Coherent Synchrotron Radiation Modelling with Discontinuous Galerkin

David Bizzozero

Outline of Talk

- Introduction of Method and Numerical Algorithm
  - Maxwell's equations with several transformations
  - Discontinuous Galerkin (DG) formulation
- Validation of Numerical Method
  - Convergence studies in test model
- Simulations of Wake Fields and CSR
  - Wake field from transitions in geometry
  - CSR in a circular bend of rectangular cross-section
  - CSR in model of DESY BC0
- Conclusions and Future Outlook

Maxwell's Equations and Coodinates

TECHNISCHE UNIVERSITÄT DARMSTADT

- Maxwell's Equations
  - Starting with Cartesian coordinates: $\mathbf{R} = (Z, X, Y), \quad \tau = ct$

$$\nabla \times \mathbf{E} = -Z_0 \frac{\partial \mathbf{H}}{\partial \tau}$$

$$\nabla \times \mathbf{H} = \frac{1}{Z_0} \frac{\partial \mathbf{E}}{\partial \tau} + \mathbf{j}$$

with

$$\nabla \times \mathbf{F} = \Big(\frac{\partial F_Y}{\partial X} - \frac{\partial F_X}{\partial Y}\Big)\mathbf{e}_Z + \Big(\frac{\partial F_Z}{\partial Y} - \frac{\partial F_Y}{\partial Z}\Big)\mathbf{e}_X + \Big(\frac{\partial F_X}{\partial Z} - \frac{\partial F_Z}{\partial X}\Big)\mathbf{e}_Y$$

  - Next consider a planar reference orbit (along $Y = 0$):

$\mathbf{R}_{\text{ref}}(s) = (Z_{\text{ref}}(s), X_{\text{ref}}(s), 0)$ parameterized by arc length s

  - Define Frenet-Serret (FS) coordinate components by:

$$\mathbf{e}_s = (Z'_{\text{ref}}(s), X'_{\text{ref}}(s), 0), \quad \mathbf{e}_x = (-X'_{\text{ref}}(s), Z'_{\text{ref}}(s), 0), \quad \mathbf{e}_y = (0, 0, 1)$$

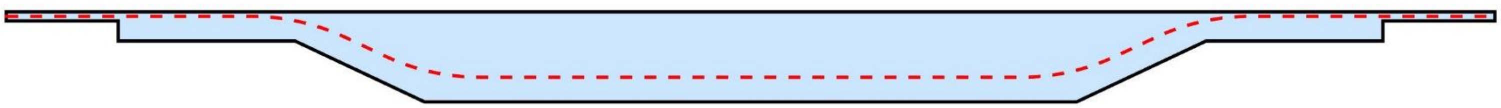
  - Also, define signed curvature κ and scale factor η by:

$$\kappa(s) = Z''_{\text{ref}}(s)X'_{\text{ref}}(s) - Z'_{\text{ref}}(s)X''_{\text{ref}}(s), \qquad \eta(s,x) = 1 + \kappa(s)x$$

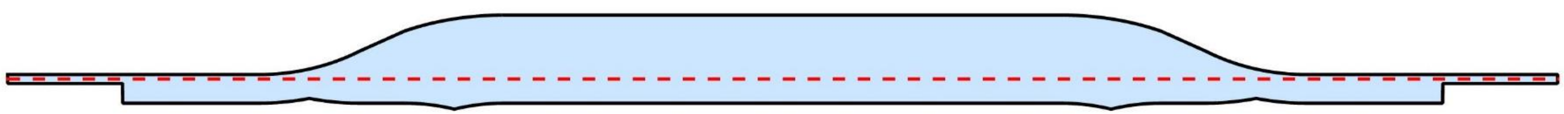
Example of Frenet-Serret Coordinates

TECHNISCHE UNIVERSITÄT DARMSTADT

- Example of mapping to FS coordinates:
  - Geometry in Cartesian coordinates: $\mathbf{R} = (Z, X, Y)$



  - Geometry in Frenet-Serret coordinates: $\mathbf{r} = (s, x, y)$



  - ✓ Advantage: source orbit is straight: allows for modelling of thin sources with DG and simpler RHS terms
  - ✘ Disadvantage: boundary may be curved, Maxwell's equations include new terms where curvature is nonzero

Maxwell's Equations in Frenet-Serret

- Maxwell's equation's in FS coordinates:

$$\frac{\partial E_s}{\partial \tau} = Z_0 \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - Z_0 j_s(\mathbf{r}, \tau)$$

$$\frac{\partial E_x}{\partial \tau} = Z_0 \left(\frac{\partial H_s}{\partial y} - \frac{1}{\eta} \frac{\partial H_y}{\partial s} \right) - Z_0 j_x(\mathbf{r}, \tau)$$

$$\frac{\partial E_y}{\partial \tau} = Z_0 \left(\frac{1}{\eta} \frac{\partial H_x}{\partial s} - \frac{1}{\eta} \frac{\partial (\eta H_s)}{\partial x} \right) - Z_0 j_y(\mathbf{r}, \tau)$$

$$\frac{\partial H_s}{\partial \tau} = \frac{-1}{Z_0} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$\frac{\partial H_x}{\partial \tau} = \frac{-1}{Z_0} \left(\frac{\partial E_s}{\partial y} - \frac{1}{\eta} \frac{\partial E_y}{\partial s} \right)$$

$$\frac{\partial H_y}{\partial \tau} = \frac{-1}{Z_0} \left(\frac{1}{\eta} \frac{\partial E_x}{\partial s} - \frac{1}{\eta} \frac{\partial (\eta E_s)}{\partial x} \right)$$

Source Term Definitions

- Charge and current density (“ribbon” model)
  - In Frenet-Serret coordinates: $\rho(\mathbf{r},\tau) = q\lambda(s-\tau)\delta(x)G(y)$

    $\mathbf{j}(\mathbf{r},\tau) = qc\lambda(s-\tau)\delta(x)G(y)\mathbf{e}_s$

    with Gaussian distributions: $\lambda(s),\ G(y)$
  - Note: $\delta(x)$ distribution avoids issue of shearing

$$q = \int_{\mathbb{R}^3} \rho_{\mathrm{C}}(Z,X,Y,\tau)\, dZdXdY = \int_{\mathbb{R}^3} \rho_{\mathrm{FS}}(s,x,y,\tau)\, \eta(s,x) ds dx dy$$

  - Rephrased: rigid source distributions (independent of $s-\tau$) in FS may not be rigid in Cartesian (this may be unphysical)

Fourier Series Decomposition

TECHNISCHE UNIVERSITÄT DARMSTADT

- Consider domain with parallel planar walls: $y = \pm h/2$
  - For PEC walls: use the Fourier series

$$f(s,x,y,\tau) = \sum_{p=1}^{\infty} b_p(s,x,\tau)\phi\left(\alpha_p(y+h/2)\right),$$

$$b_p(s,x,\tau) = \frac{2}{h}\int_{-h/2}^{h/2} f(s,x,y,\tau)\phi\left(\alpha_p(y+h/2)\right)dy,$$

$$\alpha_p = \pi p/h, \qquad \phi(\cdot) = \sin(\cdot) \text{ or } \cos(\cdot)$$

  - E_s, E_x, H_y, j_s, j_x use sine series and E_y, H_s, H_x, j_y use cosine
  - If source is symmetric about $y = 0$ then even modes vanish
  - If $\sigma_y \ll h$, more Fourier series terms may be required

