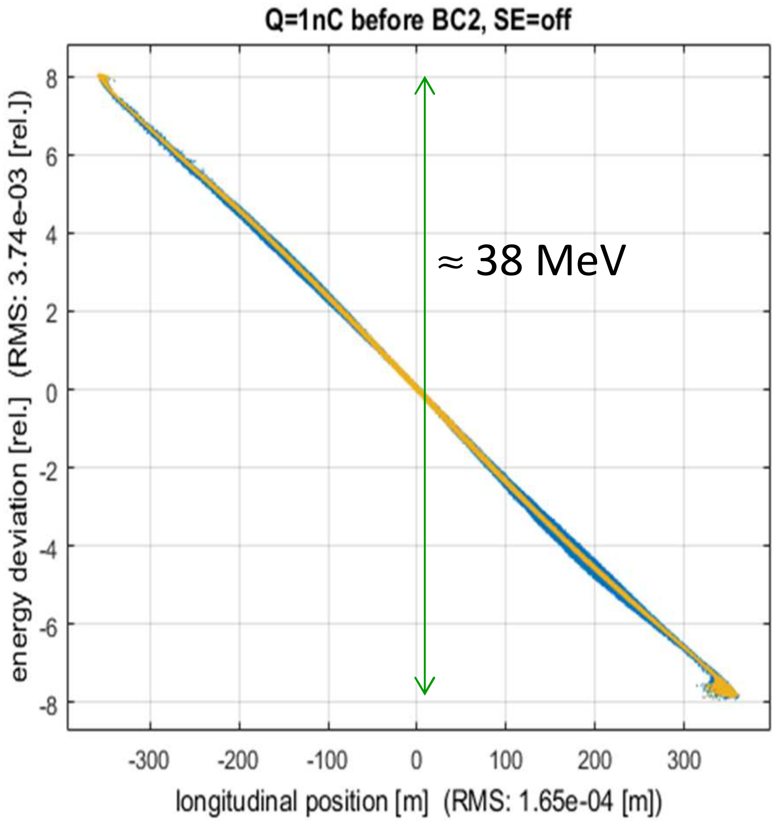

slice energy spread



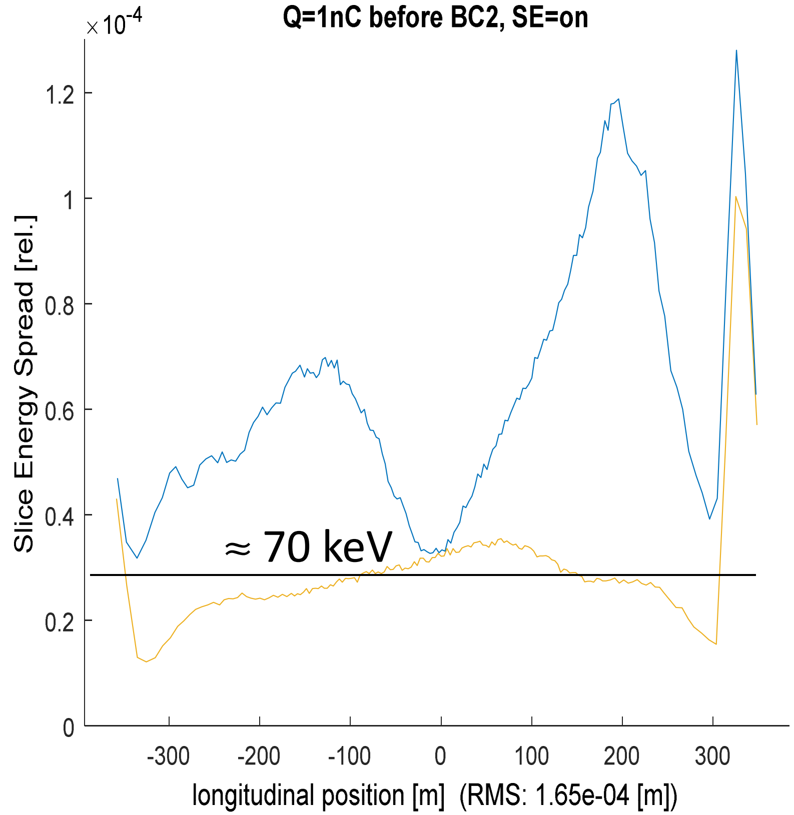
before BC2, Z=370m, 2.4 GeV, ~650A



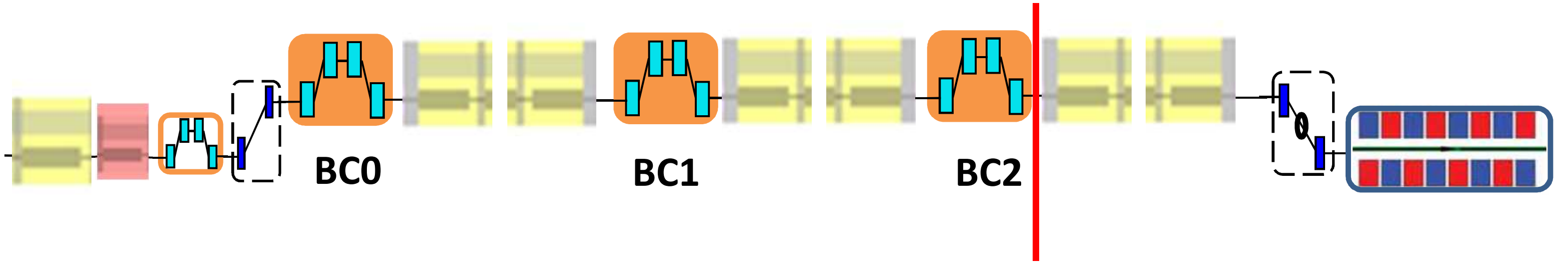

longitudinal phase space



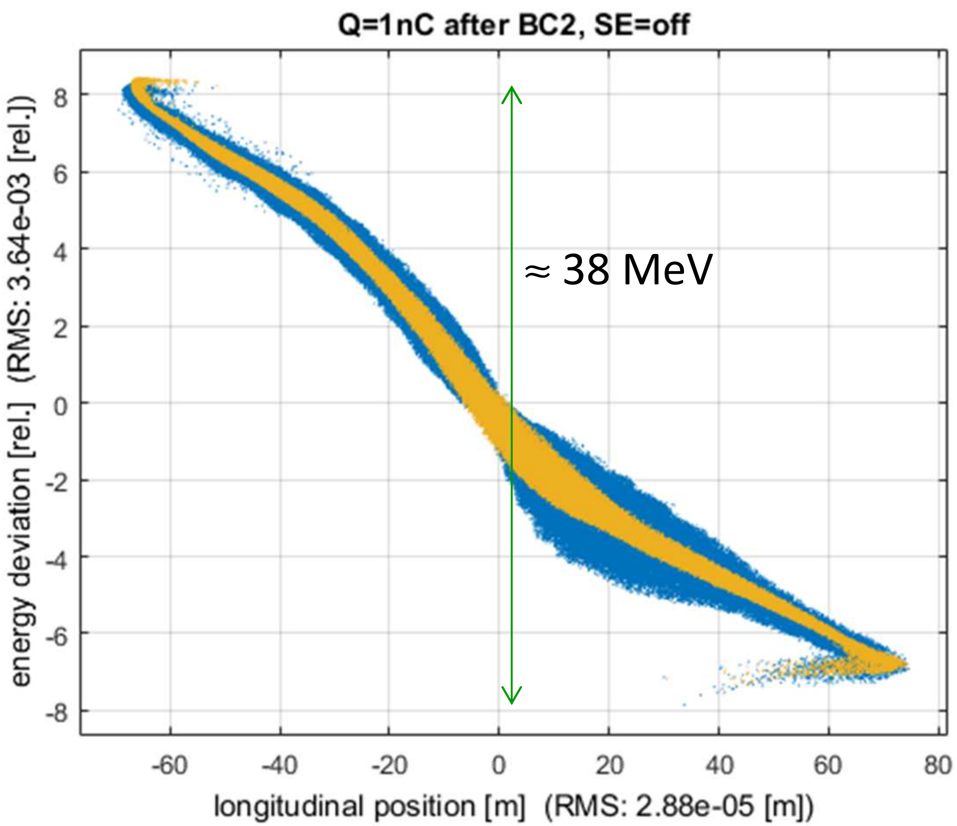

slice energy spread



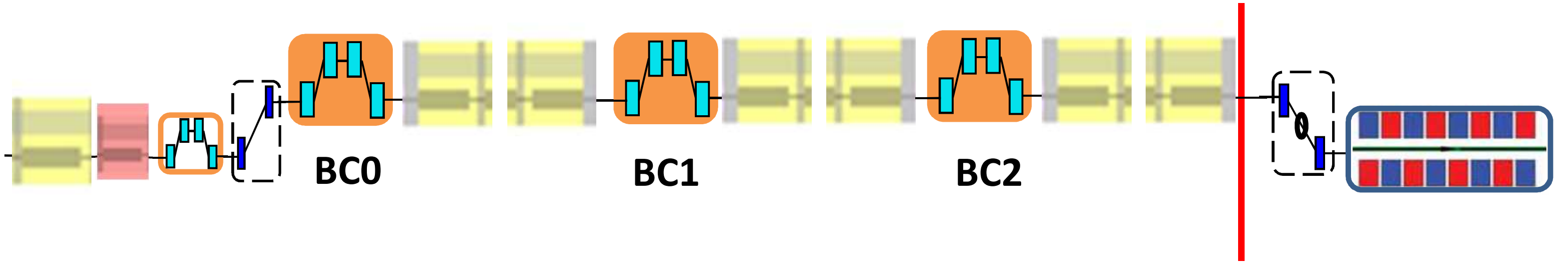
after BC2, Z=393m, 2.4 GeV, ~5kA



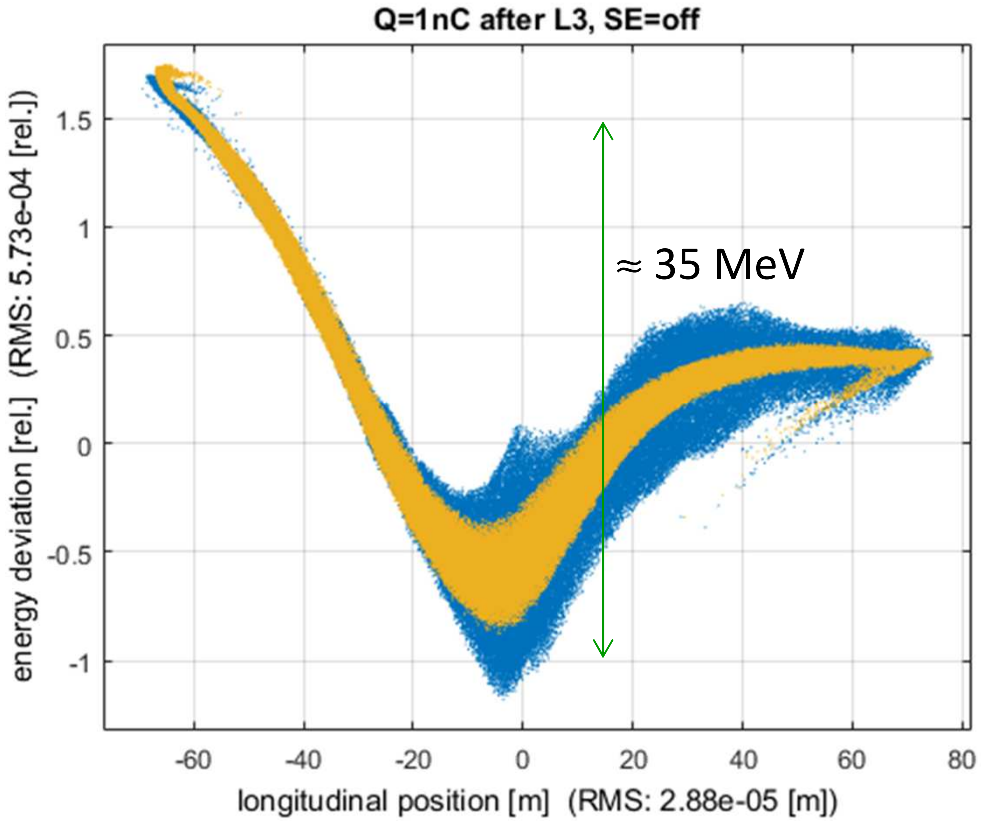

longitudinal phase space



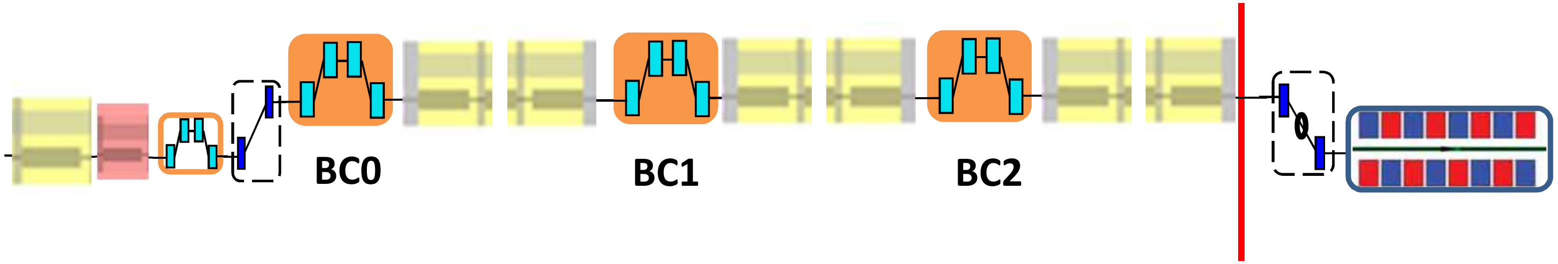
after L3, Z=1628m, 14 GeV, ~5kA



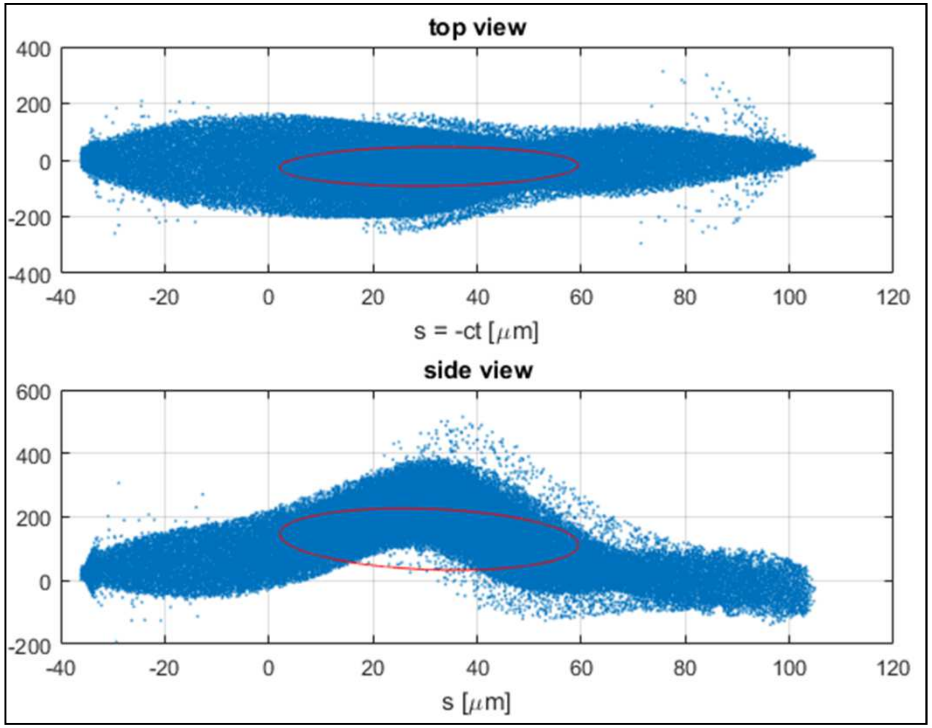

longitudinal phase space



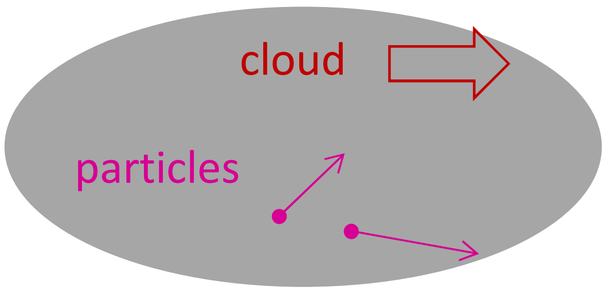
after L3, Z=1628m, 14 GeV, ~5kA




bunch shape after L3



# Collective Uniform Motion (CUM) Approach




collective motion $\mathbf{v}_c$

individual motion $\mathbf{v}_\nu$

Poisson solver $\rightarrow$ $\mathbf{E}$

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}_c \times \mathbf{E}$$

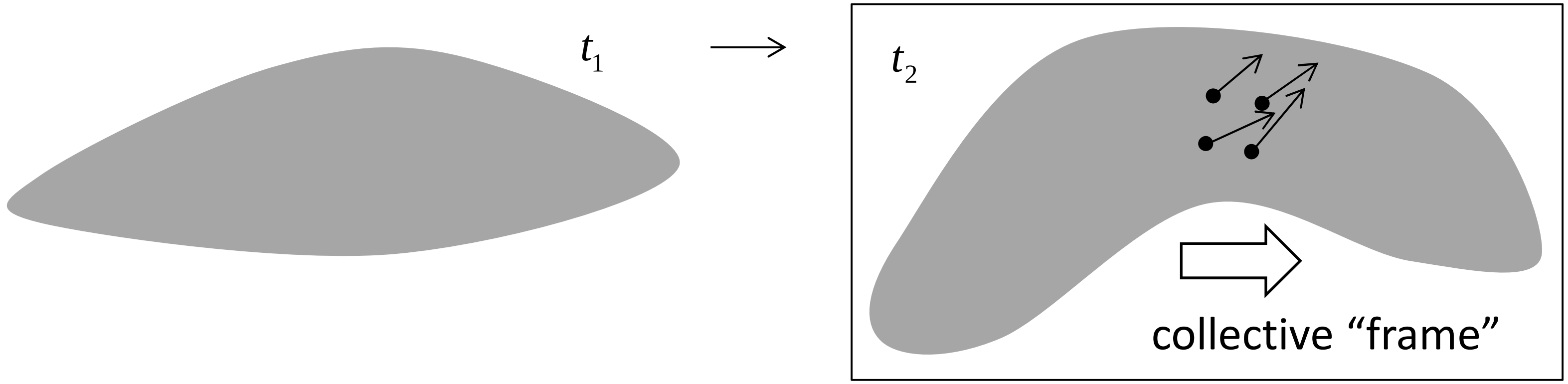
Lorentz force $$\mathbf{F}_\nu = q\left(1 + \frac{1}{c^2}\mathbf{v}_\nu \times \mathbf{v}_c \times\right)\mathbf{E}_\nu$$

in particular $\mathbf{v}_\nu \parallel \mathbf{v}_c \rightarrow$ strong suppression of transverse force

$$\mathbf{v}_\nu = \mathbf{v}_c \rightarrow \mathbf{F}_\perp = \frac{1}{\gamma^2} q\mathbf{E}_\perp$$

$$F_\parallel = qE_\parallel \sim \frac{1}{\gamma^2}$$ usually $\left|E_\parallel\right| << \left\|\mathbf{E}_\perp\right\|$

