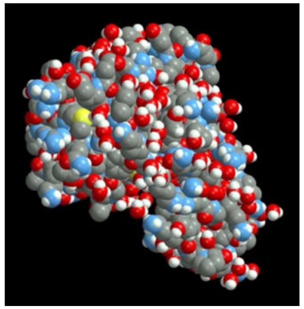
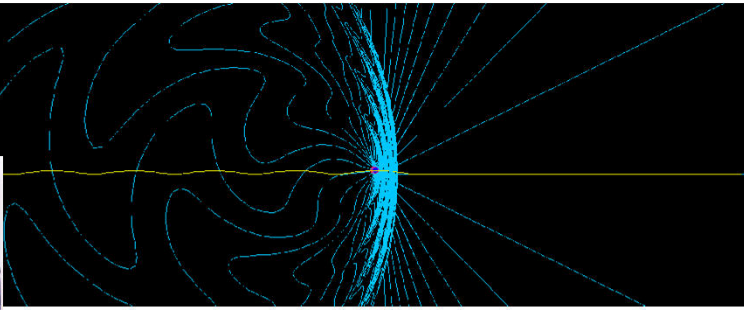
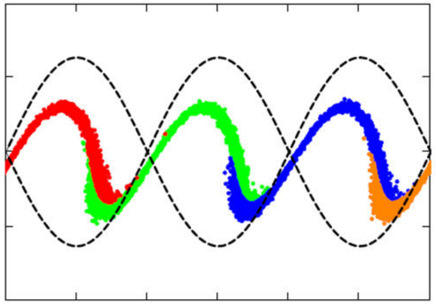
# Free-Electron Laser

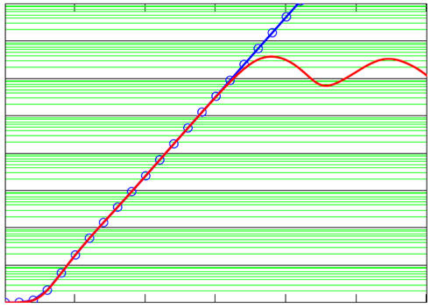
## A) Motivation and Introduction





## B) Theoretical Approach





## C) Experimental Realization / Challenges

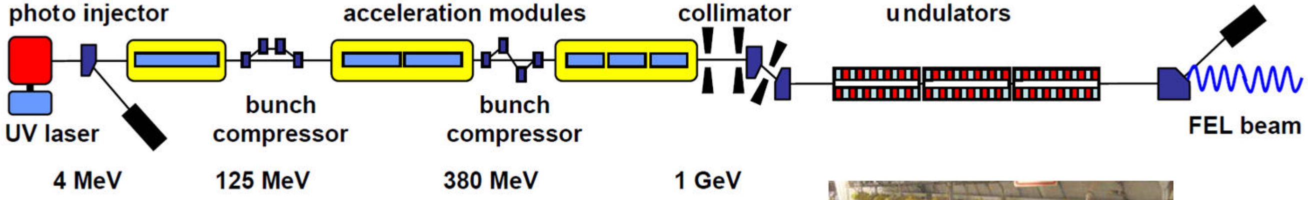


photo injector

acceleration modules

collimator

undulators

UV laser

bunch compressor

bunch compressor

FEL beam

4 MeV

125 MeV

380 MeV

1 GeV

v1

# A) Motivation and Introduction

need for short wavelengths

why FELs?

free electron ↔ wave interaction

micro-bunching

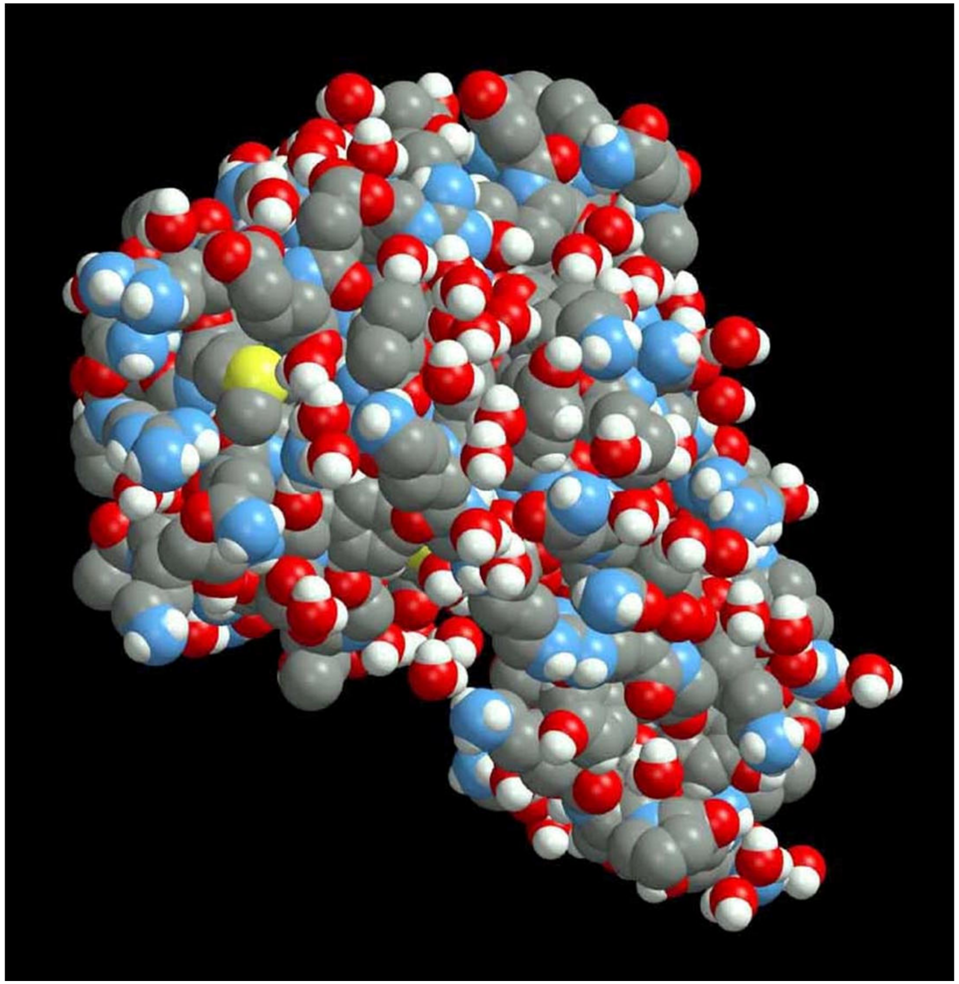
amplifier and oscillator

self amplifying spontaneous emission (SASE)

why SASE?

coherent radiation

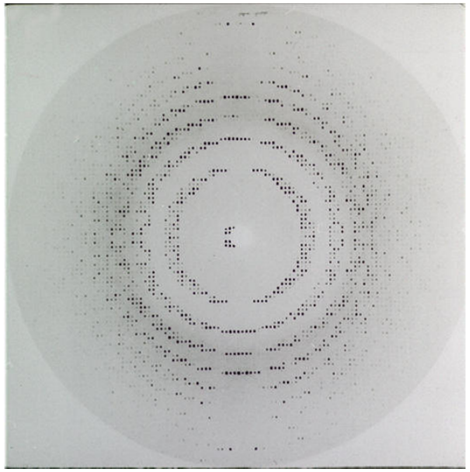
# need for short wavelengths



LYSOZYME MW=19,806

**state of the art:**
structure of biological macromolecule

reconstructed from diffraction pattern of protein crystal:



needs ≈$10^{15}$ samples

crystallized → not in life environment

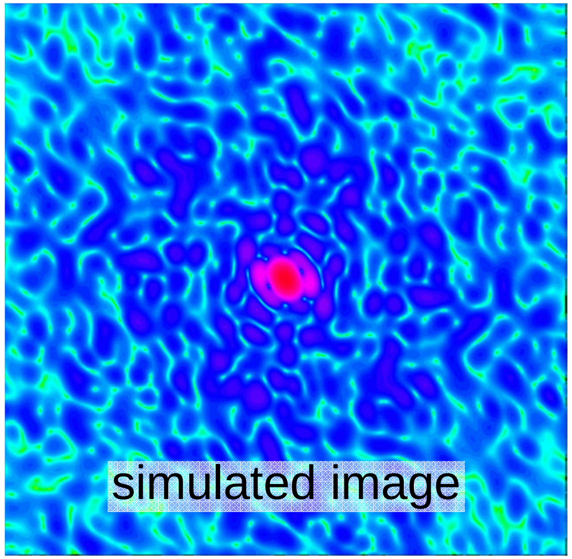
the crystal lattice imposes restrictions on molecular motion

images courtesy Janos Hajdu, slide from Jörg Rossbach

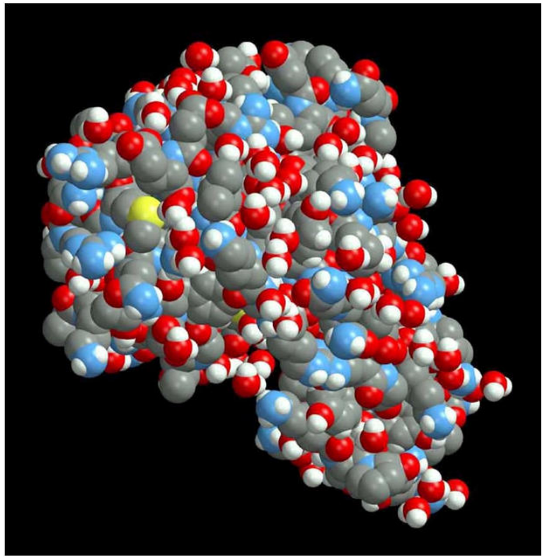
# need for short wavelengths - 2

**SINGLE**
**MACROMOLECULE**

courtesy Janos Hajdu



⇨



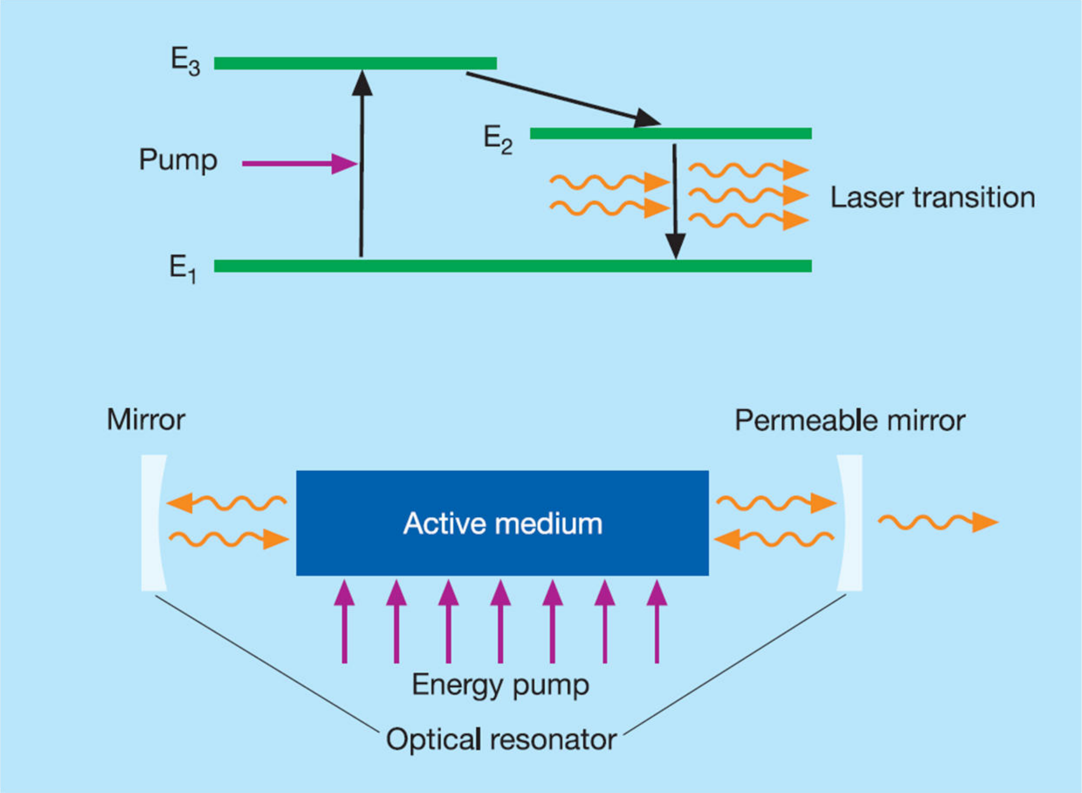
resolution does not depend on sample quality
needs very high radiation power @ $\lambda \approx 1$Å
can see dynamics if pulse length < 100 fs

we need a radiation source with

- very high peak and average power
- wavelengths down to atomic scale $\lambda \sim 1$Å
- spatially coherent
- monochromatic
- fast tunability in wavelength & timing
- sub-picosecond pulse length