Combining FS, Fourier Series, Sources

TECHNISCHE UNIVERSITÄT DARMSTADT

- The story so far:
  - ✓ Maxwell's Eqs
  - ✓ Frenet-Serret
  - ✓ Source Term
  - ✓ Fourier Series
  - ✗ Source not smooth
  - ✗ Initial Cond.?
  - ✗ Boundary Cond.?

$$\frac{1}{Z_0}\frac{\partial E_{sp}}{\partial \tau} = \frac{\partial H_{yp}}{\partial x} + \alpha_p H_{xp} - qcG_p\lambda(s-\tau)\delta(x)$$

$$\frac{1}{Z_0}\frac{\partial E_{xp}}{\partial \tau} = -\alpha_p H_{sp} - \frac{1}{\eta}\frac{\partial H_{yp}}{\partial s}$$

$$\frac{1}{Z_0}\frac{\partial E_{yp}}{\partial \tau} = \frac{1}{\eta}\frac{\partial H_{xp}}{\partial s} - \frac{\partial H_{sp}}{\partial x} - \frac{\kappa}{\eta}H_{sp}$$

$$Z_0\frac{\partial H_{sp}}{\partial \tau} = \alpha_p E_{xp} - \frac{\partial E_{yp}}{\partial x}$$

$$Z_0\frac{\partial H_{xp}}{\partial \tau} = \frac{1}{\eta}\frac{\partial E_{yp}}{\partial s} - \alpha_p E_{sp}$$

$$Z_0\frac{\partial H_{yp}}{\partial \tau} = \frac{\partial E_{sp}}{\partial x} + \frac{\kappa}{\eta}E_{sp} - \frac{1}{\eta}\frac{\partial E_{xp}}{\partial s}$$

Initial Conditions – Part 1

- Assume "steady-state" fields (satisfy $\partial/\partial s = \partial/\partial \tau$)
- Consider a rectangular straight pipe with $\kappa = 0,\ \eta = 1$
- Maxwell's wave equations with these restrictions:

$$\frac{d^2 E_{sp}}{dx^2} - \alpha_p^2 E_{sp} = 0$$

$$\frac{d^2 E_{xp}}{dx^2} - \alpha_p^2 E_{xp} = q Z_0 c G_p \lambda(s-\tau)\delta'(x)$$

$$\frac{d^2 E_{yp}}{dx^2} - \alpha_p^2 E_{yp} = q Z_0 c \alpha_p G_p \lambda(s-\tau)\delta(x)$$

$$\frac{d^2 H_{sp}}{dx^2} - \alpha_p^2 H_{sp} = 0$$

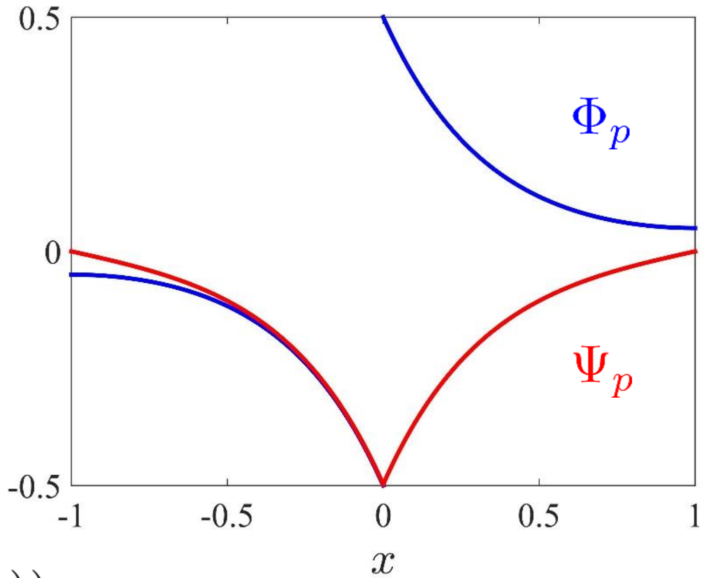
$$\frac{d^2 H_{xp}}{dx^2} - \alpha_p^2 H_{xp} = -q c \alpha_p G_p \lambda(s-\tau)\delta(x)$$

$$\frac{d^2 H_{yp}}{dx^2} - \alpha_p^2 H_{yp} = q c G_p \lambda(s-\tau)\delta'(x)$$

Initial Conditions – Part 2

- With PEC boundary conditions for $a \leq x \leq b$

$$E_{sp}(s,x,0) = 0$$
$$E_{xp}(s,x,0) = -qZ_0cG_p\lambda(s)\Phi_p(x)$$
$$E_{yp}(s,x,0) = -qZ_0cG_p\lambda(s)\Psi_p(x)$$
$$H_{sp}(s,x,0) = 0$$
$$H_{xp}(s,x,0) = qcG_p\lambda(s)\Psi_p(x)$$
$$H_{yp}(s,x,0) = -qcG_p\lambda(s)\Phi_p(x)$$



$$\Phi_p(x) = \sinh(\alpha_p b)\frac{\cosh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \cosh(\alpha_p x)\Theta(x)$$

$$\Psi_p(x) = \sinh(\alpha_p b)\frac{\sinh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \sinh(\alpha_p x)\Theta(x)$$

Smoothing the Source Term

- Issue: how to evaluate $\delta(x)$ in $\partial E_{sp}/\partial\tau$ equation?
- Fix: replace H_{yp} by $\tilde{H}_{yp} = H_{yp} - qcG_p\lambda(s-\tau)\Theta(x)$
- Result:
  - ✓ Maxwell's Eqs
  - ✓ Frenet-Serret
  - ✓ Source Term
  - ✓ Fourier Series
  - ✓ Smoother Src.
  - ✓ Initial Condition
  - \- Boundary Cond.

$$\frac{1}{Z_0}\frac{\partial E_{sp}}{\partial\tau} = \frac{\partial\widetilde{H}_{yp}}{\partial x} + \alpha_p H_{xp}$$

$$\frac{1}{Z_0}\frac{\partial E_{xp}}{\partial\tau} = -\alpha_p H_{sp} - \frac{1}{\eta}\frac{\partial\widetilde{H}_{yp}}{\partial s} - \frac{1}{\eta}qcG_p\lambda'(s-\tau)\Theta(x)$$

$$\frac{1}{Z_0}\frac{\partial E_{yp}}{\partial\tau} = \frac{1}{\eta}\frac{\partial H_{xp}}{\partial s} - \frac{\partial H_{sp}}{\partial x} - \frac{\kappa}{\eta}H_{sp}$$

$$Z_0\frac{\partial H_{sp}}{\partial\tau} = \alpha_p E_{xp} - \frac{\partial E_{yp}}{\partial x}$$

$$Z_0\frac{\partial H_{xp}}{\partial\tau} = \frac{1}{\eta}\frac{\partial E_{yp}}{\partial s} - \alpha_p E_{sp}$$

$$Z_0\frac{\partial\widetilde{H}_{yp}}{\partial\tau} = \frac{\partial E_{sp}}{\partial x} + \frac{\kappa}{\eta}E_{sp} - \frac{1}{\eta}\frac{\partial E_{xp}}{\partial s} + qZ_0cG_p\lambda'(s-\tau)\Theta(x)$$