# Time Dependent Shape + Long Bunch Approximation




long bunch estimation:

$E_z$ 2.4 → 14 GeV: ~ 10 MV / 1000m

$$E_x \approx \frac{Z_0 I}{2\pi\sigma_r} \cdot \frac{x}{\sigma_r}$$ about 10 GV / m for $I = 5$ kA, $\sigma_r = 30\ \mu$m

$$F_{v,\parallel} \approx qE_{v,z} + \boxed{x'_v \cdot \frac{qZ_0 I}{2\pi\sigma_r} \cdot \frac{x_v}{\sigma_r}}$$ strong 2nd order effect

f.i. $x'_v \sim 1\,\mu\text{rad} \rightarrow$ ~ 10 kV / m

effects of z & x components of same magnitude

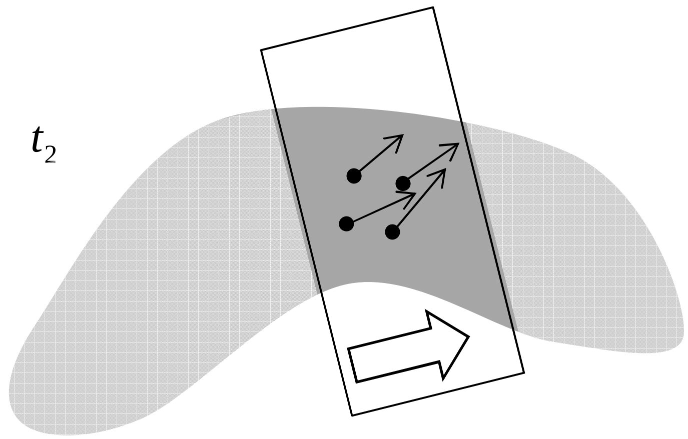
z comp.: decreasing with energy, strong correlation in slice → corr. energy spread

x comp.: ~ energy independent, weak correlation in slice → uncorr. energy spread slice correlated and uncorrelated angle $x'_\nu = \boxed{x'_{sc}(z_\nu)} + \boxed{\delta x'_\nu}$

$$F_{\nu,\parallel} \approx qE_{\nu,z} + \frac{qZ_0 I}{2\pi\sigma^2} \cdot \boxed{x'_{sc}(z_\nu)x_\nu} + \frac{qZ_0 I}{2\pi\sigma^2} \cdot \boxed{\delta x'_\nu x_\nu}$$

even the correlated part contributes to uncorrelated energy spread!

extreme case: if the bunch is infinitely long there is no slice-to-slice-interaction and one can calculate slice-self-interaction with better frames that are adjusted to the correlated angle




only the uncorrelated angel spread
would contribute to the longitudinal field!

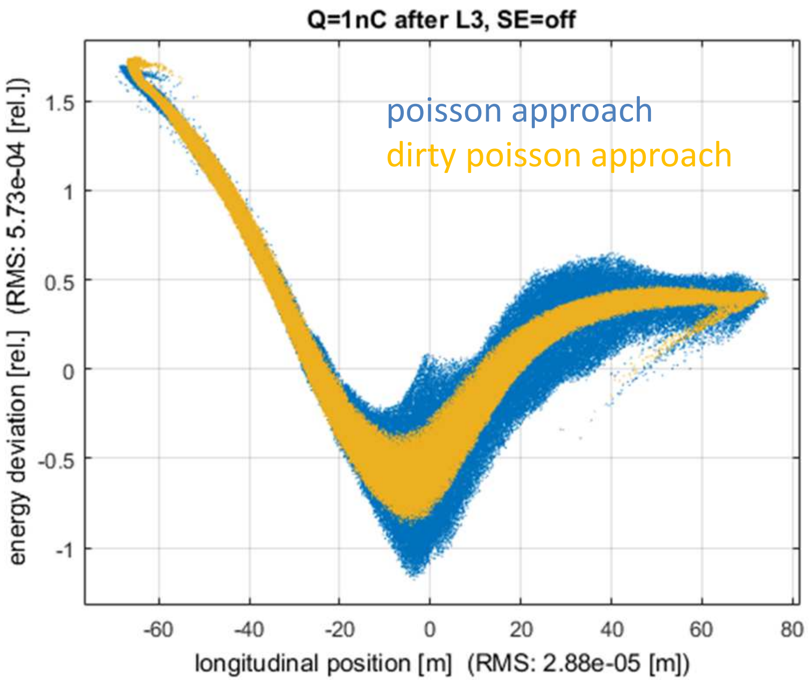
$$F_{\nu,\parallel} \approx \frac{qZ_0 I}{2\pi\sigma^2} \cdot \delta x'_\nu x_\nu$$