# why FELs?

principle of a quantum laser



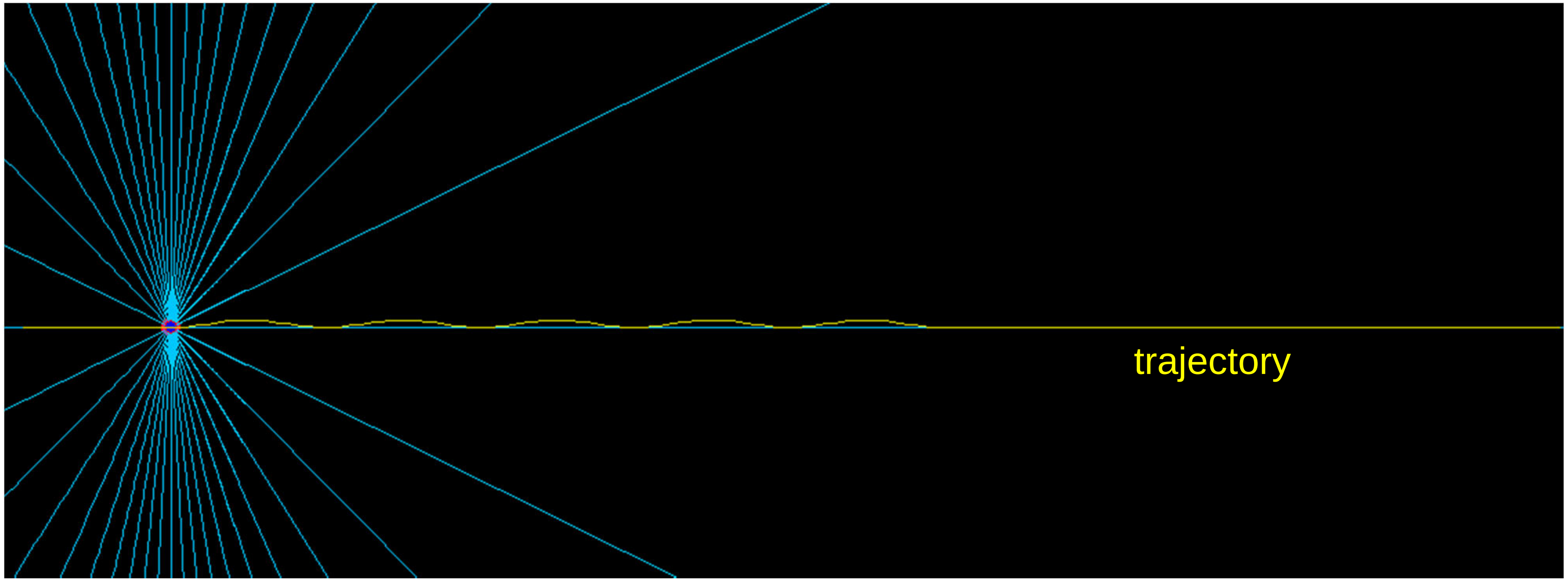
problem & solution:

active medium → free electron – EM wave interaction

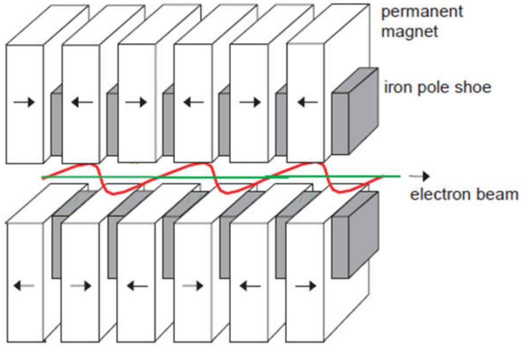
# free electron ↔ wave Interaction

free electron in uniform motion + electric field lines

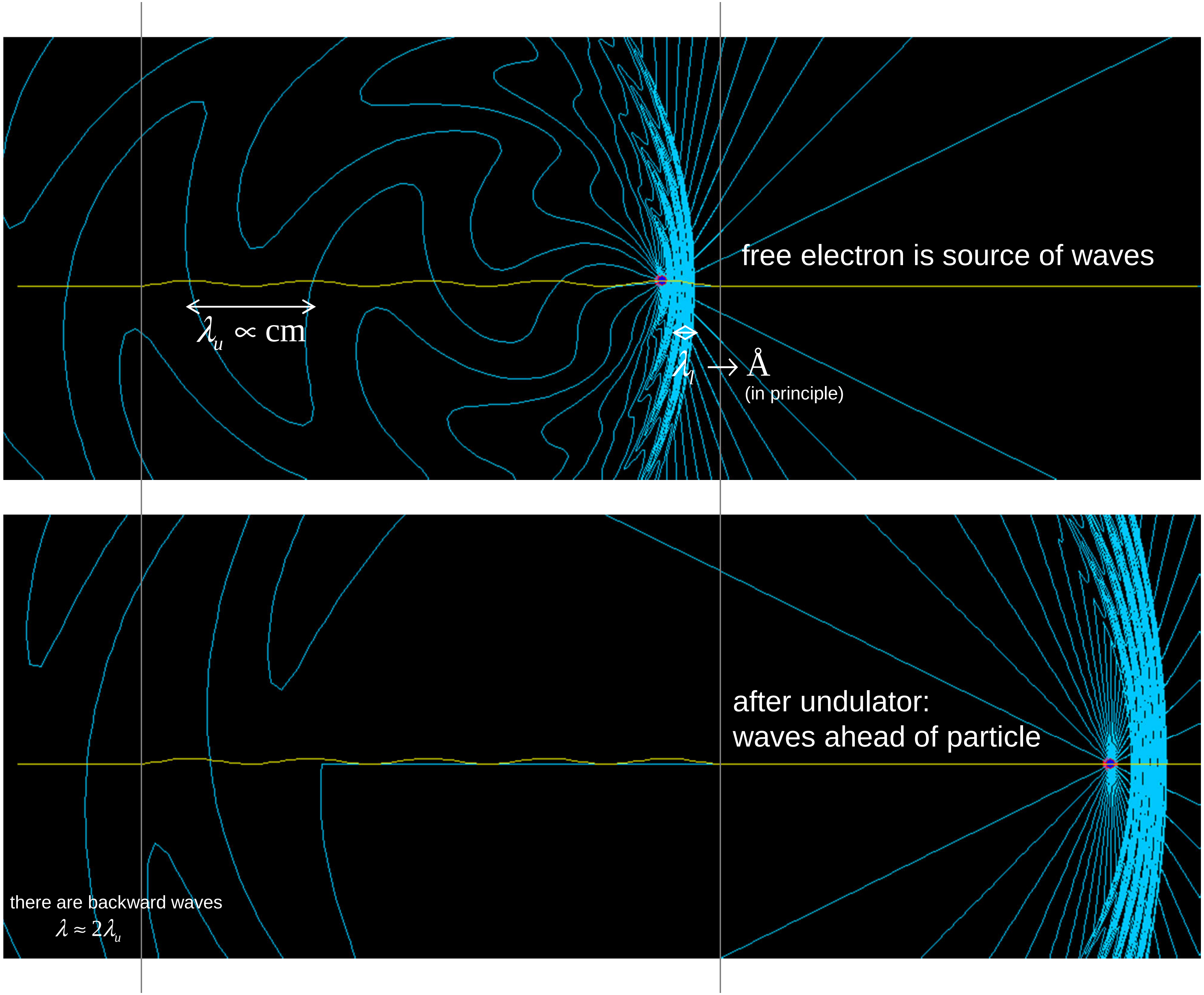
(before undulator)



courtesy T. Shintake
http://www.shintakelab.com/en/enEducationalSoft.htm



undulator



free electron is source of waves

$\lambda_u \propto \mathrm{cm}$

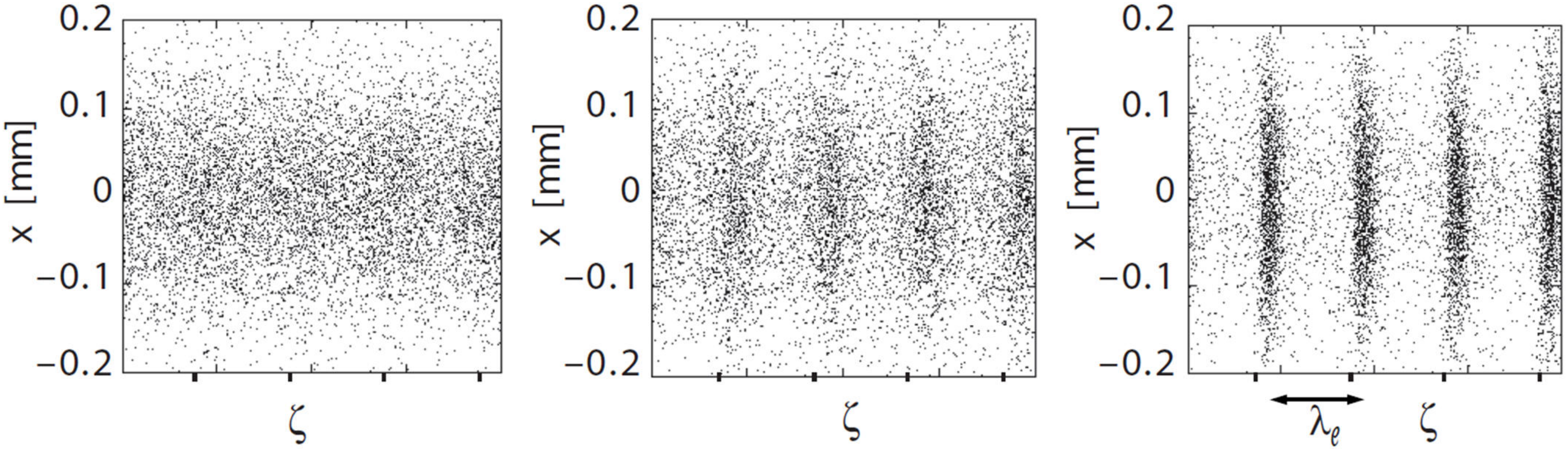
$\lambda_l \to$ Å

(in principle)

after undulator:
waves ahead of particle

there are backward waves
$\lambda \approx 2\lambda_u$

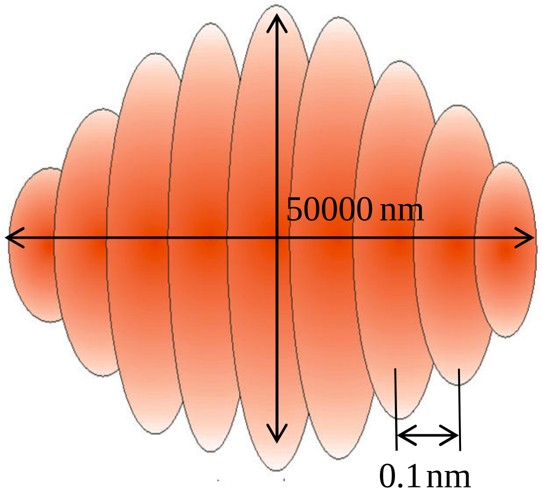
# micro-bunching



longitudinal motion to 1st order is trivial, **but**

micro-bunching is a 2nd order effect
→ coupled theory of particle motion and
wave generation

transverse bunch structure is much larger
than longitudinal sub-structure
→ 1d theory with plane waves

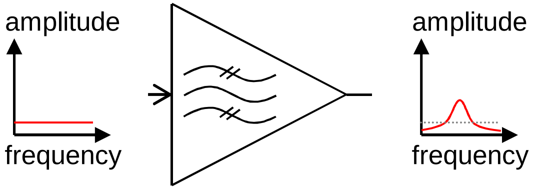


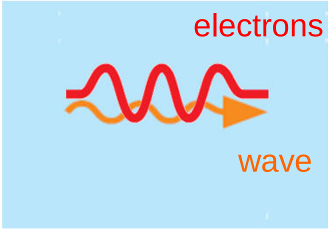
# amplifier and oscillator

in principle

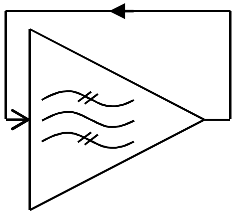
FEL

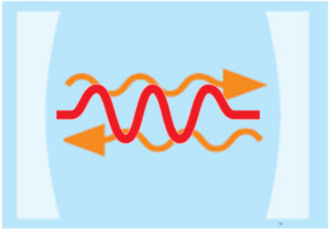
amplifier:





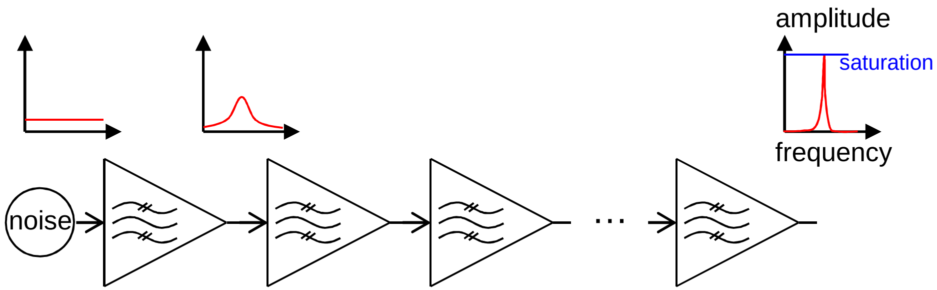
oscillator:

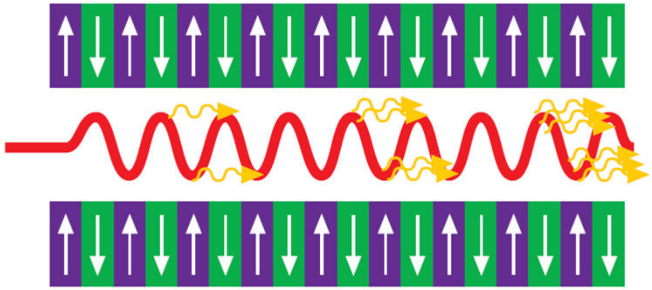




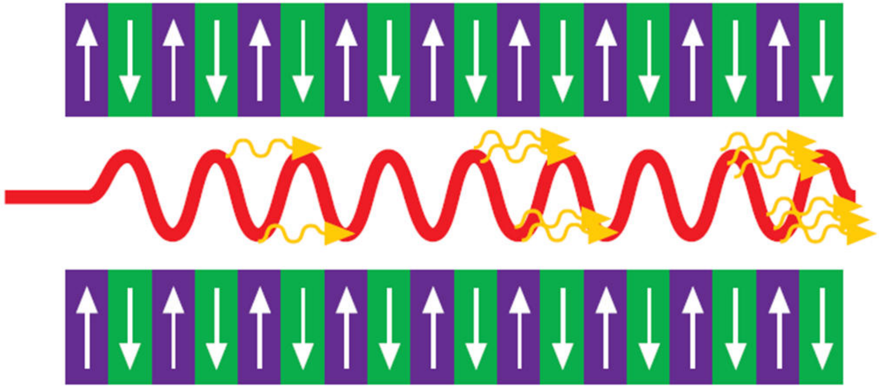
instability, driven by noise, growth until amplifier saturates

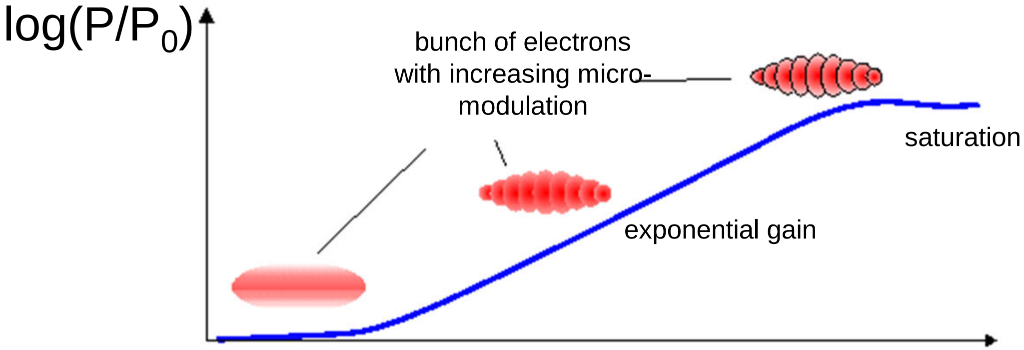
amplified noise:





# self amplifying spontaneous emission (SASE)





uniform **random** distribution of particles at entrance

**incoherent** emission of EM waves (noise, wide bandwidth)

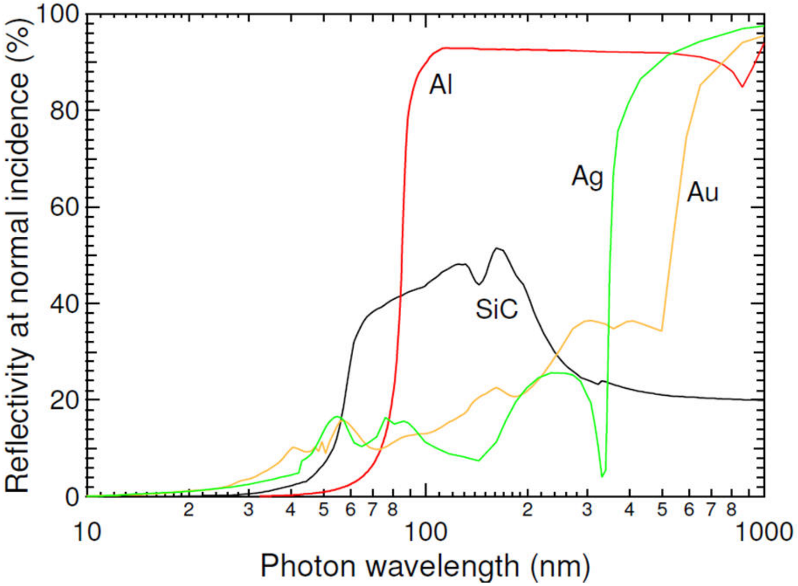
amplification (→ resonant wavelength, micro-bunching)

saturation, full **micro modulation**, **coherent** radiation

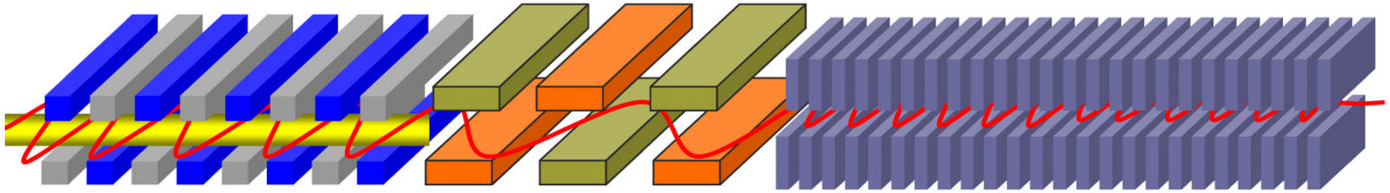
# why SASE?

oscillator needs resonator

but there are no mirrors for wavelengths < 100 nm



alternative: seed laser + harmonic generation + amplifier



seed laser

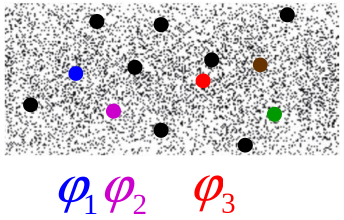
$\lambda_{\text{seed}}$

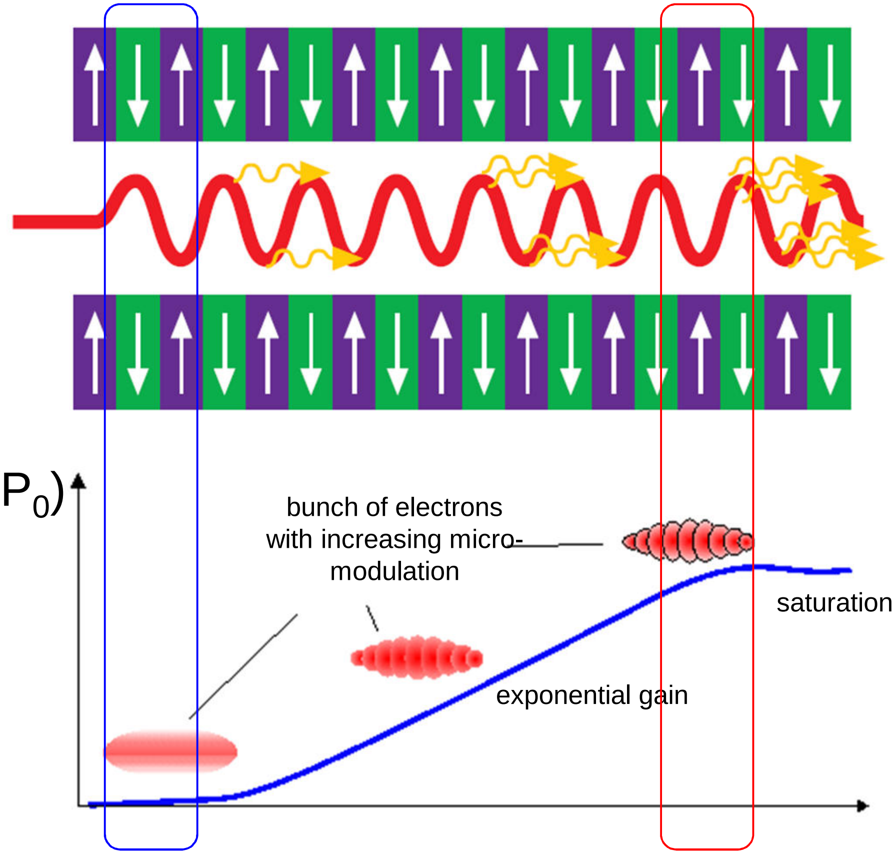
modulation of
electron beam

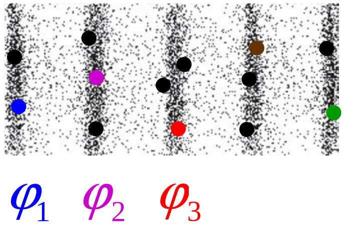
micro bunching of
a particular harmonic

amplifier $\lambda_l = n\lambda_{\text{seed}}$

# coherent radiation

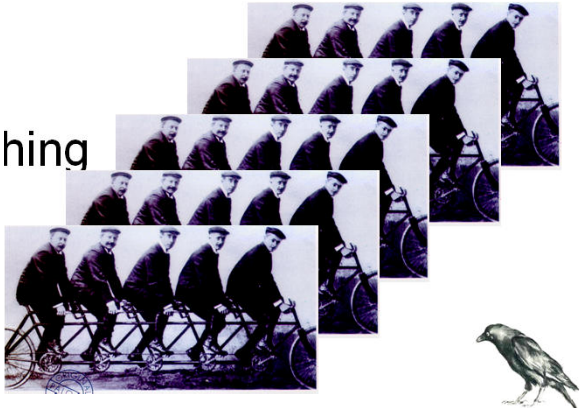






no micro-bunching
(only shot noise)
$P_0 \propto N$

saturation:
full micro-bunching
$P \propto N^2$



with $N$ = particles per $\lambda_l$

# B) Theoretical Approach

particle dynamics: undulator motion

about particles

independent parameter: z

particle dynamics: interaction with EM wave

longitudinal equation of motion

phase space and pendulum

FEL low gain theory

micro bunching

electrodynamics (1D)

FEL high gain theory (1D)

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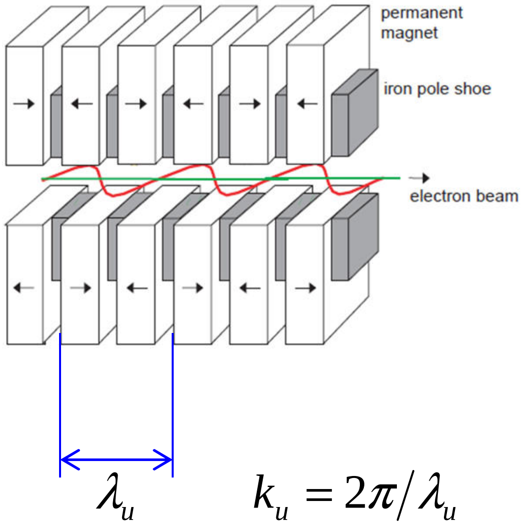
continuous phase space: Vlasov equation

FEL third order equation

# particle dynamics: undulator motion



$k_u = 2\pi/\lambda_u$

field of planar undulator $\mathbf{B}_u = -B_0 \mathbf{e}_y \sin k_u z$

equation of motion $\gamma m_e \dot{\mathbf{v}} = -e\mathbf{v} \times \mathbf{B}$

approach: $z \approx \bar{v}t - \hat{z}\cos 2\omega_u t$ with $\omega_u = 2\pi\bar{v}/\lambda_u$

$x \approx \hat{x}\sin \omega_u t$

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$$\hat{x} = \frac{e}{v\gamma m_e k_u^2} B_0 = \frac{K}{\beta\gamma k_u}$$

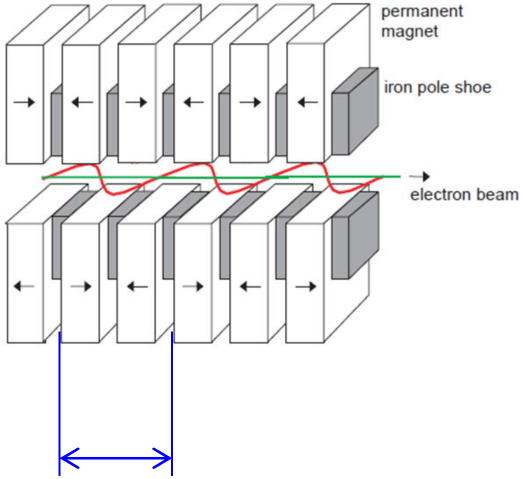
with undulator parameter $K = \dfrac{eB_0}{m_e c k_u} \propto 1$

$$\bar{v} = v - v\left(\frac{K}{2\beta\gamma}\right)^2 \approx c - \frac{c}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$$\hat{z} = \frac{K^2}{8\gamma^2 k_u}$$