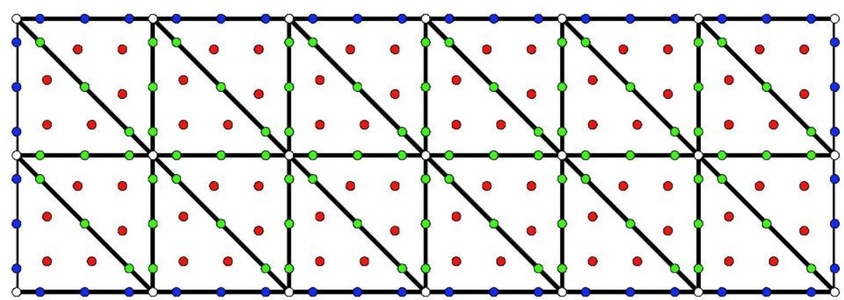
Discontinuous Galerkin – Part 1

- Nodal DG cake recipe

  - Partition domain Ω into K triangular elements
  - Approximate fields on element $D^k \subset \Omega$ by $N\mathrm{th}$ order polynomials: $\ell_j^k(s,x)$ with nodes: (s_i^k, x_i^k) where $\ell_j^k(s_i^k, x_i^k) = \delta_{ij}$ and $i,j = 1,2,...,N_p;\ N_p = \frac{(N+1)(N+2)}{2}$
  - Example field: $E_{yp}^k(s,x,\tau) = \sum_{j=1}^{N_p} E_{yp,j}^k(\tau)\ell_j^k(s,x)$
  - Example mesh:

$$N = 4,\ N_p = 15,\ K = 24$$

$$DoF = 360$$



Discontinuous Galerkin – Part 2

- Construct residuals of each field u^k :

$$\mathcal{R}(s,x,\tau) := \frac{\partial u^k}{\partial \tau} - a(s,x)\frac{\partial v^k}{\partial s} - b\frac{\partial w^k}{\partial x} - c(s,x)w^k - f(s,x,\tau)$$

- Example: $u^k = E_{yp},\ v^k = H_{xp},\ w^k = H_{sp},$

$$a(s,x) = 1/\eta(s,x),\ \ b = -1,\ \ c(s,x) = -\kappa(s)/\eta(s,x),\ \ f = 0$$

- Now set residuals to be orthogonal to each $\ell_j^k(s,x) \in \mathcal{P}^N(D^k)$

$$\int_{D^k} \mathcal{R}(s,x,\tau)\ell_j^k(s,x)dsdx = 0$$

- Integrate by parts!

$$\int_{D^k} \frac{\partial u^k}{\partial \tau}\ell_j^k + av^k\frac{\partial \ell_j^k}{\partial s} + \frac{\partial a}{\partial s}v^k\ell_j^k + bw^k\frac{\partial \ell_j^k}{\partial x} + cw^k\ell_j^k - f\ell_j^k dsdx$$

$$= \int_{\partial D^k} \mathbf{n}\cdot[av^k, bw^k]\ell_j^k dl$$

Discontinuous Galerkin – Part 3

TECHNISCHE UNIVERSITÄT DARMSTADT

- Couple elements together by introducing numerical flux
- Integrate by parts again!

$$\int_{D^k} \mathcal{R}(s,x,\tau)\ell_j^k(s,x)dsdx = -\int_{\partial D^k} \mathbf{n}\cdot[av^k-(av)^*, bw^k-(bw)^*]\ell_j^k(s,x)dl$$

- Store each field as $N_p \times K$ array
- Construct discrete matrix operators:

$$\mathcal{D}_s = \mathcal{M}^{-1}\mathcal{S}_s, \quad \mathcal{D}_x = \mathcal{M}^{-1}\mathcal{S}_x, \quad [\mathcal{M}^k]_{ij} = \int_{D^k} \ell_i^k(s,x)\ell_j^k(s,x)dsdx$$

$$[\mathcal{S}_s^k]_{ij} = \int_{D^k} \ell_i^k(s,x)\frac{\partial \ell_j^k(s,x)}{\partial s}dsdx, \quad [\mathcal{S}_x^k]_{ij} = \int_{D^k} \ell_i^k(s,x)\frac{\partial \ell_j^k(s,x)}{\partial x}dsdx$$

Discontinuous Galerkin – Part 4

- Combine everything together for each element $D^k \subset \Omega$

$$\frac{dE_{sp}}{d\tau} = Z_0 \mathcal{D}_x \widetilde{H}_{yp} + Z_0 \alpha_p H_{xp} + \frac{1}{2}(J\mathcal{M})^{-1}\Big(- Z_0 \mathbf{n}_x [\![\widetilde{H}_{yp}]\!] - [\![E_{sp}]\!] + \mathbf{n}_s(\mathbf{n}_s [\![E_{sp}]\!] + \mathbf{n}_x [\![E_{xp}]\!])\Big)$$

$$\frac{dE_{xp}}{d\tau} = - Z_0 \alpha_p H_{sp} - \frac{Z_0}{1+\kappa x}\mathcal{D}_s \widetilde{H}_{yp} - \frac{Z_0}{1+\kappa x} q c G_p \lambda'(s-\tau)\Theta(x) + \frac{1}{2}(J\mathcal{M})^{-1}\Big(\frac{Z_0}{1+\kappa x}\mathbf{n}_s [\![\widetilde{H}_{yp}]\!] - [\![E_{xp}]\!] + \mathbf{n}_x(\mathbf{n}_s [\![E_{sp}]\!] + \mathbf{n}_x [\![E_{xp}]\!])\Big)$$

$$\frac{dE_{yp}}{d\tau} = \frac{Z_0}{1+\kappa x}\mathcal{D}_s H_{xp} - Z_0 \mathcal{D}_x H_{sp} - \frac{Z_0 \kappa}{1+\kappa x} H_{sp} + \frac{1}{2}(J\mathcal{M})^{-1}\Big(- \frac{Z_0}{1+\kappa x}\mathbf{n}_s [\![H_{xp}]\!] + Z_0 \mathbf{n}_x [\![H_{sp}]\!] - [\![E_{yp}]\!]\Big)$$

With similar expressions for $dH_{sp}/d\tau,\ dH_{xp}/d\tau,\ d\tilde{H}_{yp}/d\tau$

Note: $[\![u]\!] = u^- - u^+$ defines jumps along element interfaces

Final Steps for the Numerical Method

- Boundary conditions?
  - Impose PEC with $\mathbf{n} \times [\![\mathbf{E}]\!] = 2\mathbf{n} \times \mathbf{E}^{-}$ where $\mathbf{E} = (E_{sp}, E_{xp}, E_{yp})$
  - Close end pipes with PEC (future work to add ABC)
- Evolve fields with 4th order low-storage RK
- Additional Notes:
  - Important: align elements along $x = 0$ and where κ is discontinuous (i.e. when using piecewise-defined orbits)
  - Elements along curved boundaries conformal
  - Sum over p modes for full 3D solution
  - Field values can be interpolated at any location in any element and can be averaged along element edges

Validation of Numerical Method

TECHNISCHE UNIVERSITÄT DARMSTADT

- Consider fundamental TM-mode in $\Omega = [-1, 1] \times [-1, 1]$
  - Set $\kappa = 0,\ \eta = 1$ with no source term for $p = 0$
  - Initial condition: $E_{yp}(s, x, 0) = E_0 \cos(\frac{\pi s}{2}) \cos(\frac{\pi x}{2})$
  - Solution:

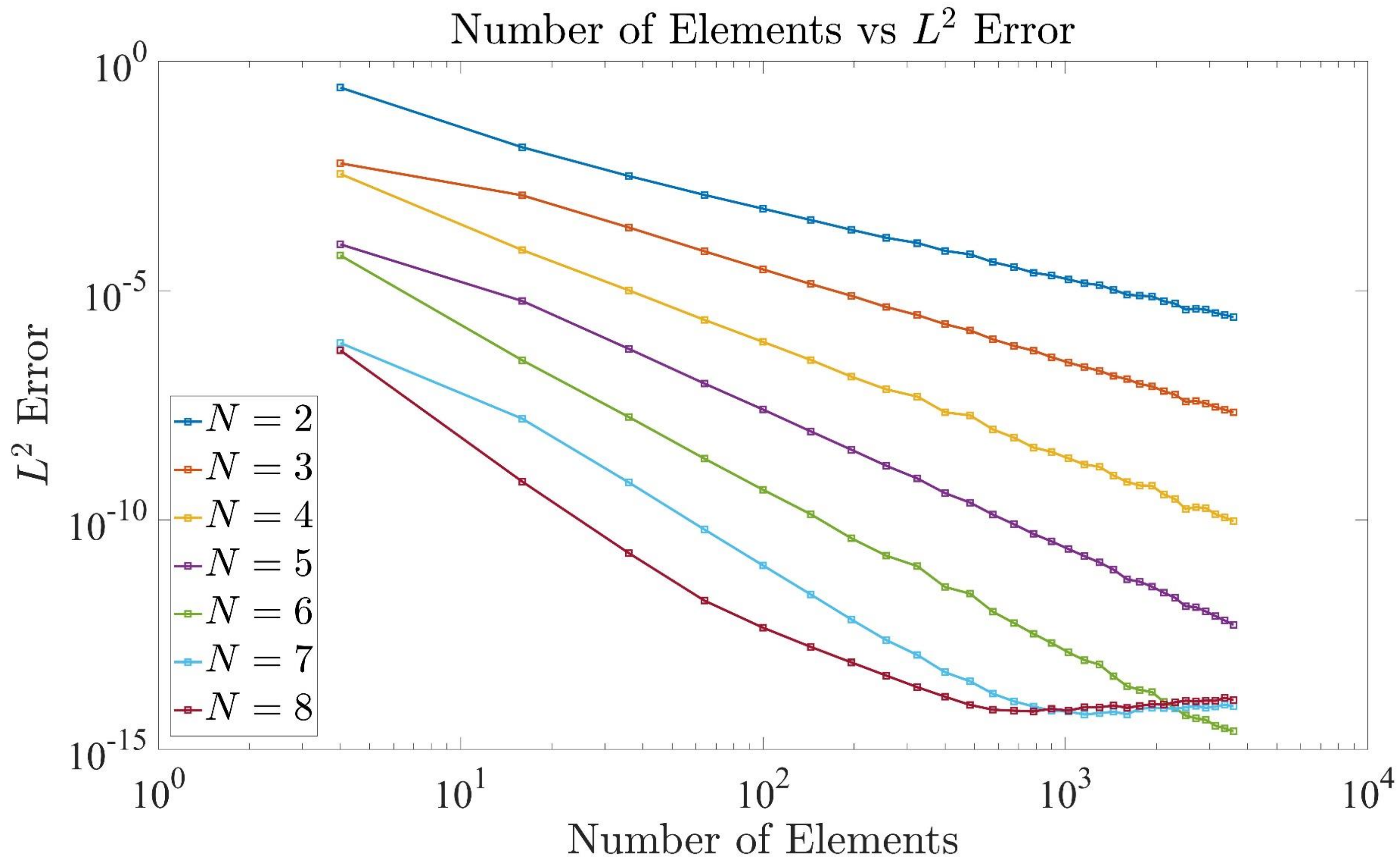
$$E_{yp}(s, x, \tau) = E_0 \cos(\frac{\pi s}{2}) \cos(\frac{\pi x}{2}) \cos(\omega \tau)$$

$$H_{sp}(s, x, \tau) = \frac{E_0 Z_0}{\sqrt{2}} \cos(\frac{\pi s}{2}) \sin(\frac{\pi x}{2}) \sin(\omega \tau)$$

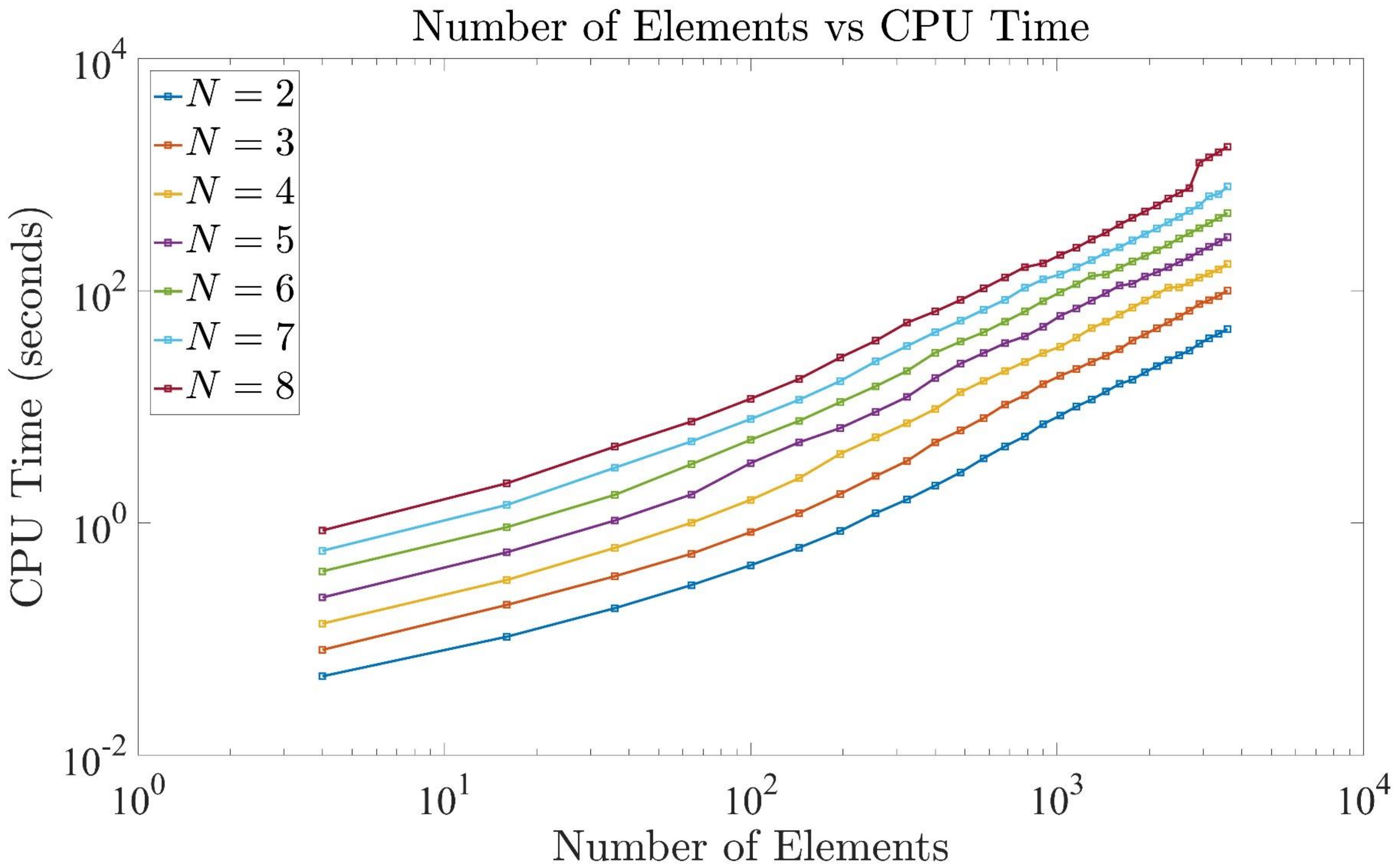
$$H_{xp}(s, x, \tau) = \frac{E_0 Z_0}{\sqrt{2}} \sin(\frac{\pi s}{2}) \cos(\frac{\pi x}{2}) \sin(\omega \tau)$$