# The Dirty Trick

Lorentz force, Poisson approach $\mathbf{F}_\nu = q\left(1 + \frac{1}{c^2}\mathbf{v}_\nu \times \mathbf{v}_c \times\right)\mathbf{E}_\nu$

slice correlated and uncorrelated motion $\mathbf{v}_\nu = \mathbf{v}_{sc}(z_\nu) + \delta\mathbf{v}_\nu$

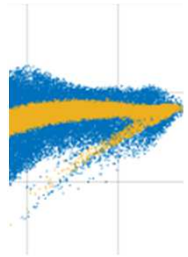
modified force $\boxed{\mathbf{F}_\nu = q\left(1 + \frac{1}{c^2}(\mathbf{v}_c + \delta\mathbf{v}_\nu) \times \mathbf{v}_c \times\right)\mathbf{E}_\nu}$



# First Conclusions

it is an empirical approach

problems with rollover part:



very different motion of particles in the same slice

perhaps it is better to consider IUM (individual uniform motion, per particle)

needs other numerical method
try to avoid quadratic scaling of effort

# but …

even the IUM approach is empirical/questionable!
full Maxwell-approaches (LW or PDE) could be better

# … high effort, ??? gain of accuracy

## !!! the Poisson approach can do better

# Two Approaches for Tracking

EoM with **E&B**:

$$\frac{d}{dt}\mathbf{r}_\nu = \mathbf{v}(\mathbf{p}_\nu)$$

$$\frac{d}{dt}\mathbf{p}_\nu = q\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]$$

Poisson approach:

$$\left[\partial_x^2 + \partial_y^2 + \gamma_0^{-2}\partial_z^2\right]V = -\rho/\varepsilon$$

$$\downarrow$$

$$V$$

$$\mathbf{A} = \mathbf{e}_z c^{-1}\beta_0 V$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\mathbf{E} = -\nabla V - \partial_t\mathbf{A} = -\nabla V + \beta_0\partial_z\mathbf{A}$$

EoM with V&**A** in canonical coordinates:

$$\frac{d}{dt}\mathbf{r}_\nu = \mathbf{v}(\mathbf{P}_\nu - \mathbf{A}(\mathbf{r}_\nu))$$

$$\frac{d}{dt}\mathbf{P}_\nu = -q\nabla\left[V - \mathbf{v}\cdot\mathbf{A}\right]$$

$$\left[\partial_x^2 + \partial_y^2 + \gamma_0^{-2}\partial_z^2\right]V = -\rho/\varepsilon$$

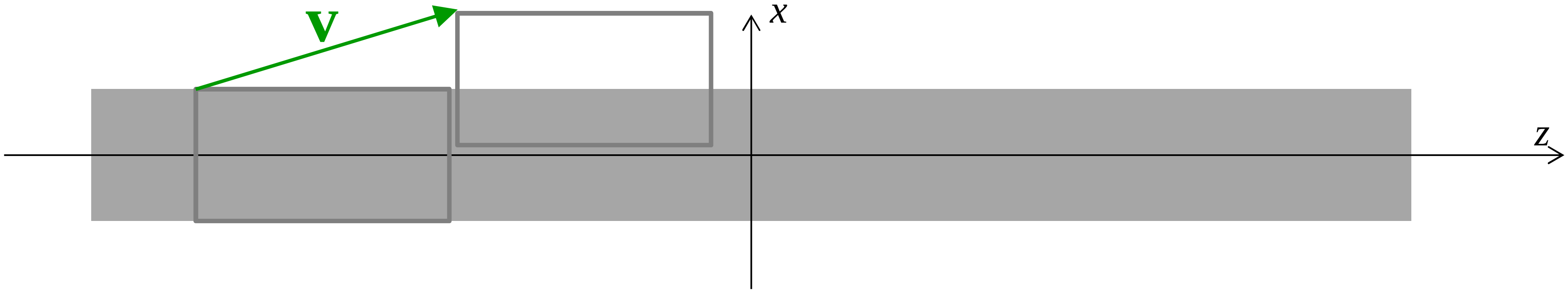
$$\downarrow$$

$$V$$

$$\mathbf{A} = \mathbf{e}_z c^{-1}\beta_0 V$$

# Simple Example

infinite charged plate in uniform motion $\rho(x,y,z,t)=\rho(x-v_x t)$ , $\rho(u)=\begin{cases}\rho_0 & \text{if} \quad |u|<a \\ 0 & \text{otherwiese}\end{cases}$




exact solution

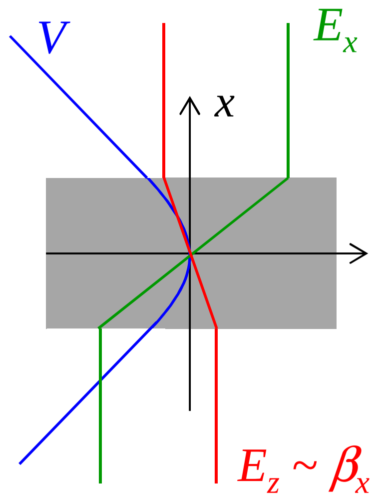
$$V = V(x - v_x t)$$

$$V(u)=\frac{\rho_0}{\left[1-\beta_x^2\right]\varepsilon}\begin{cases}-u^2 & \text{if} \quad |u|<a \\ a^2-2a|u| & \text{otherwiese}\end{cases}$$

$$c\mathbf{A}=\left[\beta_x \mathbf{e}_x+\beta_z \mathbf{e}_z\right]V(x-v_x t)$$

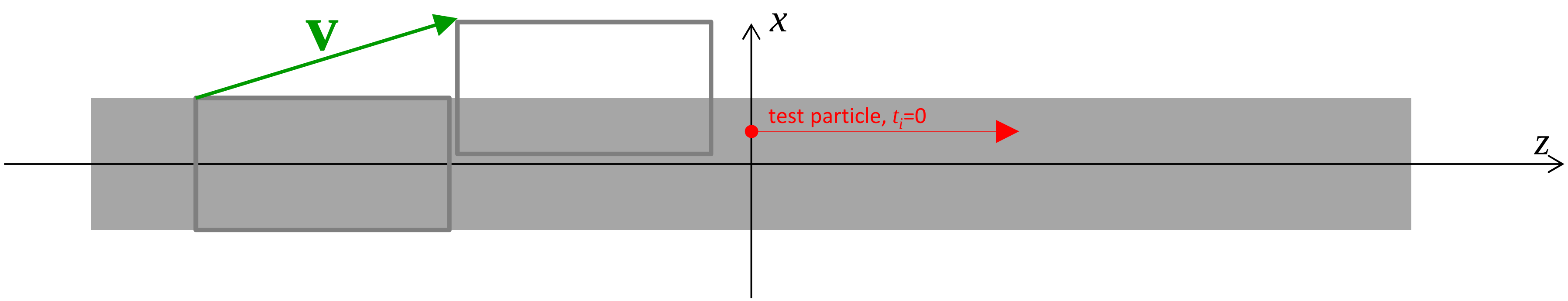
$$c\mathbf{B}=-\beta_z \mathbf{e}_y V'(x-v_x t)$$

$$\mathbf{E}=\left[\left[\beta_x^2-1\right]\mathbf{e}_x+\beta_x\beta_z \mathbf{e}_z\right]V'(x-v_x t)$$



# Simple Example

infinite charged plate in uniform motion $\rho(x,y,z,t)=\rho(x-v_x t)$ , $\rho(u)=\begin{cases}\rho_0 & \text{if} \quad |u|<a \\ 0 & \text{otherwiese}\end{cases}$




exact solution

$$V=V(x-v_x t)$$

$$V(u)=\frac{\rho_0}{\left[1-\beta_x^2\right]\varepsilon}\begin{cases}-u^2 & \text{if} \quad |u|<a \\ a^2-2a|u| & \text{otherwiese}\end{cases}$$

$$c\mathbf{A}=\left[\beta_x\mathbf{e}_x+\beta_z\mathbf{e}_z\right]V(x-v_x t)$$
$$c\mathbf{B}=-\beta_z\mathbf{e}_y V'(x-v_x t)$$
$$\mathbf{E}=\left[\left[\beta_x^2-1\right]\mathbf{e}_x+\beta_x\beta_z\mathbf{e}_z\right]V'(x-v_x t)$$

Poisson solution, for $\boldsymbol{\beta}_0=\beta_z\mathbf{e}_z\neq\boldsymbol{\beta}$

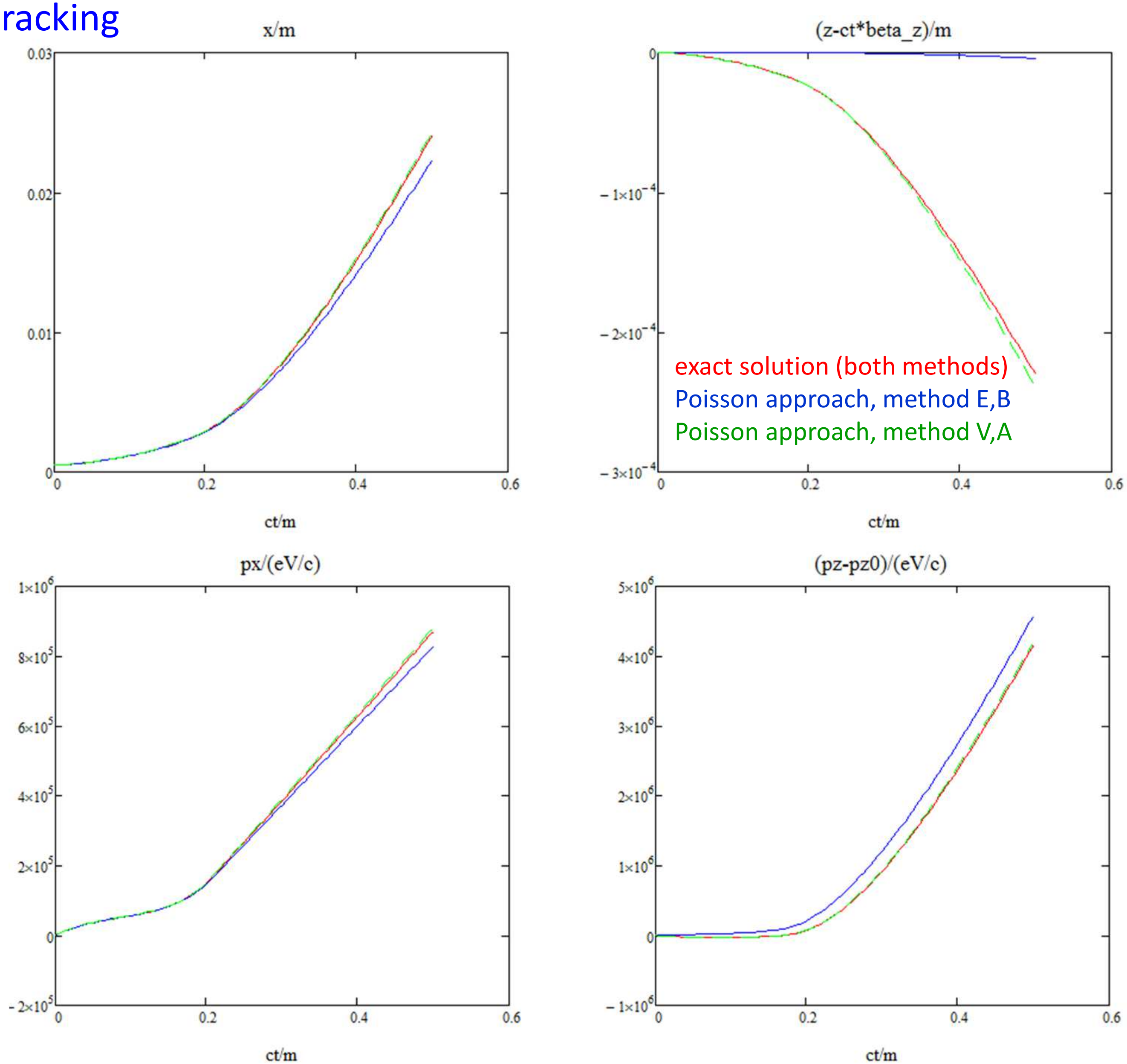
$$V_P=V_P(x-v_x t)$$

$$V_P(u)=\frac{\rho_0}{\varepsilon}\begin{cases}-u^2 & \text{if} \quad |u|<a \\ a^2-2a|u| & \text{otherwiese}\end{cases}$$

$$c\mathbf{A}_P=\beta_z\mathbf{e}_z V_P(x-v_x t)$$
$$c\mathbf{B}_P=-\beta_z\mathbf{e}_y V_P'(x-v_x t)$$
$$\mathbf{E}_P=-\mathbf{e}_x V_p'(x-v_x t)$$

# Simple Example

tracking




parameters:

$$a = 1\,\mathrm{mm}$$

$$\rho_0 = 1\,\mathrm{C/m^3}$$

$$\gamma_0 = 10$$

$$\beta_x = \beta_0 \sin(0.1/\gamma_0)$$

$$\beta_z = \beta_0 \cos(0.1/\gamma_0)$$

initial condition:

$$t_0 = 0$$

$$x = 0.5a$$

$$z = 0$$

$$v_x = 0$$

$$v_z = \beta_0 c$$

What is different?
Why is V**A**-method better?