# example: FLASH

$B_0 = 0.47\text{ T}$



$\lambda_u = 27\text{ mm}$

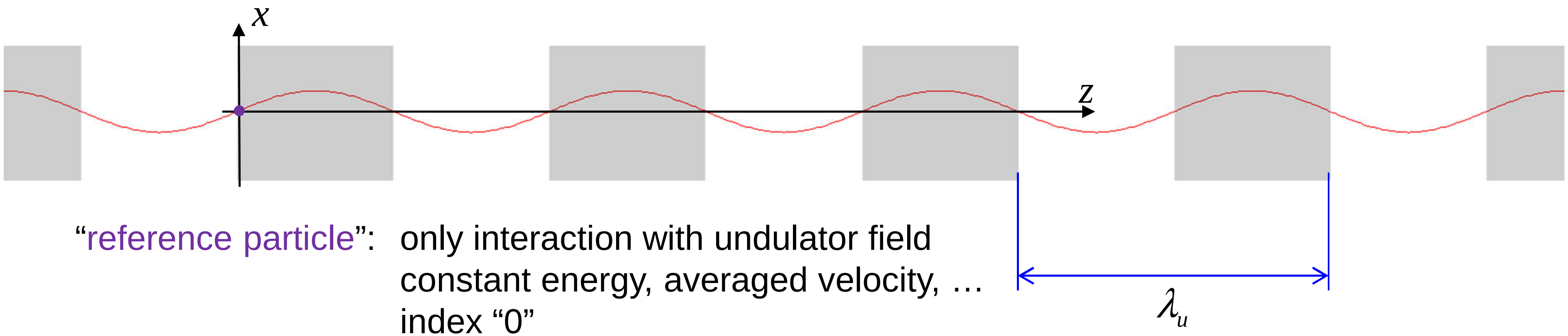
$$K \approx \frac{0.934}{\text{cmT}} B_0 \lambda_u \approx 1.2$$

$$W \approx 1\,\text{GeV} \quad \rightarrow \gamma \approx 1957$$

$$\hat{x} = \frac{K}{\gamma k_u} = 2.6\,\mu\text{m}$$

$$\hat{z} = \frac{K^2}{8\gamma^2 k_u} = 0.2\text{ nm}$$

# about particles



“reference particle”: only interaction with undulator field
constant energy, averaged velocity, …
index “0”

resonance condition

$\boxed{\gamma_0}$ $\quad v_0 = c\sqrt{1-\gamma_0^2} \quad \bar{v}_0 = v_0 - v_0\left(\frac{K}{2\gamma_0}\right)^2 \quad x_0 \approx \hat{x}_0 \sin\omega_u t \quad z_0 \approx \cdots$

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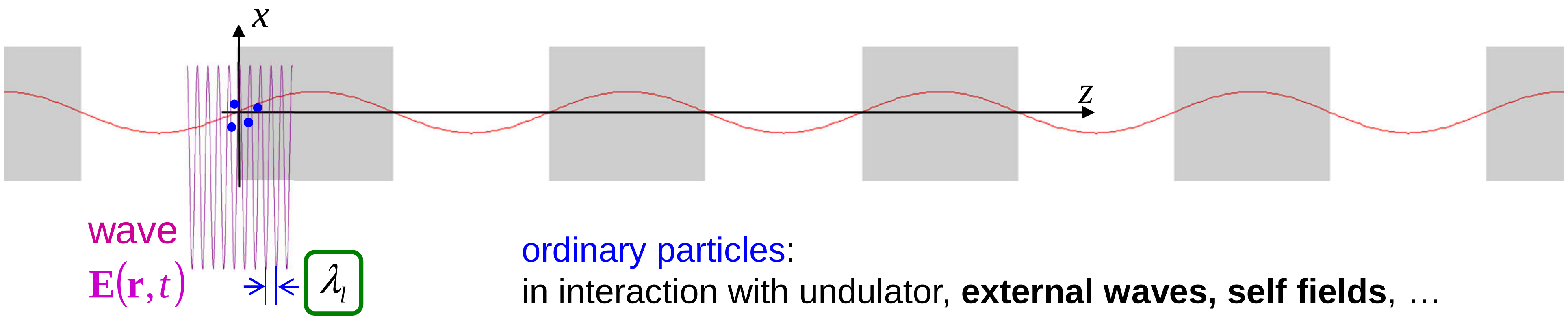

ordinary particles:
in interaction with undulator, **external waves, self fields**, …
slowly variation of energy, velocity (compared to $\lambda_u$)
index “$\nu$” or skipped

$\gamma_\nu \quad v_\nu \quad \bar{v}_\nu \quad \cdots$

# new independent parameter: $z$

"reference particle": $\gamma_0 \qquad v_0 \qquad \bar{v}_0$

$$t_0(z) = \frac{z}{\bar{v}_0} + \frac{\hat{z}_0}{\bar{v}_0} \cos 2k_u z$$

$$x_0(z) = \hat{x}_0 \sin k_u z$$

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ordinary particles:

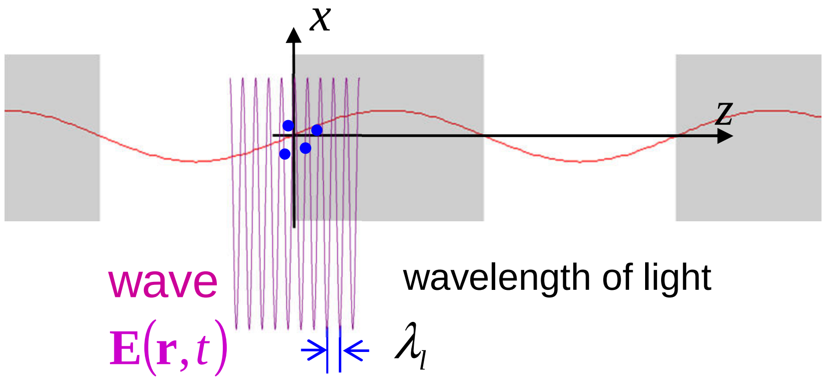
$\gamma_\nu(z) \qquad v_\nu(z) \qquad \bar{v}_\nu(z)$ approach: nearly constant on one period $\lambda_u$

$$t_\nu(z) \approx \underbrace{t_{\nu,i}}_{\text{initial condition}} + \underbrace{\int_0^z \frac{dz}{\bar{v}_\nu(z)} + \frac{\hat{z}_0}{\bar{v}_0}}_{\text{slippage effects}} \cos 2k_u z \qquad = T_\nu(z) + t_0(z)$$

$$x_\nu(z) \approx \overbrace{x_{\nu,i}} + \hat{x}_0 \sin k_u z \qquad = x_{\nu,i} + x_0(z)$$

energy parameter: $\eta_\nu = \dfrac{\gamma_\nu - \gamma_0}{\gamma_0}$

# particle dynamics: interaction with EM wave



$$dW = -e\mathbf{E}(\mathbf{r},t)\cdot d\mathbf{r}$$

plane wave with $k_l$=2π/$\lambda$
$x$ polarization

$dx/dz$

$$\frac{dW_\nu}{dz} = -e\,\boxed{E_0\cos\{k_l(z-ct_\nu(z))\}}\cdot\boxed{\frac{eK}{\gamma}\cos k_u z}$$

$t_\nu = T_\nu + \frac{z}{\bar{v}_0} + \frac{\hat{z}_0}{\bar{v}_0}\cos 2k_u z$

$$\frac{dW_\nu}{dz} = -\frac{eE_0K}{\gamma}\cos\{\boxed{k_l(1-c/\bar{v}_0)}z - k_l(cT_\nu(z)+\hat{z}_0\cos 2k_u z)\}\cdot\cos\boxed{k_u}z$$

**resonance condition** (permanent energy transfer): $k_l(1-c/\bar{v}_0)=k_u \rightarrow \boxed{\lambda_l = \frac{\lambda_u}{2\gamma_0^2}\left(1+\frac{K}{2}\right)}$

it is a condition for the energy of the reference particle or the wavelength $\lambda_l$

# particle dynamics: interaction with EM wave

averaged vs. undulator period $\left\langle \frac{dW_\nu}{dz} \right\rangle = ?$ $\quad T_\nu(z) \approx const$

estimation without longitudinal oscillation $\left\langle \cos\left(k_u z + \psi + \cancel{\hat{z}_0 \cos 2k_u z}\right) \cos k_u z \right\rangle = \frac{1}{2}\cos\psi$

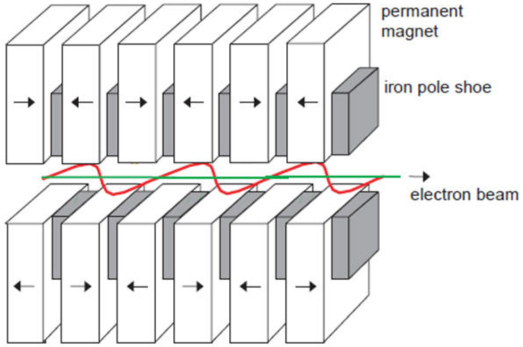
$$\boxed{\begin{aligned} \frac{dW_\nu}{dz} &= -\frac{eE_0 K}{2\gamma}\cos\psi_\nu \\ \psi_\nu &= k_l c T_\nu(z) \end{aligned}}$$

ponderomotive phase

with longitudinal oscillation: replace $K$ by $\hat{K} = K\left[J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)\right]$

(modified undulator parameter)

# example: FLASH



$$B_0 = 0.47\ \mathrm{T}$$

$$\lambda_u = 27\ \mathrm{mm}$$

$$K \approx \frac{0.934}{\mathrm{cmT}} B_0 \lambda_u \approx 1.2$$

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$$\hat{x} = \frac{K}{\gamma k_u} = 2.6\,\mu\mathrm{m}$$

$$\hat{z} = \frac{K^2}{8\gamma^2 k_u} = 0.2\ \mathrm{nm}$$

$$\lambda_l = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \approx 6\ \mathrm{nm}$$

$$\hat{K} = K\left[J_0\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right)\right]$$

$$\hat{K} = 1.06$$

# longitudinal equation of motion

ponderomotive phase

new longitudinal parameters $$t_\nu(z)=T_\nu(z)+t_0(z)=\frac{\boxed{\psi_\nu(z)}}{k_l c}+t_0(z)$$

$$\boxed{\eta_\nu(z)}=\frac{W_\nu(z)}{W_0}-1$$ relative energy deviation

with $\frac{d\psi_\nu}{dz}=k_l c\frac{dT_\nu(z)}{z}=\frac{k_l c}{\bar{v}_\nu}-\frac{k_l c}{\bar{v}_0}\approx 2k_u\eta_\nu$ and $W_0=m_e c^2\gamma_0^2$ follow

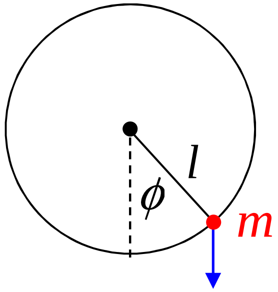
## particle equations

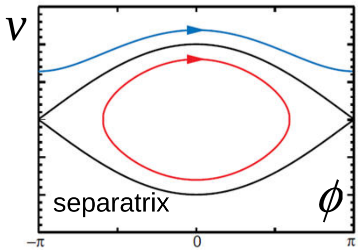
$$\frac{d\eta_\nu}{dz}\approx-\frac{eE_0\hat{K}}{2m_e c^2\gamma_0^2}\cos\psi_\nu$$

$$\frac{d}{dz}\psi_\nu\approx 2k_u\eta_\nu$$

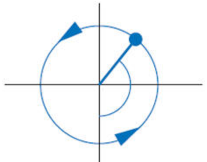
# phase space and pendulum

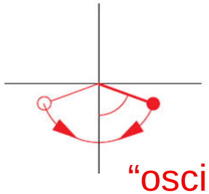
equations are analog to mathematical pendulum





two types of solution:


“rotation”


“oscillation”

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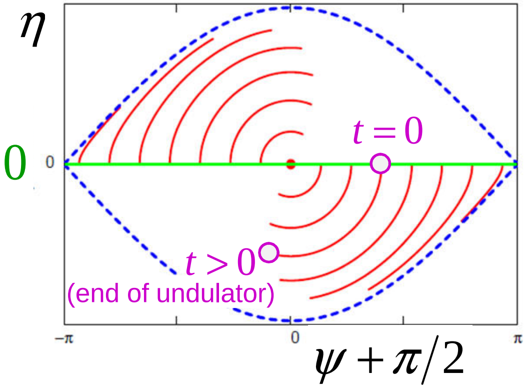
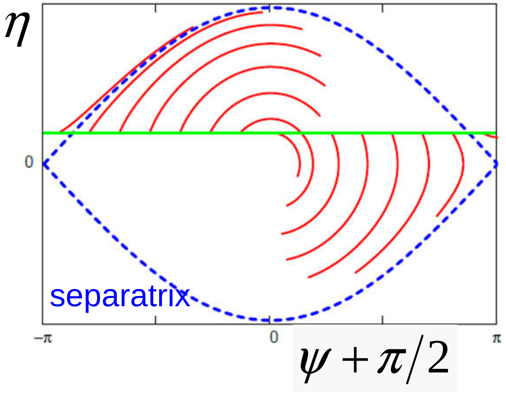
trajectories in phase space

$\eta_\nu(0)=0$





$\eta_\nu(0)>0$

or

$\gamma_\nu(0)>\gamma_0$

15 particles with different initial conditions

# FEL low gain theory

neglect change of field amplitude

indirect gain calculation

$$G = \frac{\text{gain of field energy}}{\text{initial field energy}} = \frac{\text{loss of particle energy}}{\text{initial field energy}} = \frac{W_\Sigma(\text{in}) - W_\Sigma(\text{out})}{\text{initial field energy}}$$

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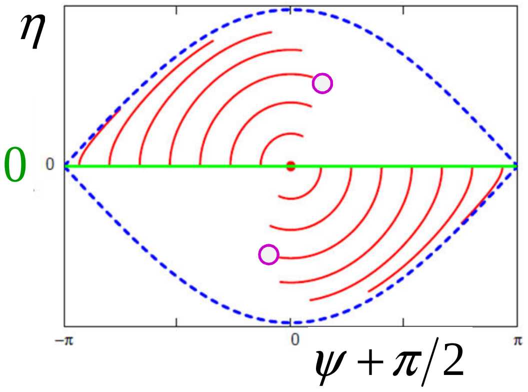
$$W_\Sigma(\text{in}) = W_\Sigma(\text{out})$$