  - Examine CPU times and errors for $K = 4 \sim 3600$ and $N = 2 \sim 8$ for one period $T = 2\sqrt{2}$

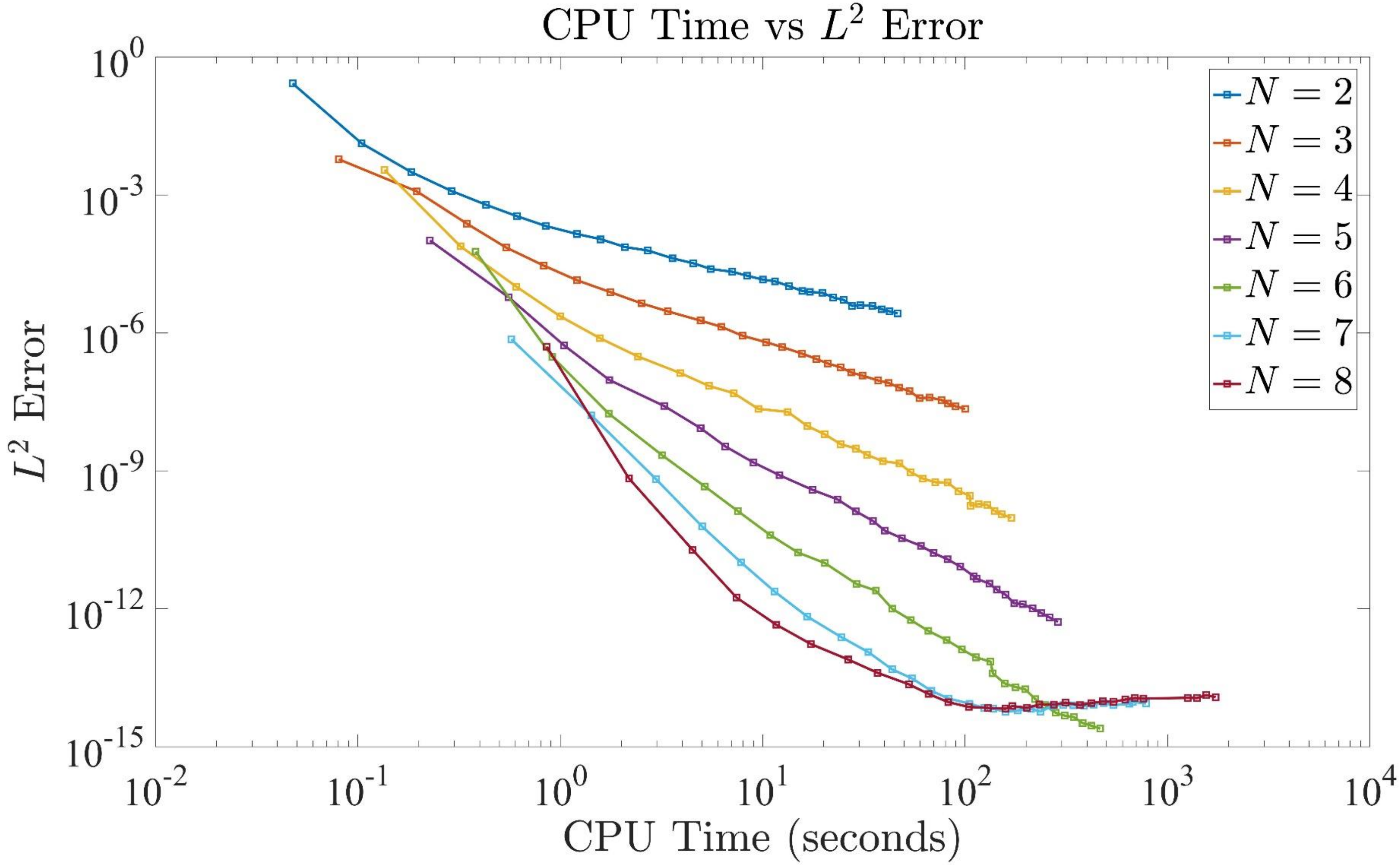
Convergence and CPU Time Scaling



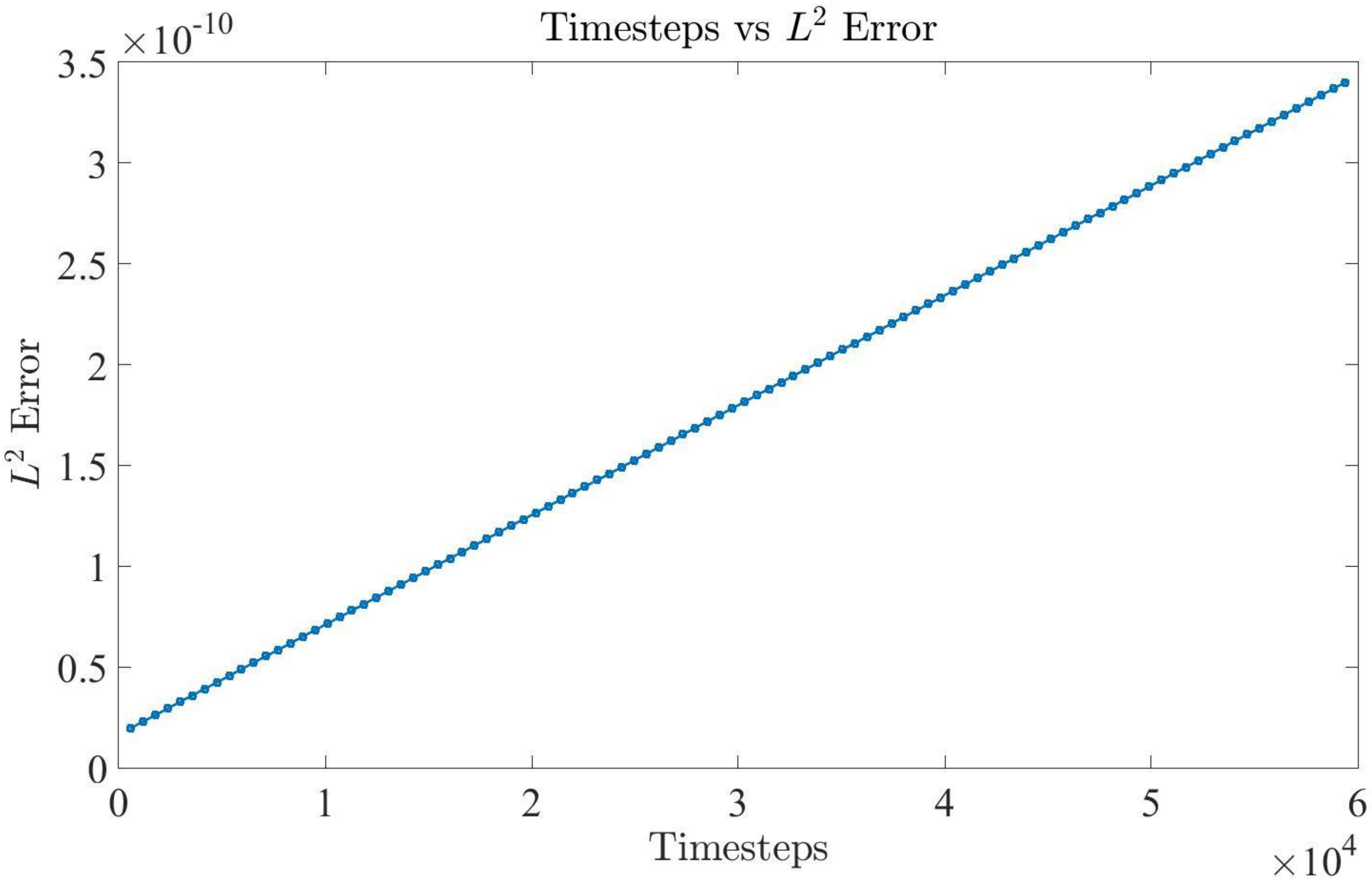
Convergence and CPU Time Scaling



Convergence and CPU Time Scaling



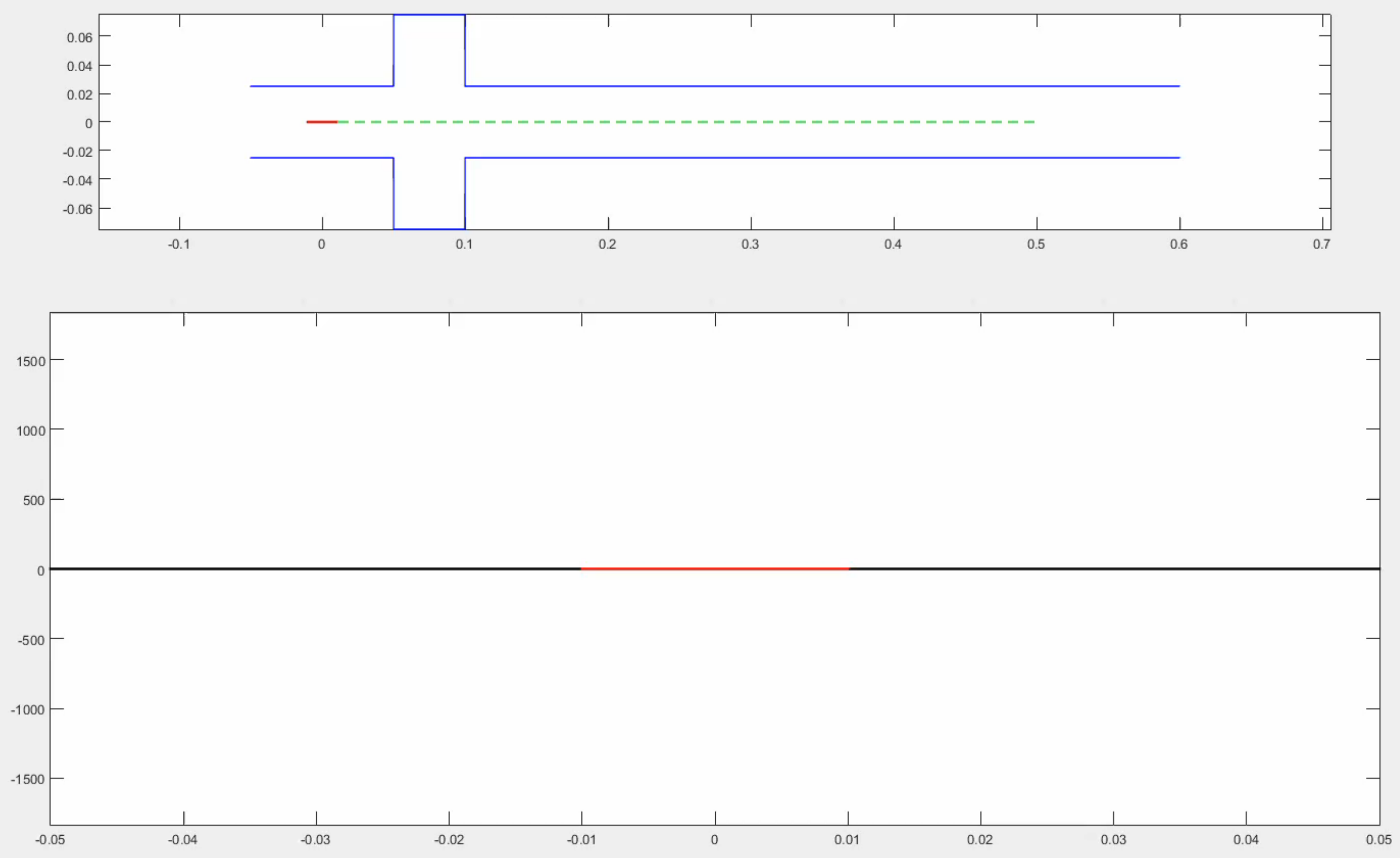
Stability Analysis for $(N, K) = (6, 100)$



Wake Field and CSR Simulations

- Experiment 1
  - No curvature – straight trajectory
  - Geometry generates wake
  - E_{sp} sampled along $x = 0$ near source for $p = 1$
  - Source size: $\sigma_s = \sigma_y = 5\,\mathrm{mm}$
  - Additional parameters: $(N, K) = (6, 927)$
    - Total length: $L = 650\,\mathrm{mm}$
    - End pipe width: $w = 50\,\mathrm{mm}$
    - Protrusion width: $d = 150\,\mathrm{mm}$
    - Vertical height: $h = 50\,\mathrm{mm}$