It is not because coordinates are canonical!
It is because the field approximation is better:

still Poisson, but

# Two Approaches for Field Calculation

$$\left[\partial_x^2 + \partial_y^2 + \gamma_0^{-2}\partial_z^2\right]V = -\rho/\varepsilon$$

$$\downarrow$$

$$V$$

$$\mathbf{A} = \mathbf{e}_z c^{-1}\beta_0 V$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla V - \partial_t \mathbf{A} = -\nabla V + \beta_0 \partial_z \mathbf{A}$$

= “P1 approach"

$\partial_t \mathbf{A} = -v_z\, \partial_z \mathbf{A}$ assumes $\mathbf{A} = \mathbf{A}(z\text{-}v_z t)$

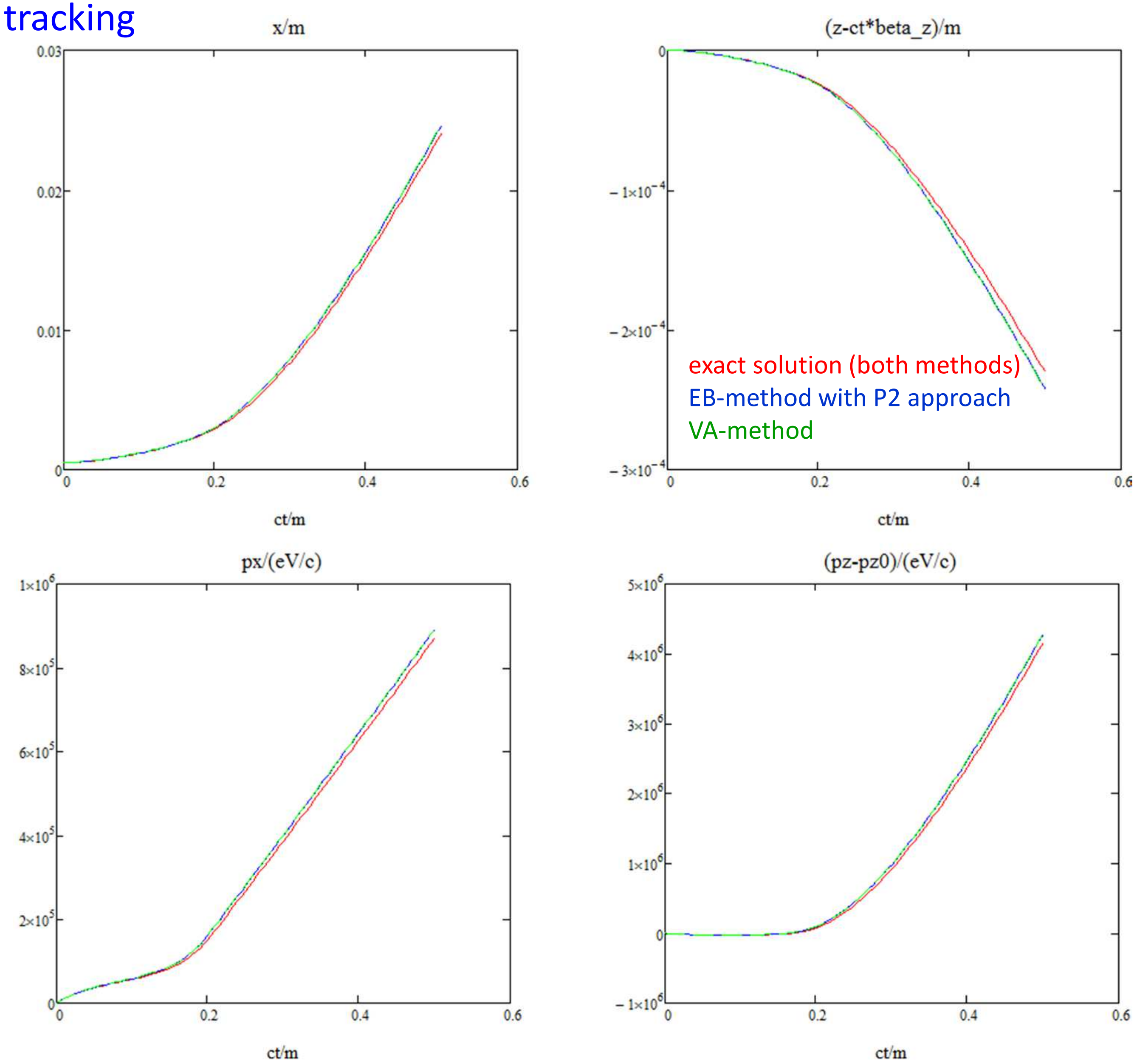
use the same approach as VA-method:

= “P2 approach"

$$\mathbf{E} = -\nabla V - \partial_t \mathbf{A} = -\left[\nabla + \mathbf{e}_z c^{-1}\beta_0 \partial_t\right]V$$

# Again: Simple Example

tracking




parameters:

$a = 1\,\text{mm}$

$\rho_0 = 1\,\text{C/m}^3$

$\gamma_0 = 10$

$\beta_x = \beta_0 \sin(0.1/\gamma_0)$

$\beta_z = \beta_0 \cos(0.1/\gamma_0)$

initial condition:

$t_0 = 0$

$x = 0.5a$

$z = 0$

$v_x = 0$

$v_z = \beta_0 c$

# Point Particle

exact (UM)

$$\mathbf{E} = \frac{q}{4\pi\varepsilon} \frac{\mathbf{r}\gamma_q}{\left[\mathbf{r}^2 + \left[\mathbf{r}\cdot\frac{\mathbf{p}_q}{m_0 c}\right]^2\right]^{3/2}}$$

$$c\mathbf{B} = \frac{\mathbf{p}_q}{m_0 c} \times \mathbf{E}$$

$$\mathbf{p}_q \neq \mathbf{p}_0 = p_0 \mathbf{e}_z$$

P1 approach ($\partial_t \mathbf{A} = -v_z\, \partial_z \mathbf{A}$ )

$$\mathbf{E}_1 = \frac{q}{4\pi\varepsilon} \frac{\mathbf{r}\gamma_0}{\left[x^2 + y^2 + \gamma_0^2 z^2\right]^{3/2}}$$

$$c\mathbf{B}_1 = \beta_0 \mathbf{e}_z \times \mathbf{E}_1$$

P2 approach ($\partial_t \mathbf{A}$ )

$$\mathbf{E}_2 = \frac{q\gamma_0}{4\pi\varepsilon} \frac{\mathbf{r} - \mathbf{e}_z \beta_0 \left[x\beta_x + y\beta_y + \gamma_0^2 z[\beta_z - \beta_0]\right]}{\left[x^2 + y^2 + \gamma_0^2 z^2\right]^{3/2}}$$

$$c\mathbf{B}_2 = \beta_0 \mathbf{e}_z \times \mathbf{E}_2 = c\mathbf{B}_1$$

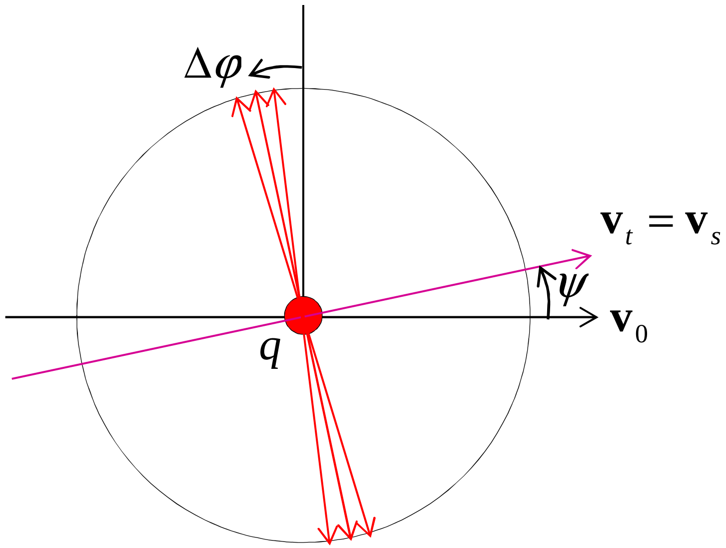
# Point Particle

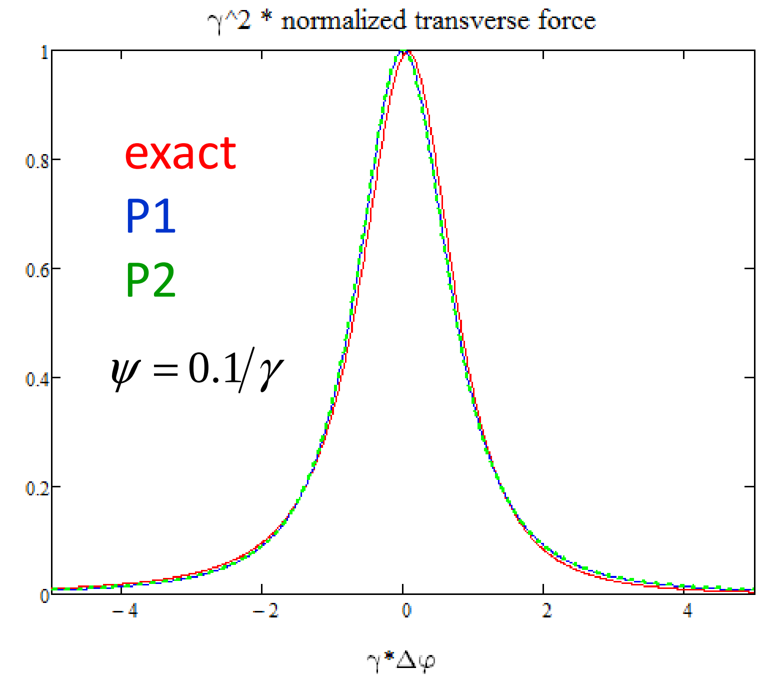
normalized Lorentz force

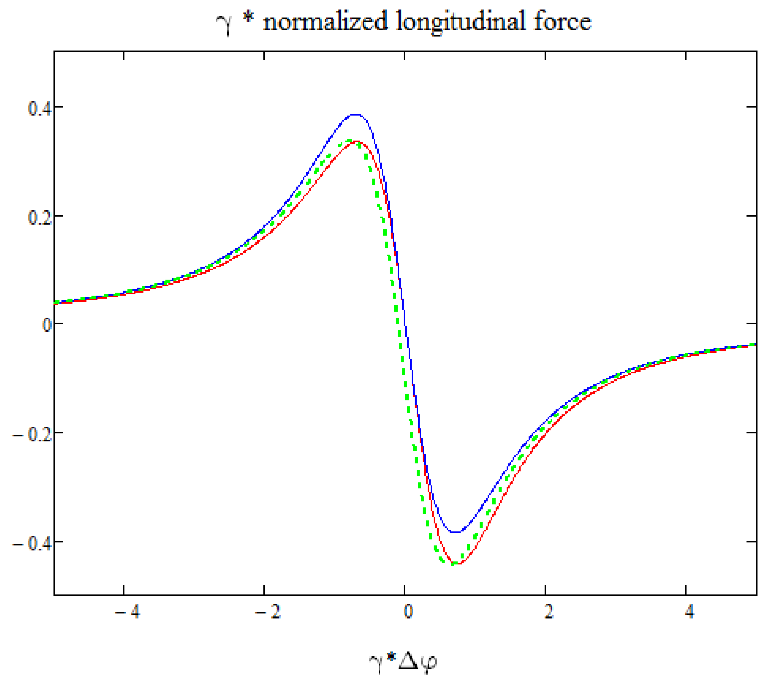
$$\mathbf{f} = \frac{4\pi\varepsilon}{r^2}\left(\mathbf{E} + \mathbf{v}_t \times \mathbf{B}\right)$$