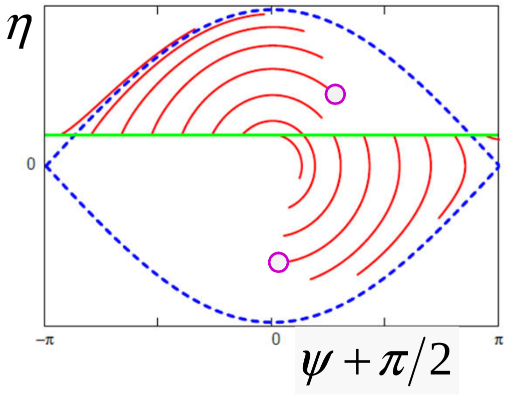
$$G = 0$$

$$\eta_\nu(0) = 0$$



$$W_\Sigma(\text{in}) > W_\Sigma(\text{out})$$

$$G > 0$$



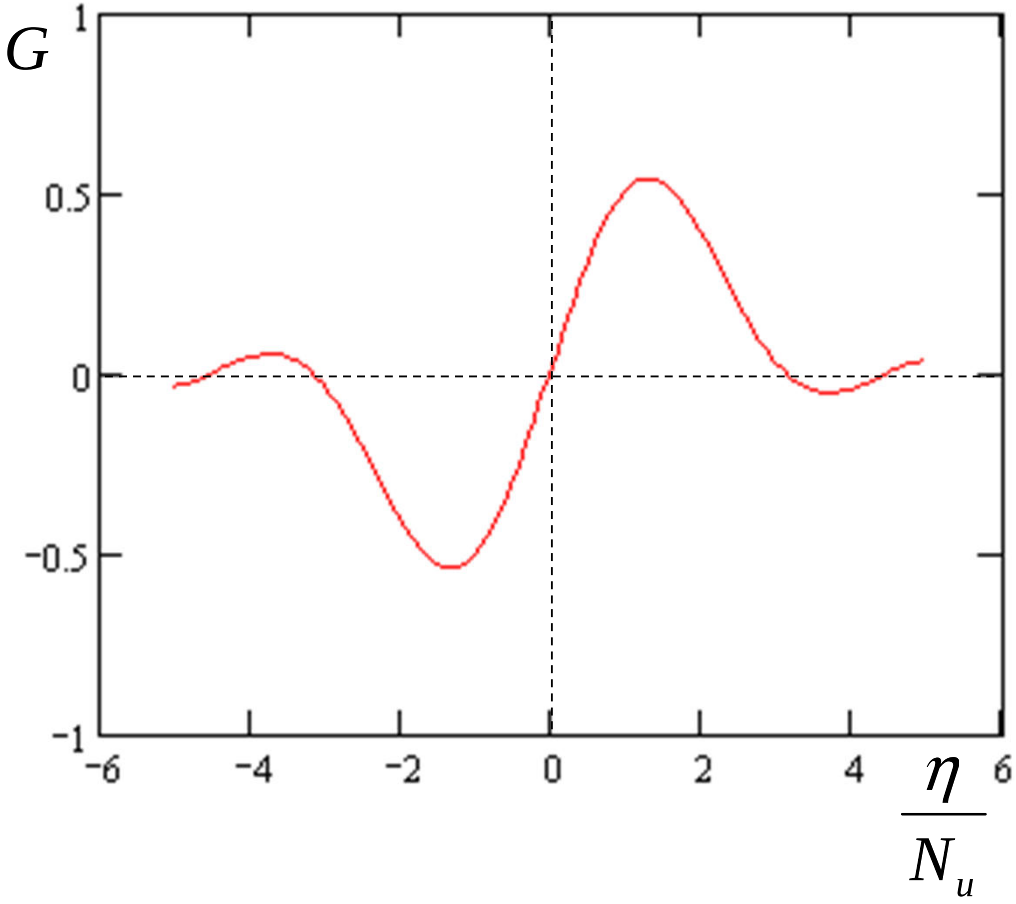
$$\eta_\nu(0) > 0$$

or

$$\gamma_\nu(0) > \gamma_0$$

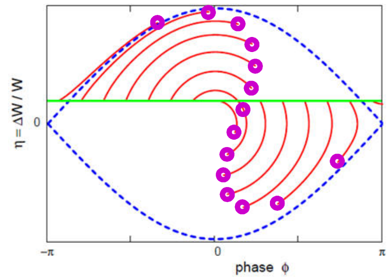
# FEL low gain theory

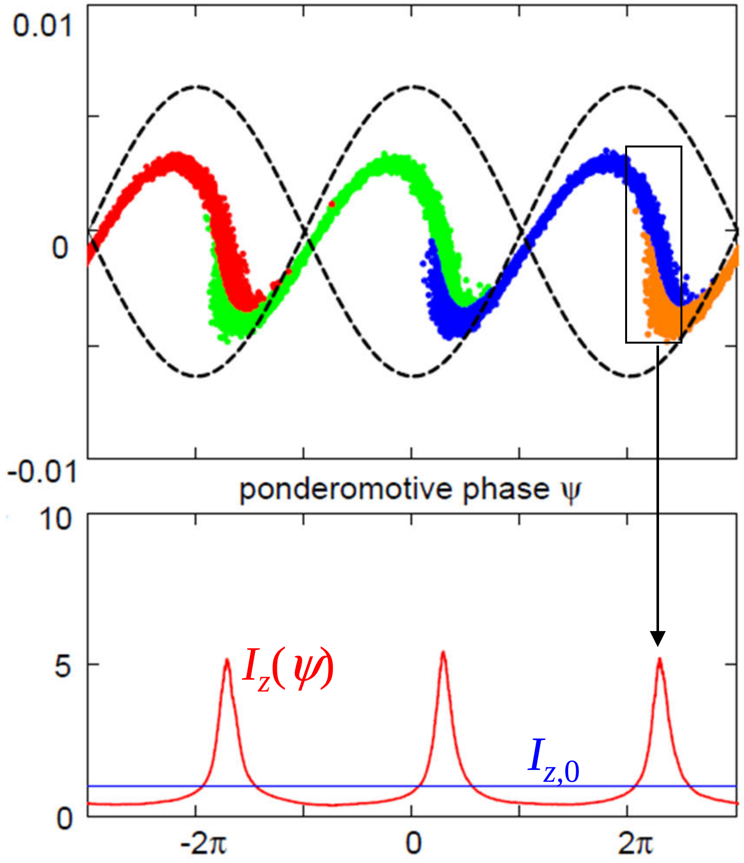
neglect change of field amplitude



$N_u =$ periods of undulator

# micro-bunching





Fourier analysis
→ amplitude of micro modulation

$$\boxed{\hat{I} \propto \sum \exp(-i\psi_\nu)}$$

(fundamental mode)

# electrodynamics (1D)

Maxwell equations

↓

wave equation

↓

1D wave equation

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_x=\mu_0\frac{\partial}{\partial t}J_x$$

$$J_x(z,t)=\frac{\overbrace{v_x(z)}^{\hat{x}_0 k_u \cos k_u z}}{v_z(z)}J_z(z,t)$$

approach with slowly varying amplitude

$$J_z(z,t)\sim \mathrm{Re}\left\{\hat{I}_1(z)\exp\left(ik_l\left(z-t/c\right)\right)\right\}+\cdots$$

$$E_x(z,t)\sim \mathrm{Re}\left\{\hat{E}_x(z)\exp\left(ik_l\left(z-t/c\right)\right)\right\}+\cdots$$

$$\boxed{\frac{d}{dz}\hat{E}_x(z)=-\frac{\mu_0 c\hat{K}}{4\gamma_0}\hat{I}_1}$$

# FEL high gain theory (1D)

$$\frac{d\eta_\nu}{dz} \approx -\frac{e\hat{K}}{2m_e c^2 \gamma_0^2}\mathrm{Re}\left\{\hat{E}_x \exp\left(i\psi_\nu\right)\right\}$$

particle equations

$$\frac{d}{dz}\psi_\nu \approx 2k_u\eta_\nu$$

$$\hat{I} \propto \sum \exp\left(-i\psi_\nu\right)$$

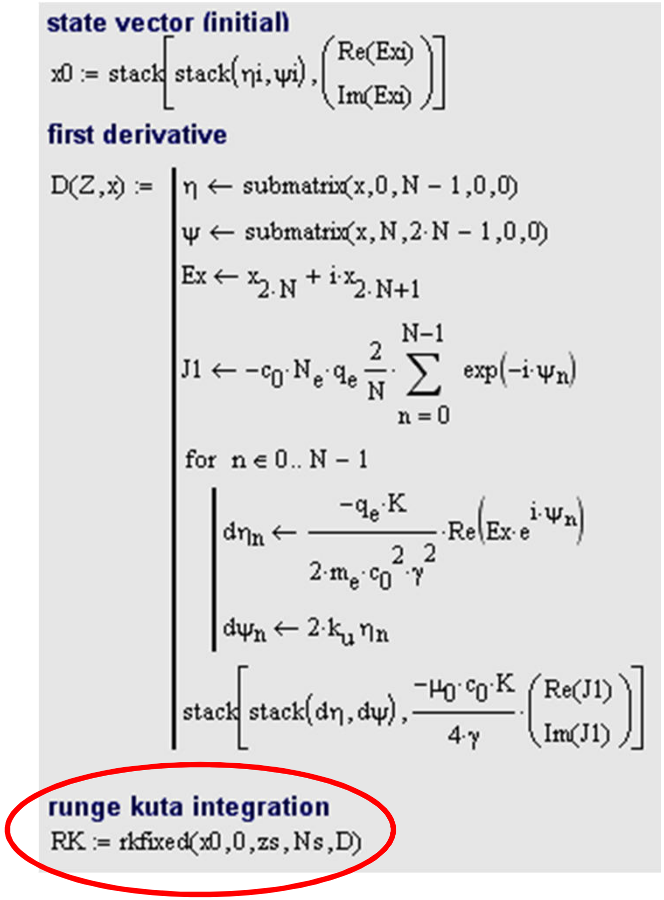
micro modulation

$$\frac{d}{dz}\hat{E}_x \approx -\frac{\mu_0 c\hat{K}}{4\gamma_0}\hat{I}_1$$

electrodynamics

# FEL high gain theory (1D)

numerical solution
(Mathcad)



**state vector (initial)**

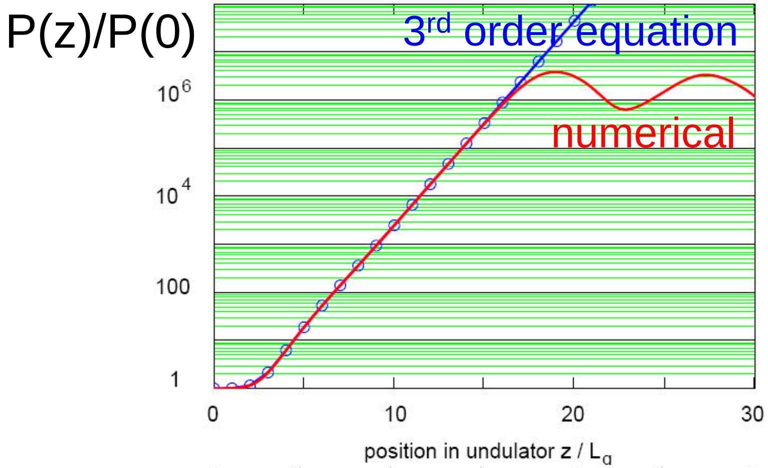
$$x0 := \mathrm{stack}\left[\mathrm{stack}(\eta i, \psi i), \begin{pmatrix} \mathrm{Re}(Exi) \\ \mathrm{Im}(Exi) \end{pmatrix}\right]$$