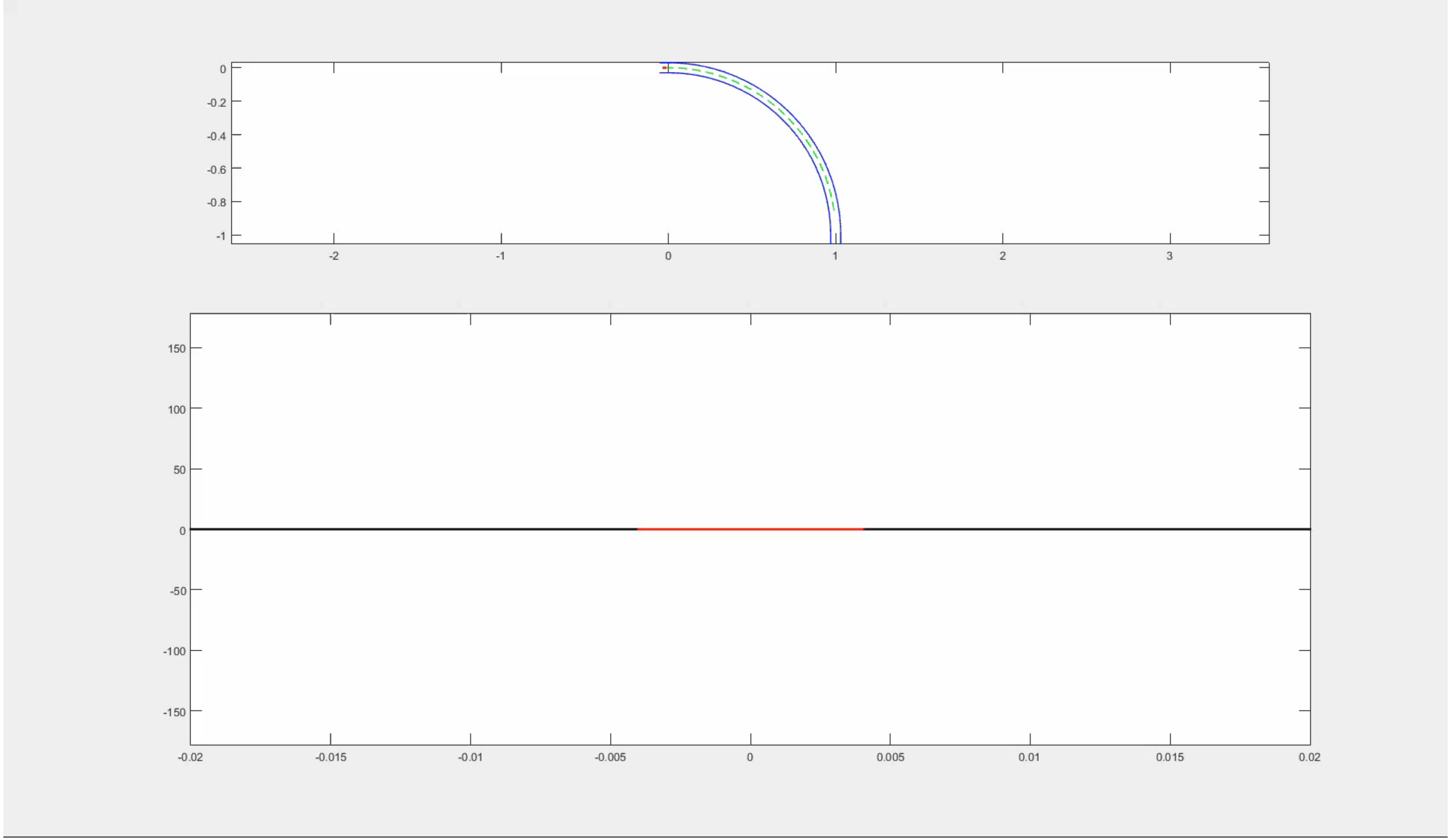
Experiment 1 Simulation



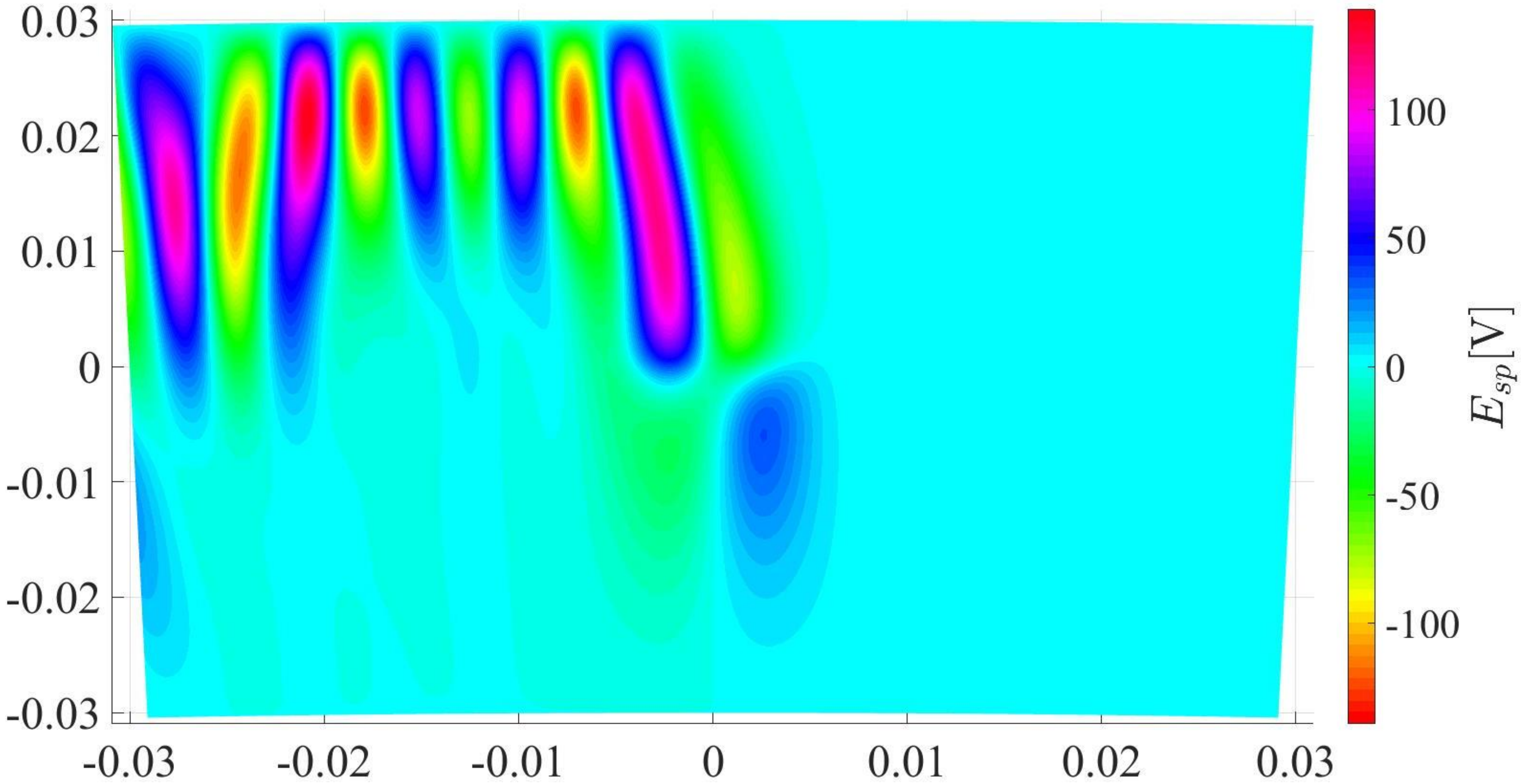
Wake Field and CSR Simulations

- Experiment 2
  - Constant curvature – circular trajectory
  - CSR generates wake
  - E_{sp} sampled along $x = 0$ near source for $p = 1$
  - Source size: $\sigma_s = 2\,\mathrm{mm},\ \sigma_y = 1\,\mathrm{mm}$
  - Additional parameters: $(N, K) = (8, 15026)$
    - Bending radius: $R = 1000\,\mathrm{mm}$
    - Horizontal width: $w = 60\,\mathrm{mm}$
    - Vertical height: $h = 20\,\mathrm{mm}$
    - Total angle: $\theta_{\max} = 90°$