$$\left|\mathbf{v}_t\right| = \left|\mathbf{v}_s\right| = \left|\mathbf{v}_0\right|$$

$$\mathbf{v}_t = \mathbf{v}_s$$



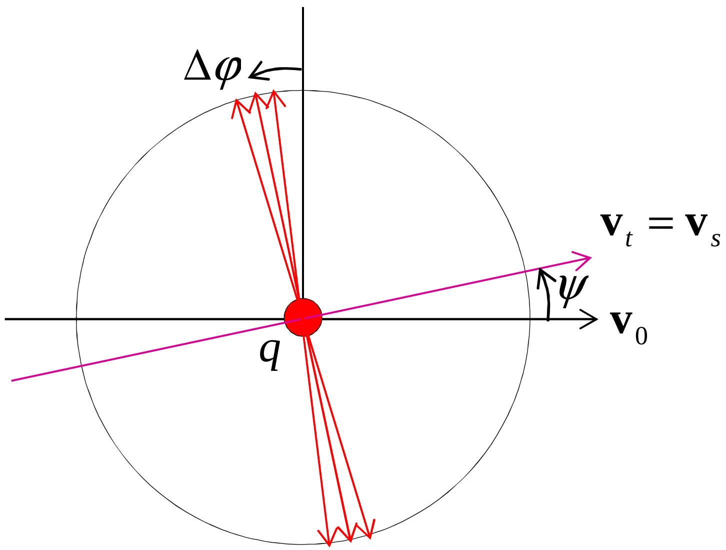
# Point Particle

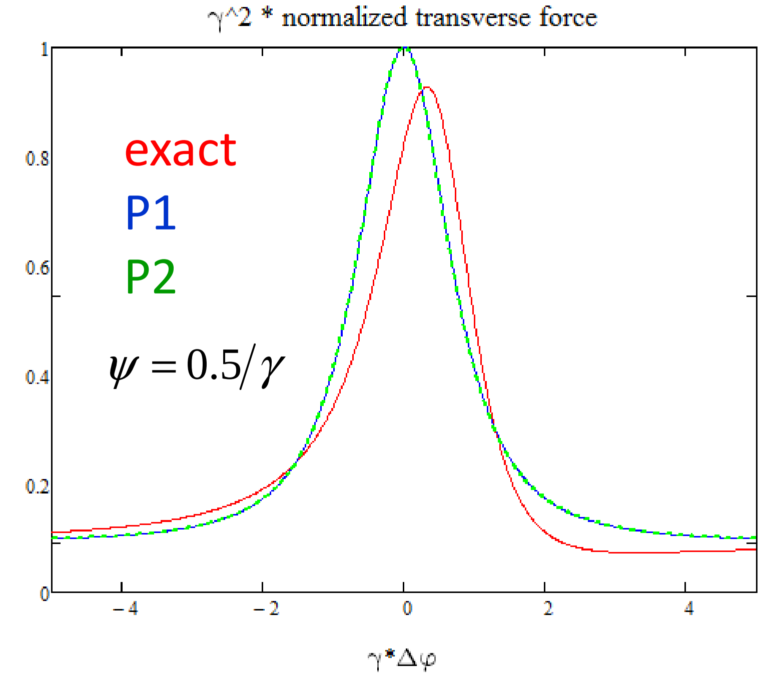
normalized Lorentz force

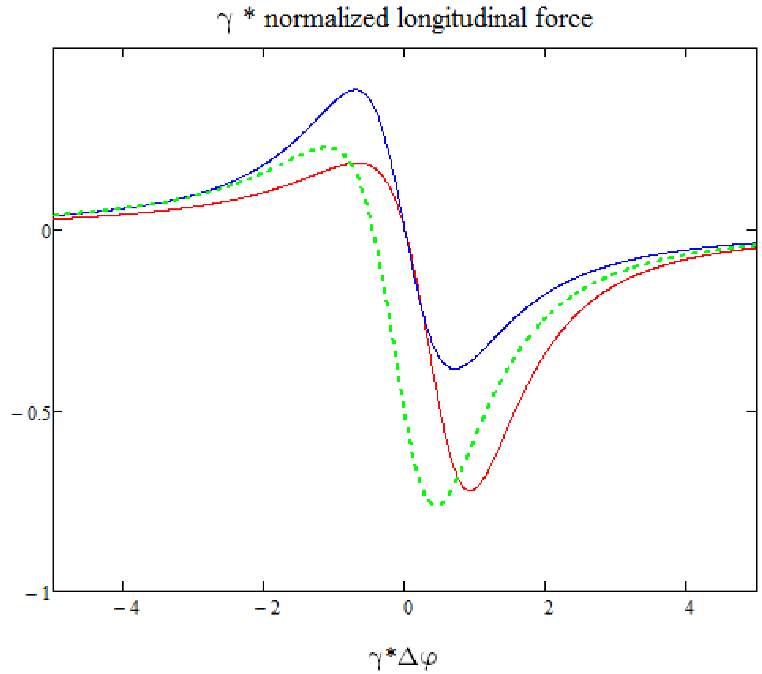
$$\mathbf{f} = \frac{4\pi\varepsilon}{r^2}\left(\mathbf{E} + \mathbf{v}_t \times \mathbf{B}\right)$$

$$\left|\mathbf{v}_t\right| = \left|\mathbf{v}_s\right| = \left|\mathbf{v}_0\right|$$

$$\mathbf{v}_t = \mathbf{v}_s$$



# Gaussian Bunch

6D phase space distribution $f(x,y,z,x',y',\delta) = f_x(x,x')f_y(y,y')f_z(z,\delta)$

with $f_x(x,x') = \frac{1}{2\pi\varepsilon_x}\exp\left\{\frac{x^2\gamma_x + 2xx'\boxed{\alpha_x} + x'^2\boxed{\beta_x}}{-2\varepsilon_x}\right\}$

$f_y(y,y') = \cdots \qquad f_z(z,\delta) = \cdots$

6D integration $\mathbf{E} = \frac{q}{4\pi\varepsilon}\int \frac{\mathbf{r}_q\gamma_q}{\left[\mathbf{r}_q^2 + \left[\mathbf{r}_q\cdot\frac{\mathbf{p}_q}{m_0c}\right]^2\right]^{3/2}} f(\mathbf{r}-\mathbf{r}_q,\mathbf{p}_q)dX_q$

linearization for $\mathbf{p}_q=\mathbf{p}_0+\Delta\mathbf{p}$ in denominator

$$\mathbf{E} \approx \frac{q}{4\pi\varepsilon}\int \mathbf{r}_q\gamma_0\left\{\frac{1}{\left[\mathbf{r}_q^2 + \left[\mathbf{r}_q\cdot\frac{\mathbf{p}_0}{m_0c}\right]^2\right]^{3/2}} - \frac{3\left[\mathbf{r}_q\cdot\frac{\mathbf{p}_0}{m_0c}\right]\left[\mathbf{r}_q\cdot\frac{\boxed{\Delta\mathbf{p}}}{m_0c}\right]}{\left[\mathbf{r}_q^2 + \left[\mathbf{r}_q\cdot\frac{\mathbf{p}_0}{m_0c}\right]^2\right]^{5/2}}\right\} f(\mathbf{r}-\mathbf{r}_q,\mathbf{p}_q)dX_q$$

for $\frac{1}{64}\frac{1}{\gamma_0}\frac{\beta_x}{\alpha_x^2} >> \varepsilon_{x,n}$, ...

analytic integration of momenta coordinates $\longrightarrow$ 3D integral

$$\mathbf{E} \approx \frac{q}{4\pi\varepsilon}\int \mathbf{r}_q\gamma_0\left\{\frac{1}{[\cdots]^{3/2}} - \frac{2\gamma_0^2 z_q\left(-\boxed{\frac{\alpha_x}{\beta_x}}x_q(x-x_q) - \frac{\alpha_y}{\beta_y}\cdots\right)}{[\cdots]^{5/2}}\right\} f_r(\mathbf{r}-\mathbf{r}_q)dV_q$$

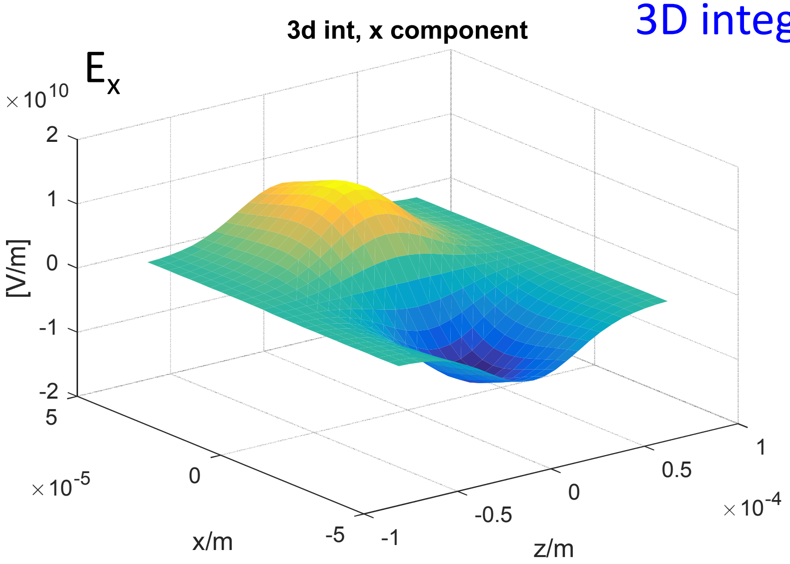
```
Q=1e-9;
% LONGITUDINAL
pz=2.4e9;
sigz=24E-6;
emitz=0;
% HORIZONTAL
emitx=1e-6/gam;
alphax= 0.2;
betax = 1.0;
% VERTICAL
emity=emitx;
alphay= 1.0;
betay = 0.322;
```

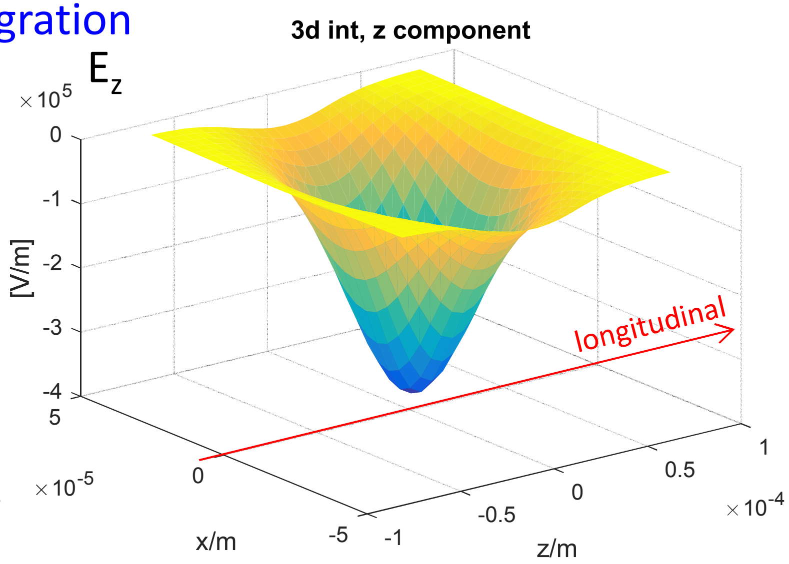
$$\frac{\alpha_y}{\beta_y}=3.1$$

$$\left|\frac{\alpha_x}{\beta_x}\right| \ll \left|\frac{\alpha_y}{\beta_y}\right|$$