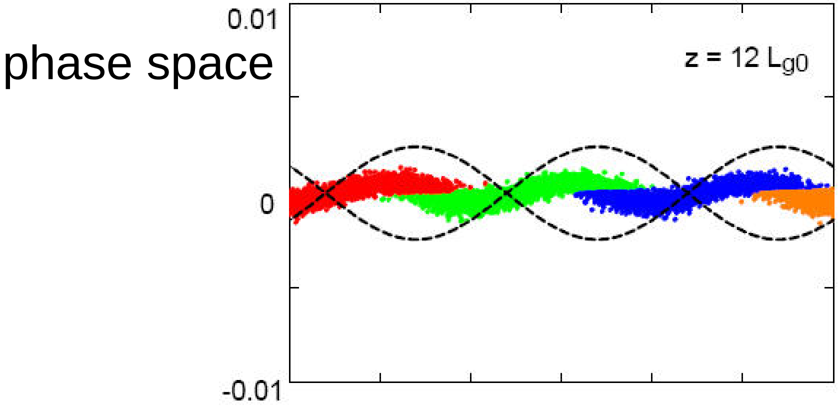
**first derivative**

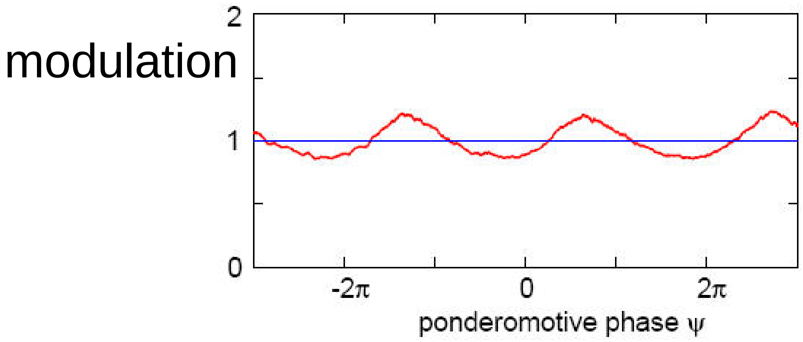
$$D(Z,x) := \left| \begin{array}{l} \eta \leftarrow \mathrm{submatrix}(x,0,N-1,0,0) \\ \psi \leftarrow \mathrm{submatrix}(x,N,2\cdot N-1,0,0) \\ Ex \leftarrow x_{2\cdot N} + i\cdot x_{2\cdot N+1} \\ J1 \leftarrow -c_0\cdot N_e\cdot q_e\cdot \frac{2}{N}\cdot \sum_{n=0}^{N-1} \exp(-i\cdot\psi_n) \\ \text{for } n \in 0..N-1 \\ \quad \left| \begin{array}{l} d\eta_n \leftarrow \frac{-q_e\cdot K}{2\cdot m_e\cdot c_0^2\cdot\gamma^2}\cdot \mathrm{Re}\left(Ex\cdot e^{i\cdot\psi_n}\right) \\ d\psi_n \leftarrow 2\cdot k_u\cdot\eta_n \end{array} \right. \\ \mathrm{stack}\left[\mathrm{stack}(d\eta, d\psi), \frac{-\mu_0\cdot c_0\cdot K}{4\cdot\gamma}\cdot\begin{pmatrix} \mathrm{Re}(J1) \\ \mathrm{Im}(J1) \end{pmatrix}\right] \end{array} \right.$$

**runge kuta integration**

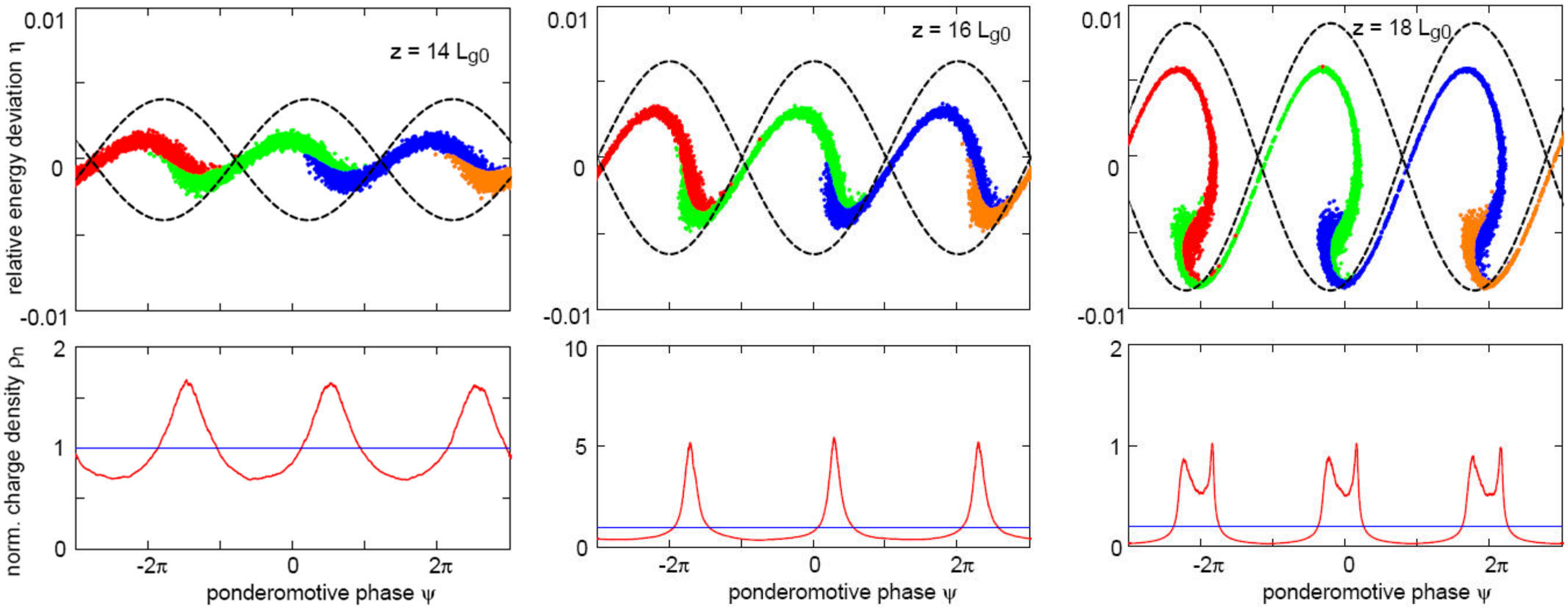
$$RK := \mathrm{rkfixed}(x0,0,zs,Ns,D)$$





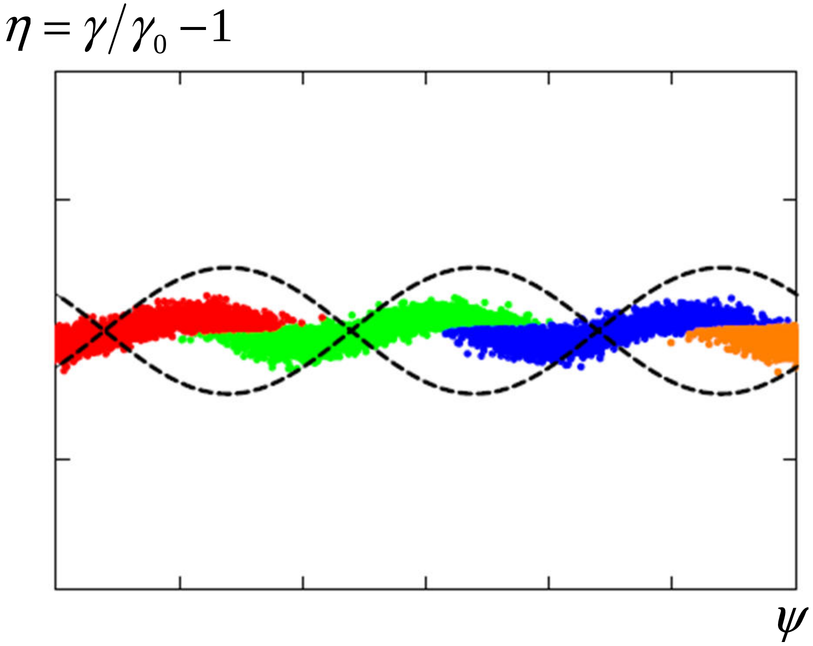


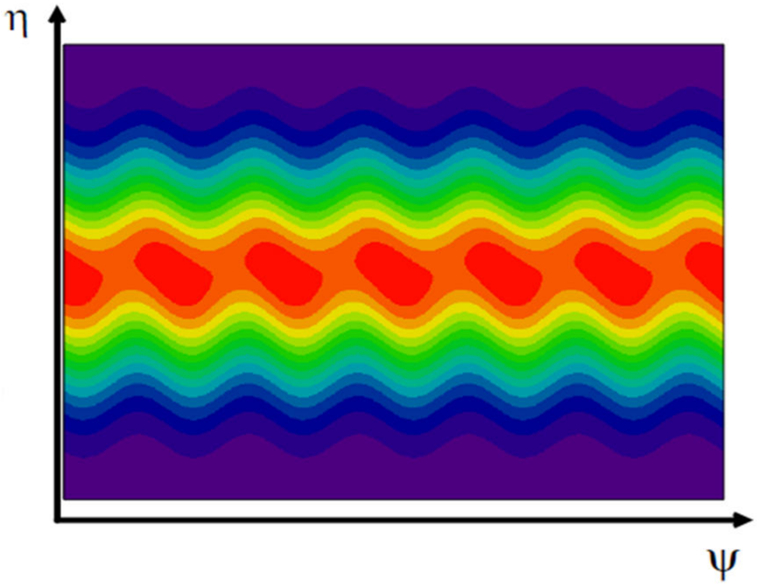
# FEL high gain theory (1D)



# continuous phase space, Vlasov equation

many point particles $\psi_\nu,\eta_\nu$ → continuous density distribution $F(\psi,\eta,z)$





# continuous phase space, Vlasov equation

phase space density: $\boxed{F(\psi,\eta,z)}$

continuity equation: $\nabla\cdot\mathbf{J}(\psi,\eta,z)+\frac{\partial F}{\partial z}=0$

with “current density” $\mathbf{J}(\psi,\eta,z)=\begin{pmatrix}\psi'\\ \eta'\end{pmatrix}F(\psi,\eta,z)$

therefore $\frac{\partial(\psi' F)}{\partial\psi}+\frac{\partial(\eta' F)}{\partial\eta}+\eta'\frac{\partial F}{\partial\eta}+\frac{\partial F}{\partial z}=0$

with particle equations $\psi'=\frac{\partial\psi}{\partial z}=f(\eta,z)$

$\eta'=\frac{\partial\eta}{\partial z}=g(\psi,z)$

Vlasov equation: $\boxed{\psi'\frac{\partial F}{\partial\psi}+\eta'\frac{\partial F}{\partial\eta}+\frac{\partial F}{\partial z}=0 \qquad \text{or} \qquad \frac{dF}{dz}=0}$

# FEL 3rd order equation

perturbation approach: $F(\psi,\eta,z) \approx F_0(\eta-\eta_{\mathrm{off}}) + \mathrm{Re}\left\{\hat{F}_1(\eta,z)\exp(i\psi)\right\}$

$$\boxed{\frac{d^3\hat{E}_x}{dz^3} + 4ik_u\eta_{\mathrm{off}}\frac{d^2\hat{E}_x}{dz^2} - 4k_u^2\eta_{\mathrm{off}}^2\frac{d\hat{E}_x}{dz} - i\Gamma^3\hat{E}_x = 0}$$

with energy offset $\eta_{\mathrm{off}} = \langle\eta_\nu\rangle = \dfrac{\langle\gamma_\nu\rangle - \gamma_0}{\gamma_0}$

gain parameter $\Gamma = \sqrt[3]{\dfrac{1}{4}\dfrac{e}{m_e}\dfrac{\mu_0}{c}\dfrac{I}{A}\dfrac{k_u\hat{K}^2}{\gamma_0^3}}$

beam current $I$

beam cross-section $A$

solution $\hat{E}_x(z) = A_1\exp\alpha_1 z + A_2\exp\alpha_2 z + A_3\exp\alpha_3 z$

# FEL 3rd order equation

$$\hat{E}_x(z) = A_1 \exp \alpha_1 z + A_2 \exp \alpha_2 z + A_3 \exp \alpha_3 z$$

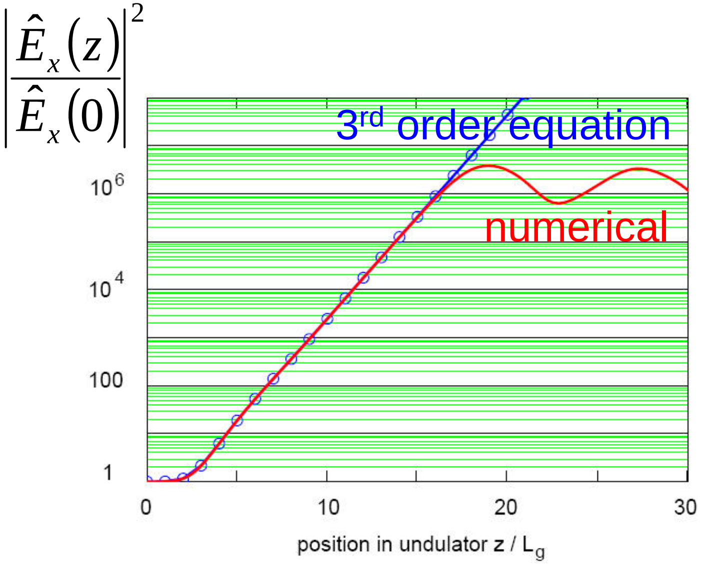
no energy offset: $\eta_{\mathrm{off}} = 0$ or $\langle \gamma_\nu \rangle = \gamma_0$

$$\alpha_1 = \frac{i + \boxed{\sqrt{3}}}{\boxed{2}}\Gamma$$

$$\alpha_2 = \frac{i - \sqrt{3}}{2}\Gamma$$

$$\alpha_3 = -i\Gamma$$

positive real part → exponential growth !



power gain length: $P(z) \to \left|A_1^2 \exp 2\alpha_1 z\right| \propto \exp\left(z \overset{1/L_g}{\boxed{\sqrt{3}\Gamma}}\right)$