Experiment 2 Simulation



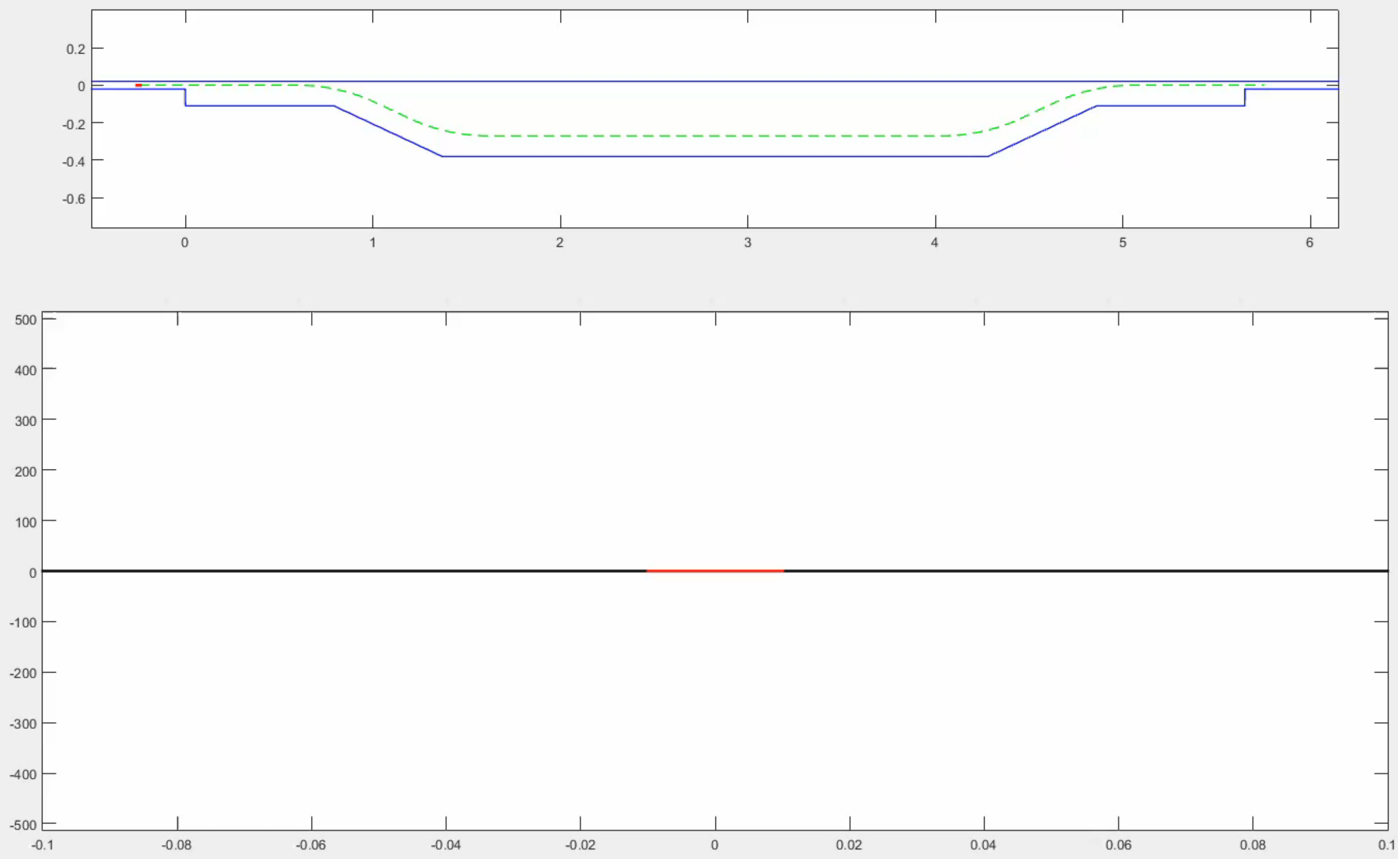
Longitudinal Electric Field View



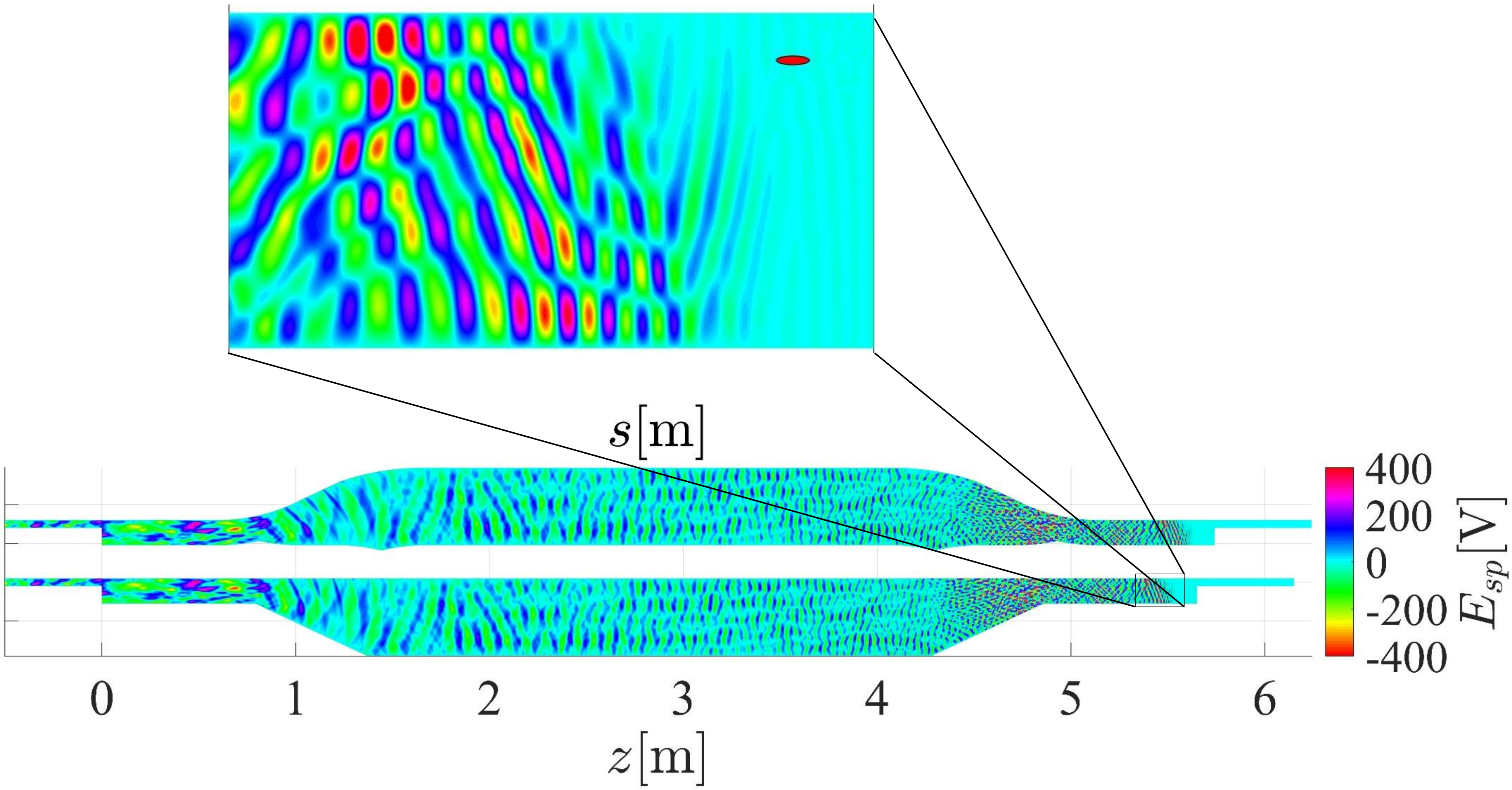
Wake Field and CSR Simulations

- Experiment 3
  - DESY BC0 – Piecewise constant curvature
  - CSR and geometry generates wake
  - E_{sp} sampled along $x = 0$ near source, sum over $p = 1,\ 3,\ 5$
  - Source size: $\sigma_s = 5\,\mathrm{mm},\ \sigma_y = 1\,\mathrm{mm}$
  - Additional parameters: $(N, K) = (8, 28718)$
    - Bending radius: $R = 1000\,\mathrm{mm}$
    - Deflection angle: $\theta_\mathrm{d} = 24.04°$
    - BC chamber height: $h = 20.3\,\mathrm{mm}$
    - BC chamber length: $L = 5650\,\mathrm{mm}$
    - BC chamber max width: $w = 400\,\mathrm{mm}$

Experiment 3 Simulation



Longitudinal Electric Field View



Conclusions and Comments

- Developed DG Method based on “Nodal Discontinuous Galerkin Methods” by *J. Hesthaven, T. Warburton*
- Demonstrated stability and convergence
- Computed geometrical and CSR wakes
- GPU-enabled MATLAB code
- Performed calculations for Canadian Light Source
- Other available codes:
  - Moving-window code for simple geometries
  - Paraxial frequency-domain code with DG

Future Directions

- Implement surface impedance boundary conditions (SIBCs) to examine wall losses
- Export code to C++ or Fortran
  - Increase performance
  - Interface with existing PIC codes
  - Improve ease-of-use
- Compute transverse momentum kicks with Panofsky-Wenzel Theorem
- Test with other geometries and experiments

TECHNISCHE UNIVERSITÄT DARMSTADT

Thank you for your attention!