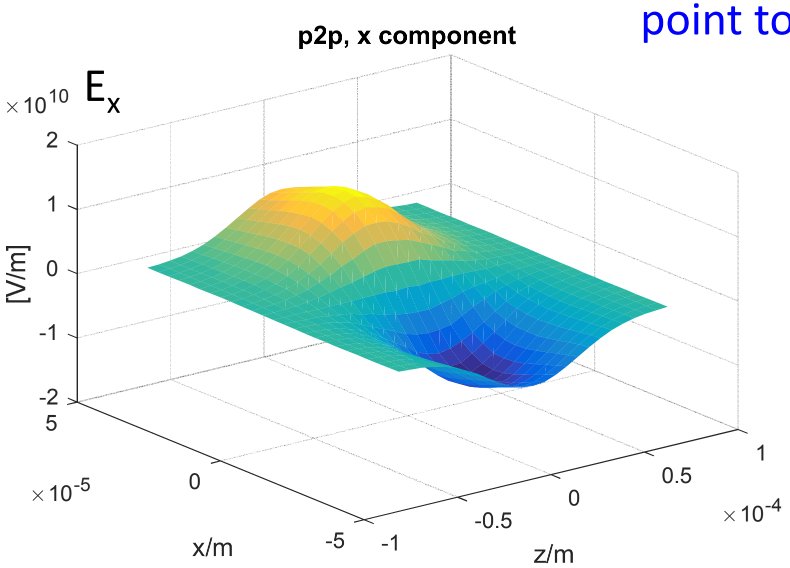
# gaussian bunch, E-field in y=0 plane

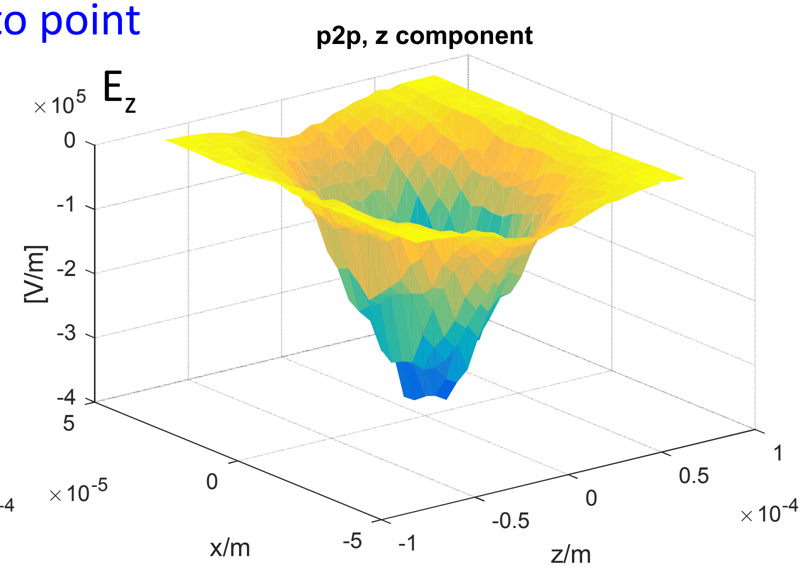
## 3D integration




## point to point



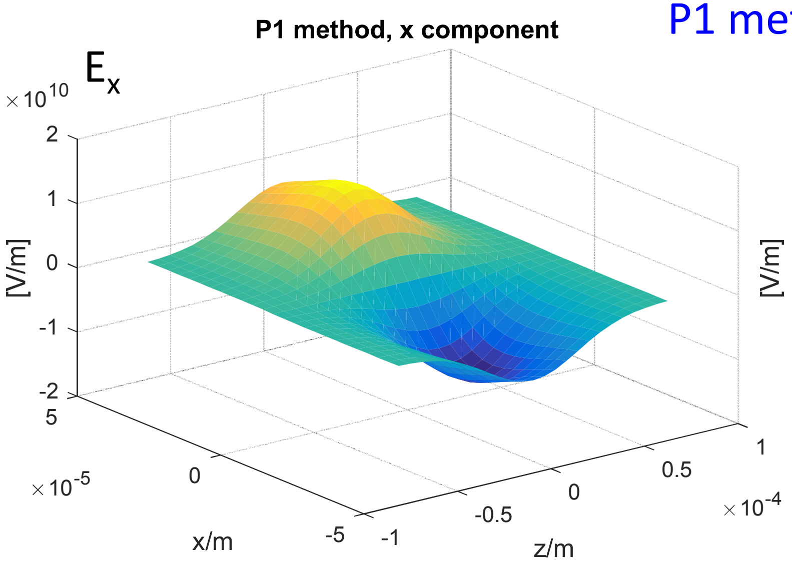

```
Q=1e-9;
% LONGITUDINAL
pz=2.4e9;
sigz=24E-6;
emitz=0;
% HORIZONTAL
emitx=1e-6/gam;
alphax= 0.2;
betax = 1.0;
% VERTICAL
emity=emitx;
alphay= 1.0;
betay = 0.322;
```

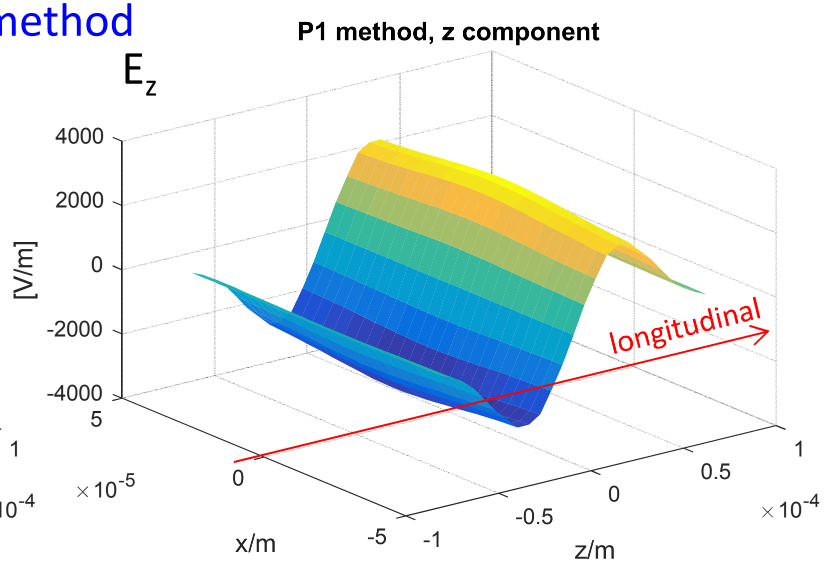
$$\frac{\alpha_y}{\beta_y}=3.1$$

$$\left|\frac{\alpha_x}{\beta_x}\right| \ll \left|\frac{\alpha_y}{\beta_y}\right|$$

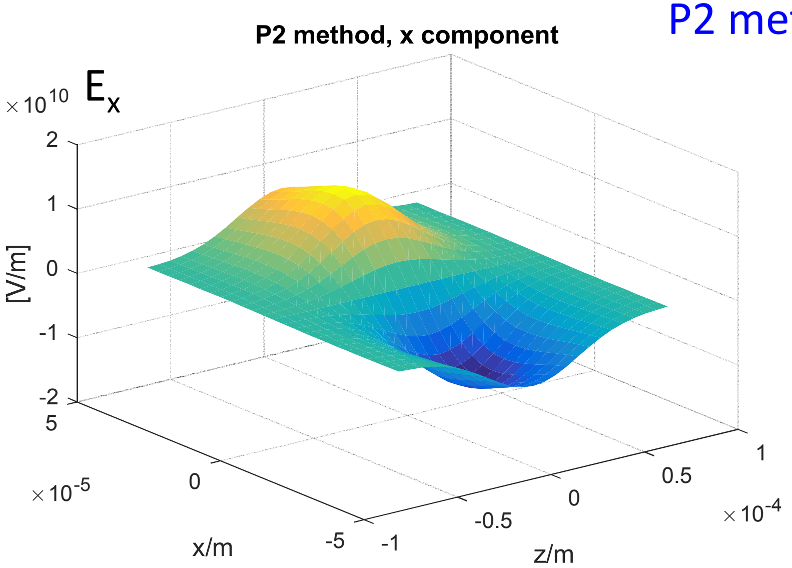
# gaussian bunch

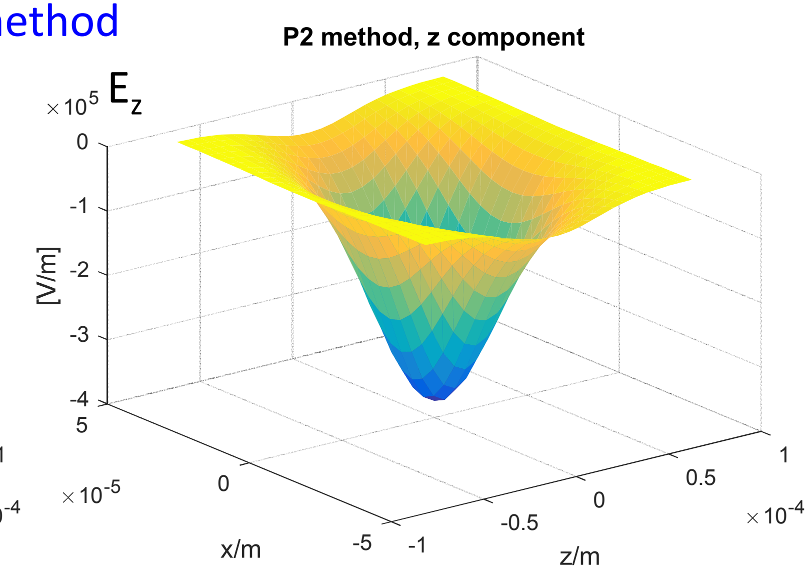
## P1 method




## P2 method



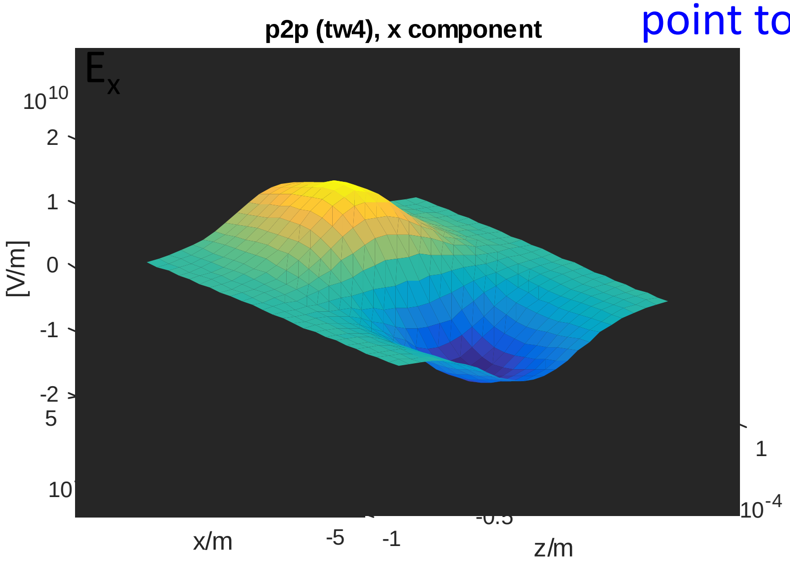



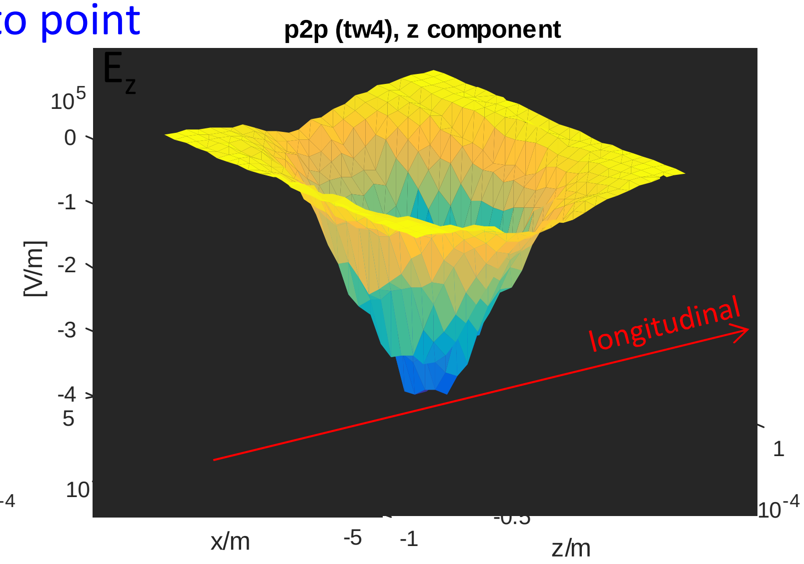

# gaussian bunches with different α/β

```
Q=1e-9;
% LONGITUDINAL
pz=2.4e9;
sigz=24E-6;
emitz=0;
% HORIZONTAL
emitx=1e-6/gam;
alphax= 0.2;
betax = 1.0;
% VERTICAL
emity=emitx;
alphay= 1.0;
betay = 0.322;
```