# C) Experimental Realization / Challenges

Linac Coherent Light Source - LCLS

scales

challenges

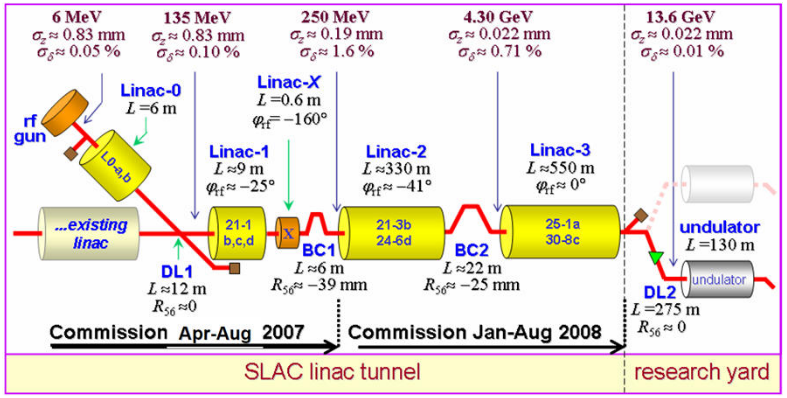
rf gun

bunch compression

European X-FEL

# Linac Coherent Light Source- LCLS

SLAC mid-April 2009 – first lasing at 1.5 Å



injector

$q \approx 0.25\,\mathrm{nC}$

linac &
bunch compression

$L \propto 10^3\,\mathrm{m}$

$E = 13.6\,\mathrm{GeV}$

$I = 3\,\mathrm{kA}$

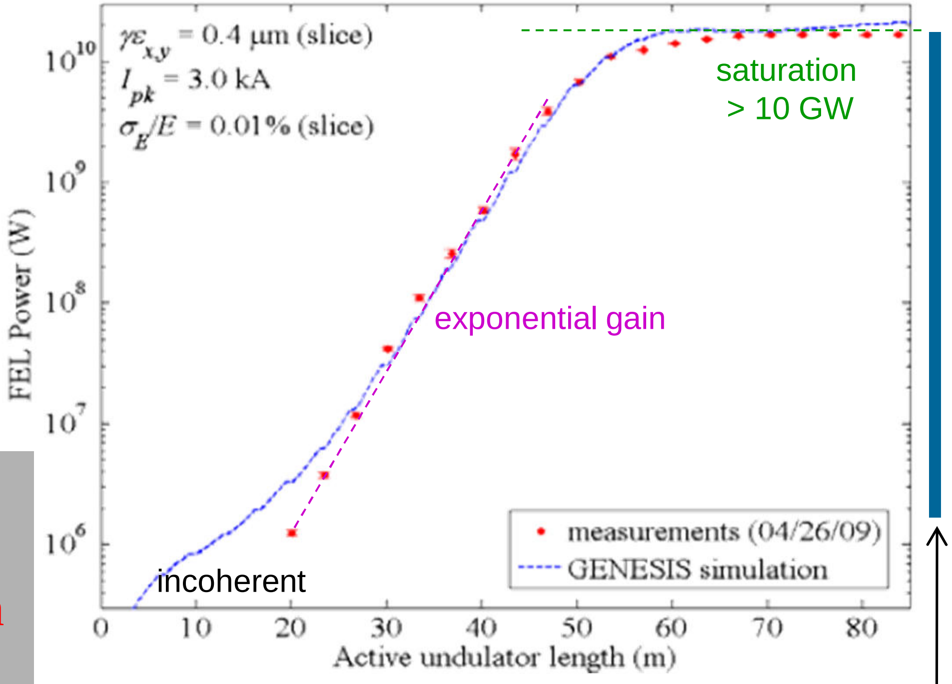
undulator

$L \propto 100\,\mathrm{m}$

$\lambda_u \approx 3\,\mathrm{cm}$

$L_g \approx 3.3\,\mathrm{m}$

$\boxed{\lambda_l = 1.5\,\text{Å}}$



particles per $\lambda_l$

$$N = \frac{I\lambda_l}{ec} \approx \boxed{10^4}$$

# scales

| | | |
|---|---|---|
| photon wavelength | $\lambda_l \propto 10^{-10}\,\mathrm{m} \quad \propto \lambda_u/\gamma^2$ | |
| cooperation length | $L_l \propto 10^{-8}\,\mathrm{m}$ | |
| transverse oscillation | $\hat{x} \propto 10^{-6}\,\mathrm{m}$ | (undulator trajectory) |
| bunch length | $L_b \propto 10^{-5}\,\mathrm{m}$ | |
| bunch width | $\sigma_w^{\mathrm{bunch}} \propto 10^{-5}\,\mathrm{m}$ | width of photon beam $\sigma_w^{\mathrm{wave}} \propto \sqrt{\lambda_l L_R}$ |
| undulator period | $\lambda_u \propto 10^{-2}\,\mathrm{m}$ | |
| power gain length | $L_g \approx 1..10\,\mathrm{m}$ | |
| Rayleigh length | $L_R$ | (scale of widening of photon beam) |
| saturation length | $L_s \approx 10L_g..20L_g < L_u$ | |
| undulator length | $L_u \propto 100\,\mathrm{m}$ | |
| total length | $L \propto 10^3\,\mathrm{m}$ | |

overlap of particle beam
with photon beam

# challenges

$$\lambda_l = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

$$L_g = \frac{1}{\sqrt{3}}\left(\frac{4mc}{\mu e}\frac{\gamma^3\lambda_u}{K^2}\boxed{\frac{\sigma_r^2}{I}}\right)^{1/3}$$

$$\sigma_r^2 \propto \lambda_l L_g$$

- $\lambda_l \rightarrow$ Å
- Energy → 10 .. 20 GeV
- gain length $L_g$ <~ 10 m
- high peak current >~ kA
- transverse beam size $\sigma_r \propto 10\ \mu\mathrm{m}$
- energy spread
- overlap electron-photon beam

(undulator parameter K ∝ 1)

---

transverse: generate low emittance beam
preservation of emittance

longitudinal: compression

acceleration

diagnostic and steering

undulator alignment

$$\boxed{\frac{\sigma_r^2}{I}} = \frac{\boxed{\sigma_r^2 L_b}}{\boxed{q}c}$$

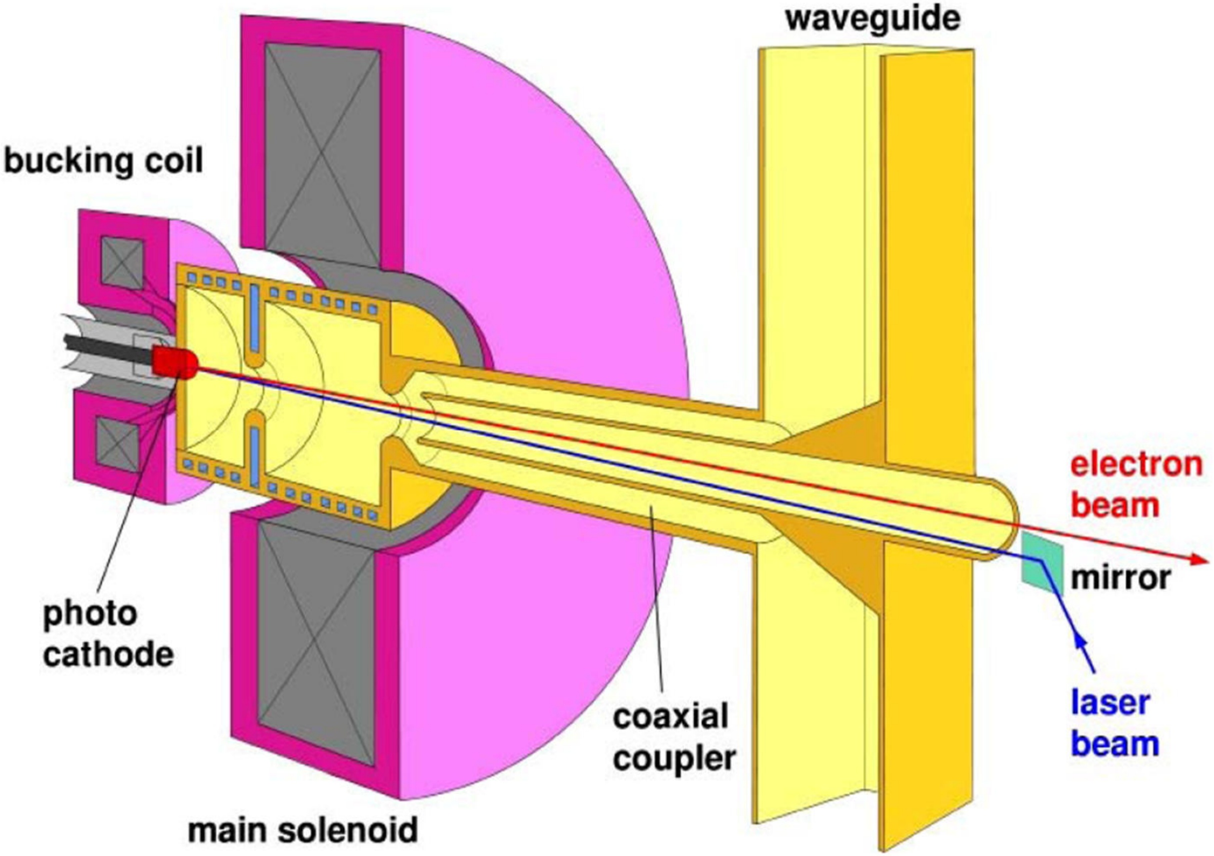
volume

bunch charge

space charge forces:

$$E_{sq} \propto \frac{1}{\gamma^2}\frac{q}{\sigma_r^2}$$

# rf gun



typical parameters of FLASH & European XFEL:

$$q \propto 0.1\,\mathrm{nC} \qquad E \propto 5\,\mathrm{MeV} \qquad I \propto 5\,\mathrm{A}$$

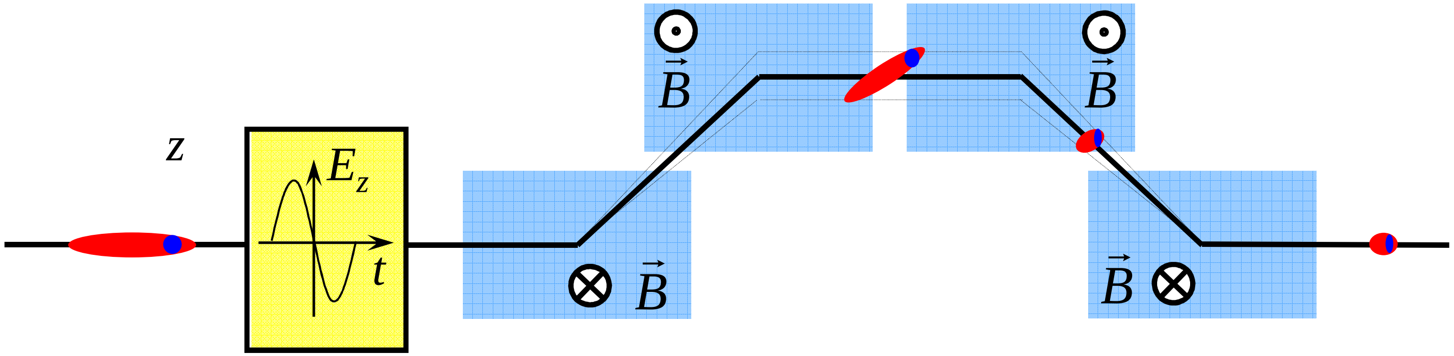
$$\gamma \propto 10$$

longitudinal compression 1 → 0.001 needed !   $(5\,\mathrm{kA})$

# magnetic bunch compression

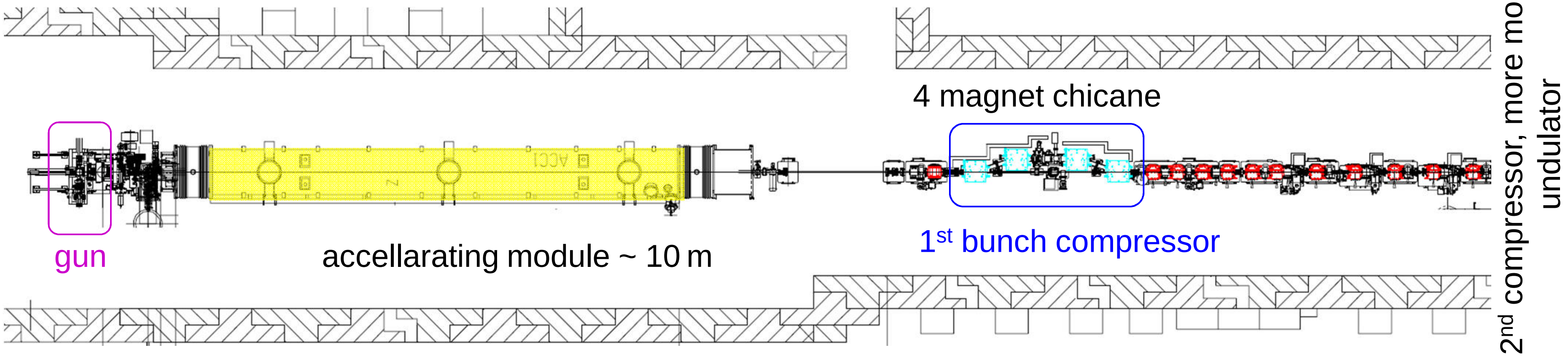
$\gamma >> 1$ → velocity differences are too small for effective compression

magnetic compression: path length depends on energy



accelleration “off crest” →
head particle with less energy than tail

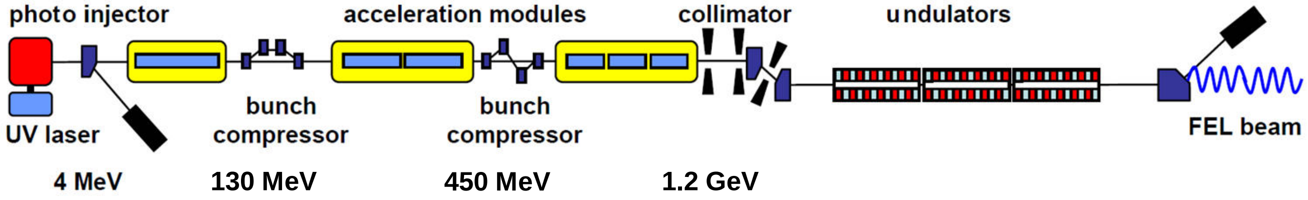
FLASH:



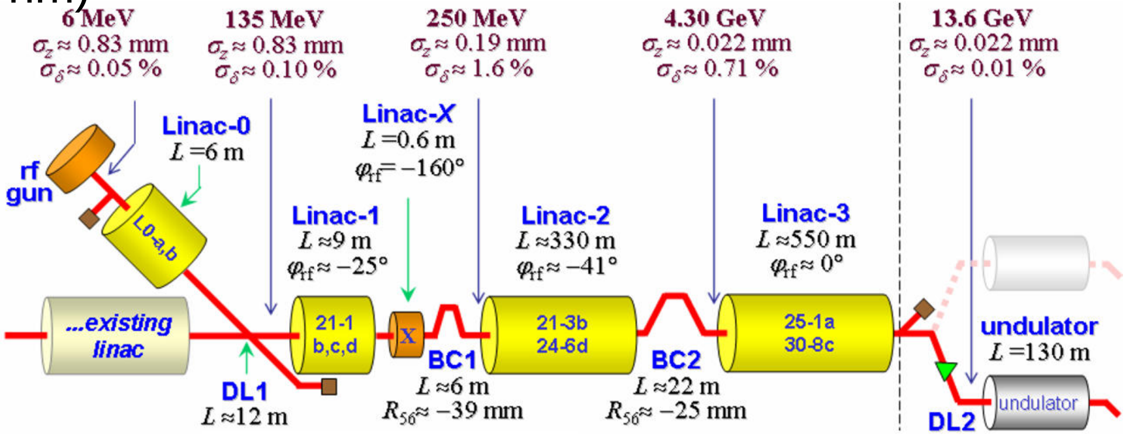
beam dynamics with space charge and CSR effects

# magnetic bunch compression - 2

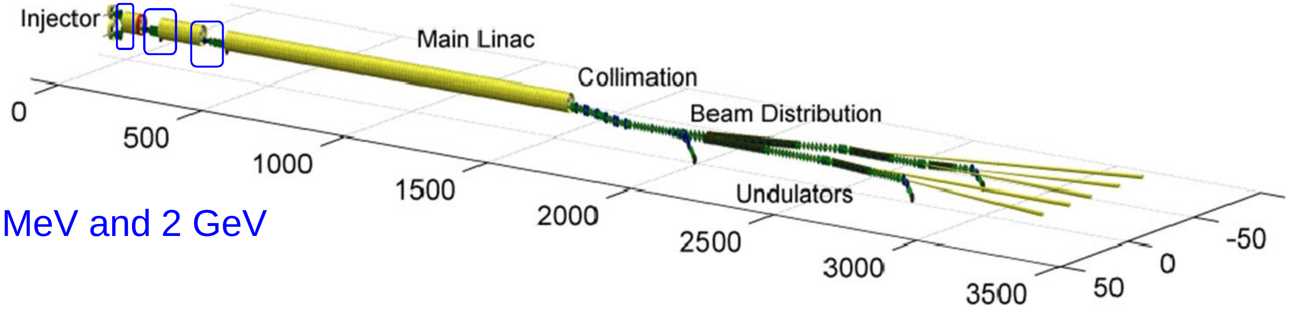
FLASH (1.2 GeV, 4 nm)



LCLS (14 GeV, 0.15 nm)

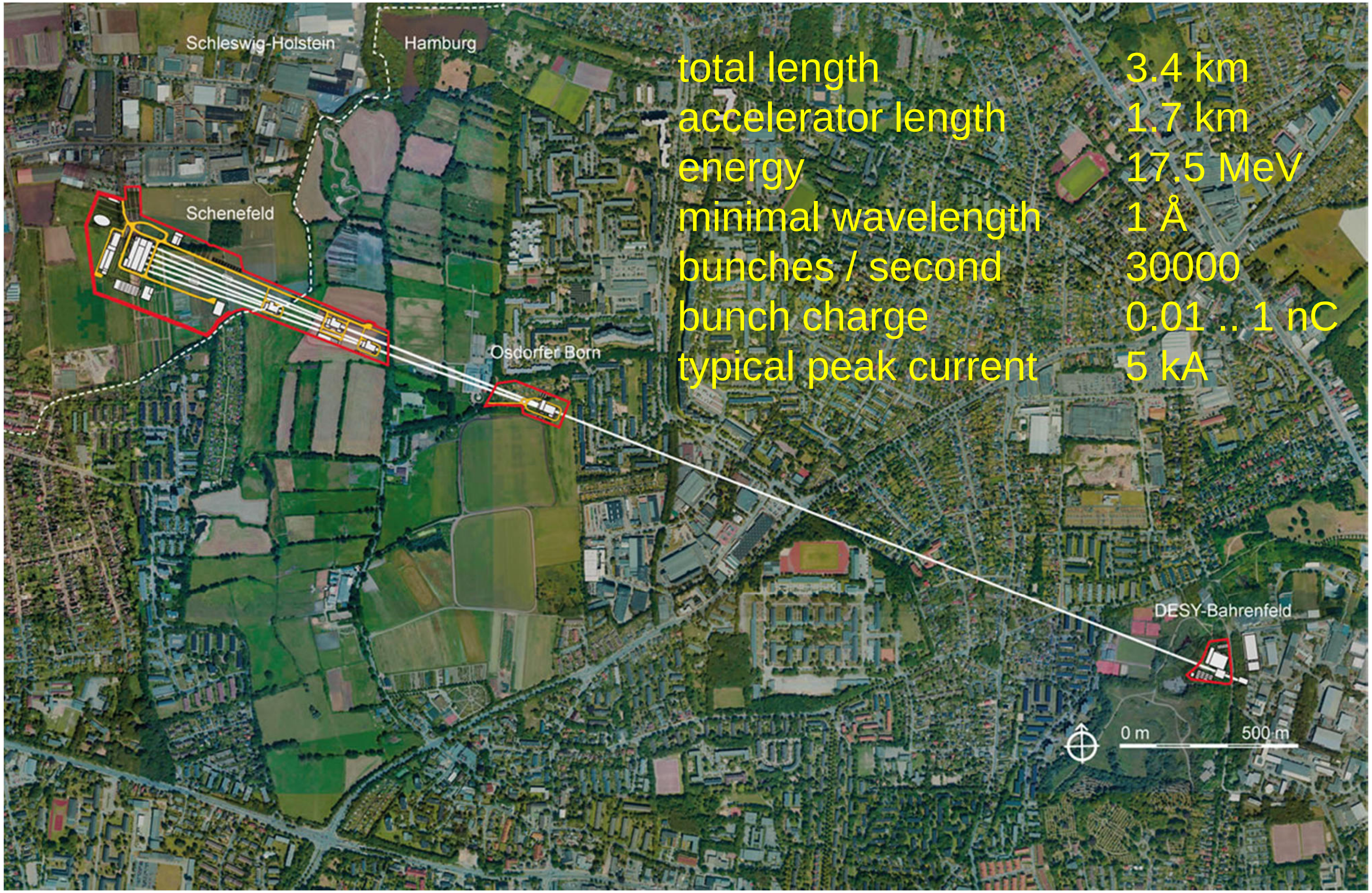


European XFEL (0.1 nm)



BCs at 130 MeV, 500 MeV and 2 GeV

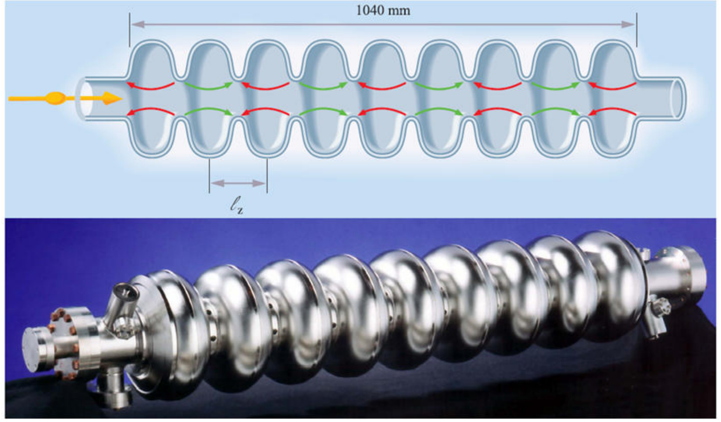
# European XFEL

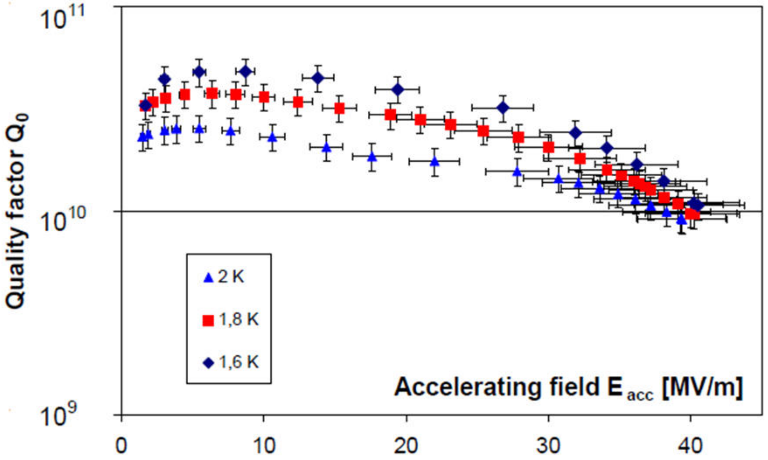


| | |
|---|---|
| total length | 3.4 km |
| accelerator length | 1.7 km |
| energy | 17.5 MeV |
| minimal wavelength | 1 Å |
| bunches / second | 30000 |
| bunch charge | 0.01 .. 1 nC |
| typical peak current | 5 kA |

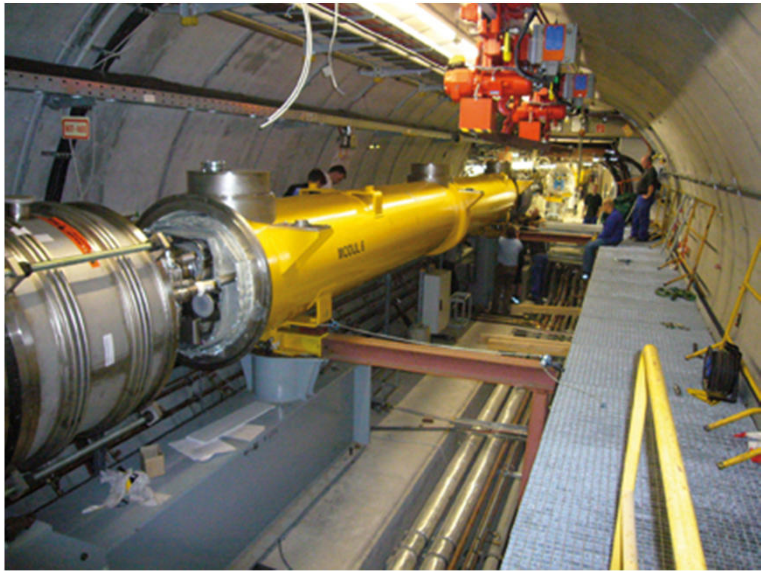
# European X-FEL - 2

superconducting cavity, 1.3 GHz $E_{\mathrm{acc}} \to$ 40 MeV/m
23.5 MeV/m are needed





FLASH tunnel: cryo module



undulator

# European X-FEL - 3

| | LCLS | SCSS | European XFEL |
|---|---|---|---|
| **Abbreviation for** | Linac Coherent Light Source | Spring-8 Compact SASE Source | European X-Ray Free-Electron Laser |
| **Location** | California, USA | Japan | Germany |
| **Start of commissioning** | 2009 | 2010 | 2014 |
| **Accelerator technology** | normal conducting | normal conducting | superconducting |
| **Number of light flashes per second** | 120 | 60 | 30 000 |
| **Minimum wavelength of the laser light** | 0.15 nanometres | 0.1 nanometres | 0.1 nanometres |
| **Maximum electron energy** | 14.3 billion electron volts (14.3 GeV) | 6-8 billion electron volts (6-8 GeV) | 17.5 billion electron volts (17.5 GeV) |
| **Length of the facility** | 3 Kilometer | 750 Meter | 3.4 Kilometer |
| **Number of undulators (magnet structures for light generation)** | 1 | 3 | 5 |
| **Number of experiment stations** | 3-5 | 4 | 10 |
| **Peak brilliance [photons / s / mm$^2$ / mrad$^2$/ 0.1% bandwidth]** | $8.5 \cdot 10^{32}$ | $5 \cdot 10^{33}$ | $5 \cdot 10^{33}$ |

multi bunch operation

beamlines