$$\frac{\alpha_y}{\beta_y}=+3.1$$

point to point

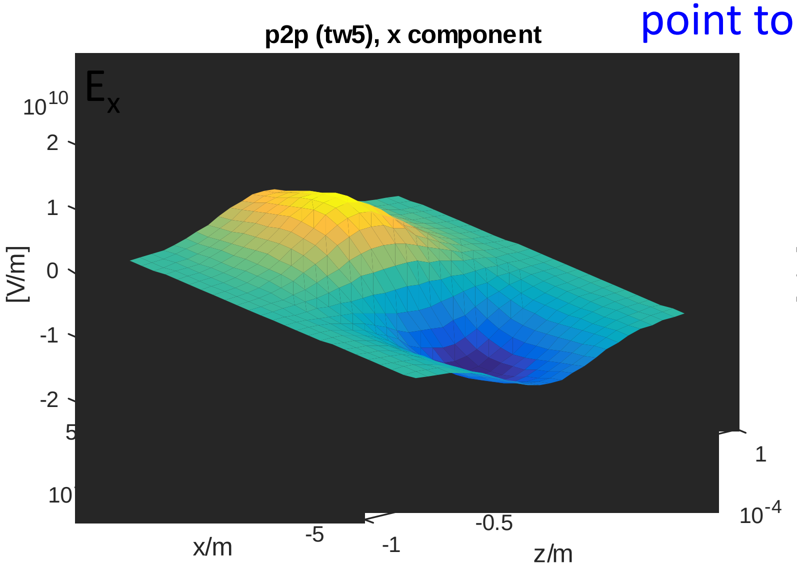


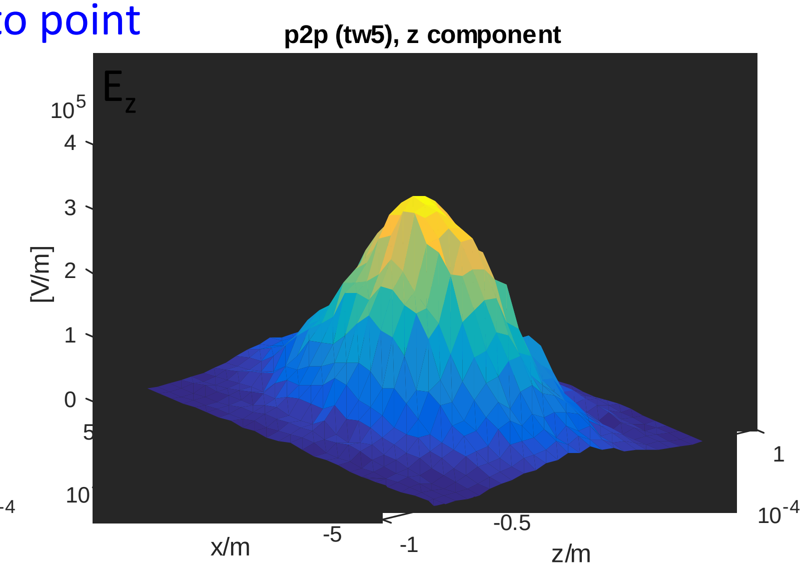




```
Q=1e-9;
% LONGITUDINAL
pz=2.4e9;
sigz=24E-6;
emitz=0;
% HORIZONTAL
emitx=1e-6/gam;
alphax= 0.058;
betax = 1.75;
% VERTICAL
emity=emitx;
alphay=-1.0;
betay = 0.194;
```

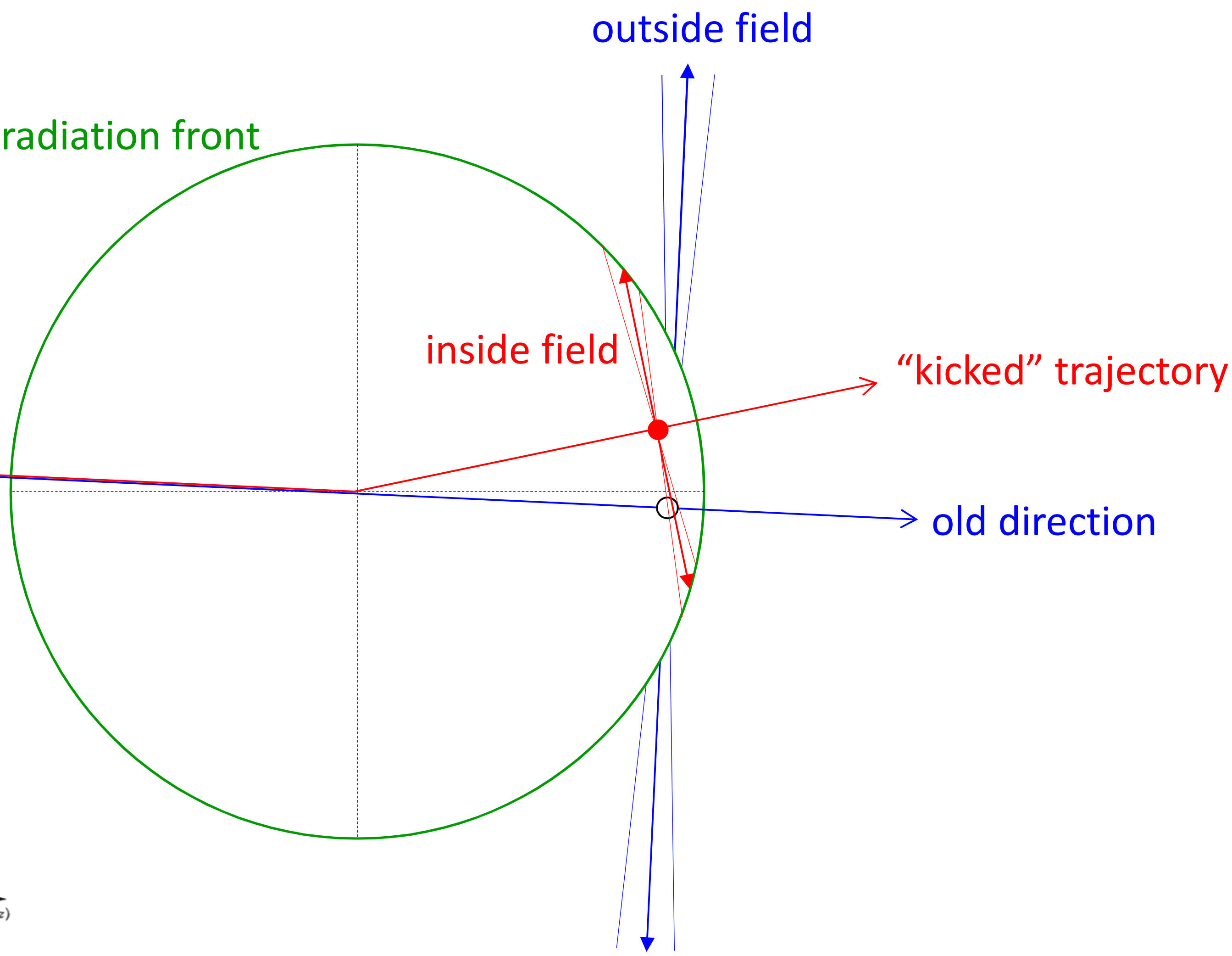
$$\frac{\alpha_y}{\beta_y}=-5.1$$

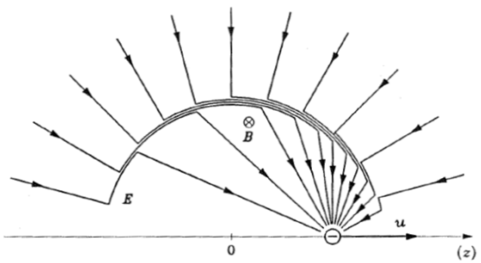
point to point



# Kicked Point Particle



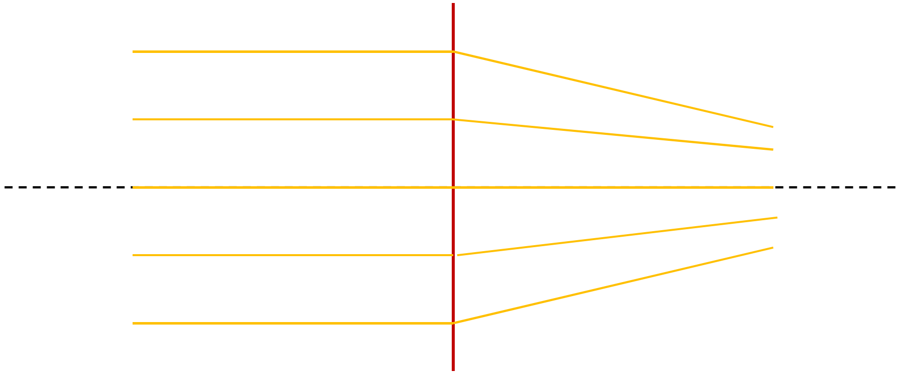




outside field and inside fields are as from charge in uniform motion
only the $\mathbf{r}$ and $\mathbf{v}$ are different

# Self-Field due to a Discrete Quadrupole

## without radiation part

kick ~ offset

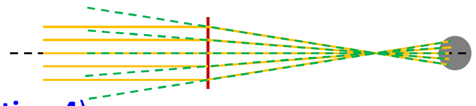


point to point as for individual uniform motion

$$\mathbf{E} = \frac{q}{4\pi\varepsilon} \sum_{\nu} \frac{\Delta\mathbf{r}_\nu \gamma_q}{\left[\Delta\mathbf{r}_\nu^2 + \left[\Delta\mathbf{r}_\nu \cdot \mathbf{p}_\nu \frac{1}{m_0 c}\right]^2\right]^{3/2}}$$

but: $\mathbf{r}_\nu$, $\mathbf{p}_\nu$ are either the actual properties (after the quadrupole kick) or the properties without quadrupole; one has to distinguish if the retarded source is observed before or after the quadrupole; this depends on the location of the observer!

# same (new) distribution, but different history

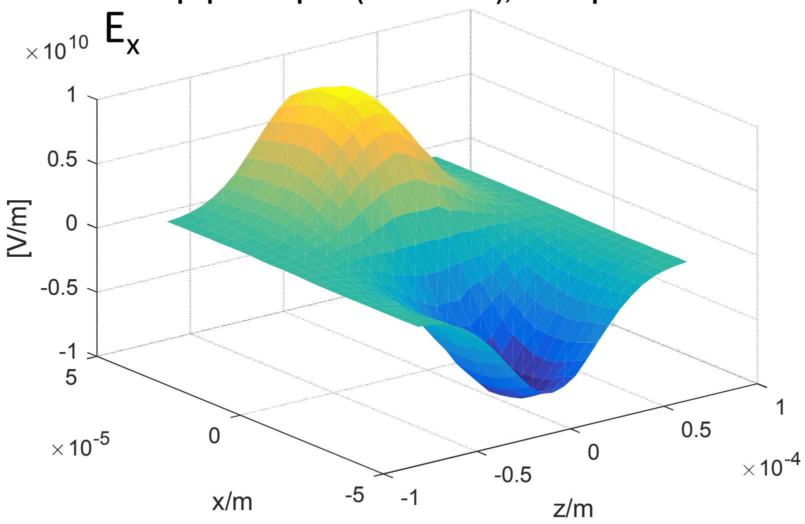


(distribution 3) drift --> discrete quad --> drift (**disribution 4**)

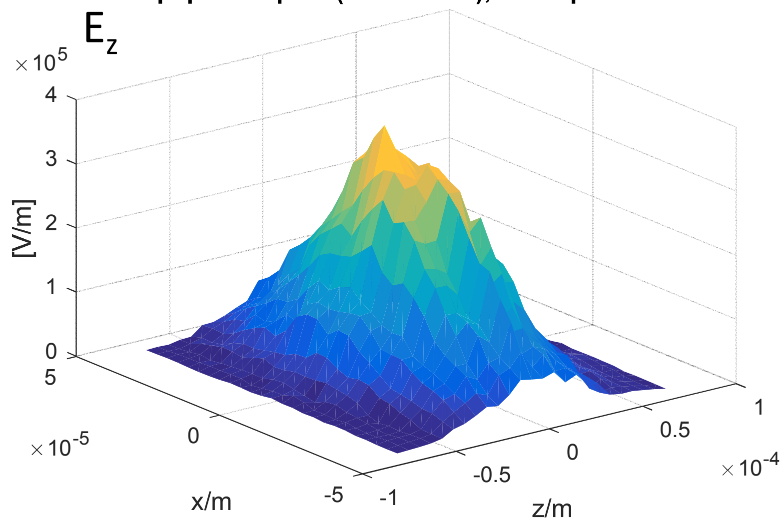
```
distribution 4
% HORIZONTAL
alphax= 0.2;
betax = 1.0;
% VERTICAL
alphay= 1.0;
betay = 0.322;
```

$\frac{\alpha_y}{\beta_y}$=+3.1

**p2p with quad (tw3 --> tw4), x component**

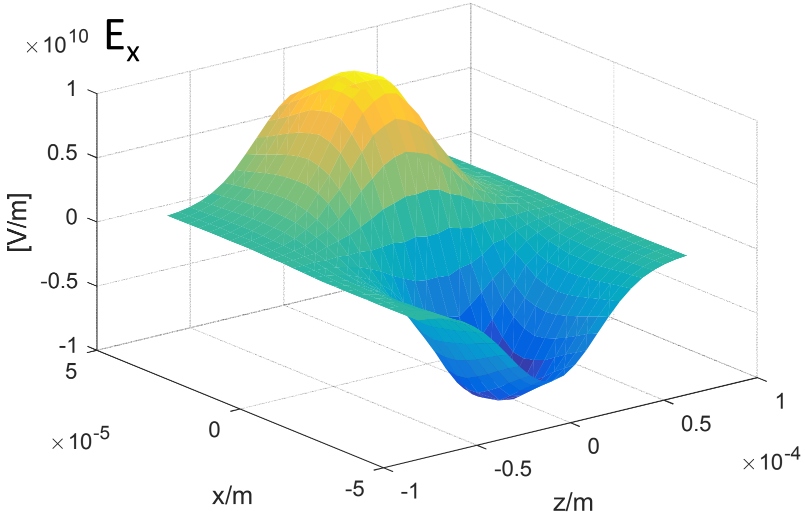



**p2p with quad (tw3 --> tw4), z component**

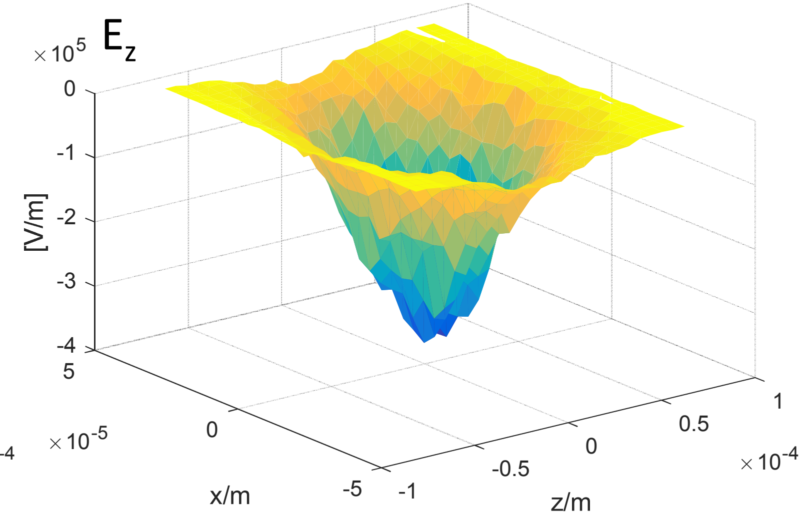



--> drift (**distribution 4**)

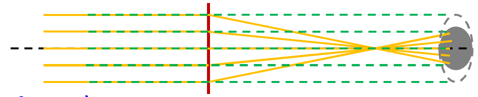
**p2p (xxx --> tw4), x component**




**p2p (xxx --> tw4), z component**




# different bunches, but same history (before q.)

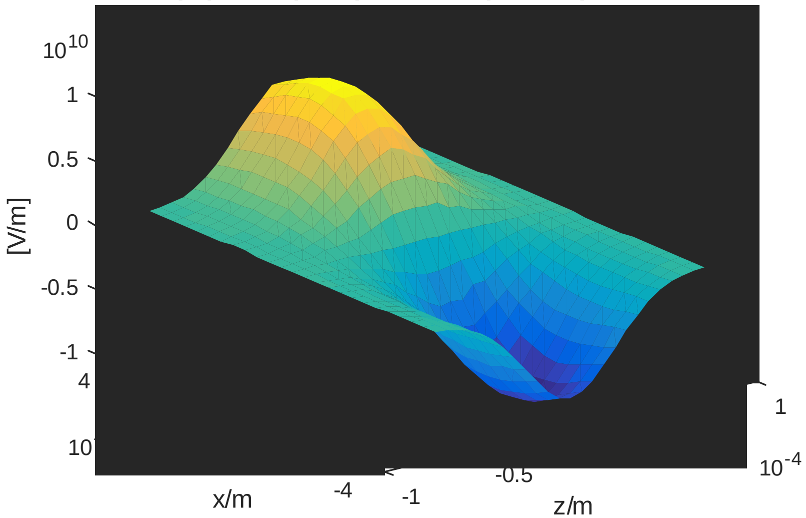


(distribution 3) drift --> discrete quad --> drift (**distribution 4**)

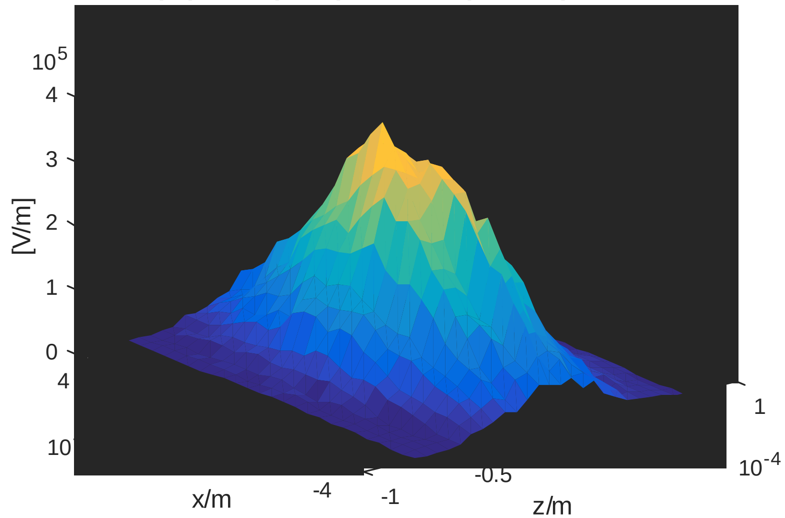
```
distribution 4
% HORIZONTAL
alphax= 0.2;
betax = 1.0;
% VERTICAL
alphay= 1.0;
betay = 0.322;
```

$$\frac{\alpha_y}{\beta_y}=+3.1$$

**p2p with quad (tw3 --> tw4), x component**




**p2p with quad (tw3 --> tw4), z component**

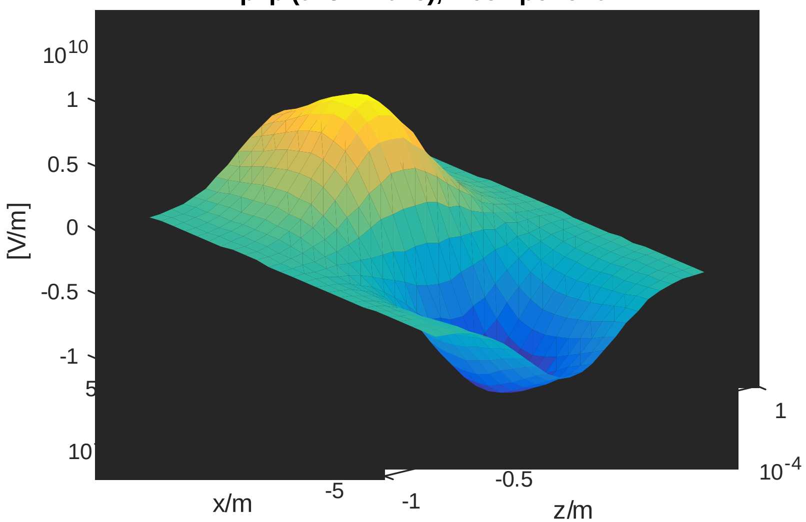



(distribution 3) --> drift (**distribution 5**)

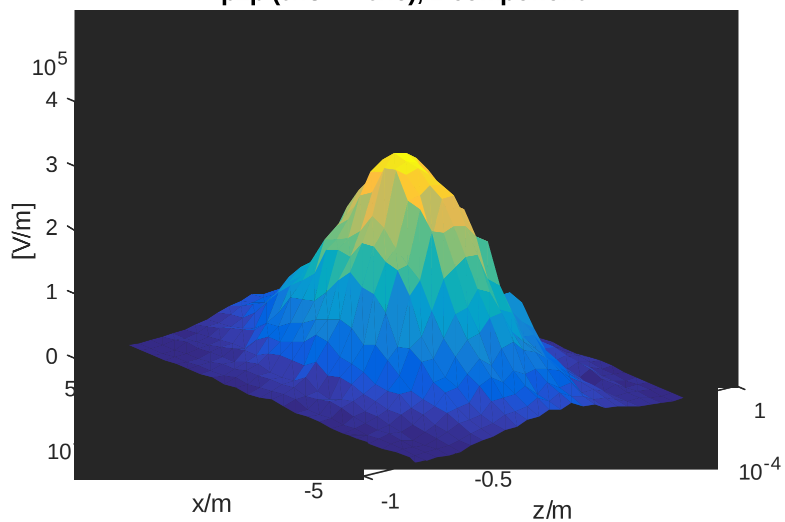
```
distribution 5
% HORIZONTAL
alphax= 0.058;
betax = 1.75;
% VERTICAL
alphay=-1.0;
betay = 0.194;
```

$$\frac{\alpha_y}{\beta_y}=-5.1$$

**p2p (tw3 --> tw5), x component**




**p2p (tw3 --> tw5), z component**




# Some Conclusions 2

Xtrack uses P1 approach

results (uncorrelated longitudinal energy spread) are not satisfying

P1 approach does not consider transient shape variations

there are better methods f.i.

- P2 approach
- IUM (individual uniform motion)
- Taylor expansion around $\mathbf{p}_0$

are not perfect (f.i. long bunch in undulator, nun-IUM shape variations)

P2 is for free with $\mathbf{rP}$-state-variables, needs $\partial V/\partial t$ with $\mathbf{rp}$-state-variables

examples of IUM-type: infinite plate with transverse motion
point particle
6D gaussian bunch

--> P2 significantly better than P1, close to IUM
P2 slightly better than P1