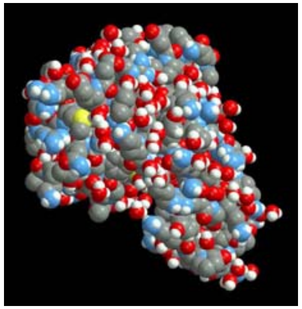
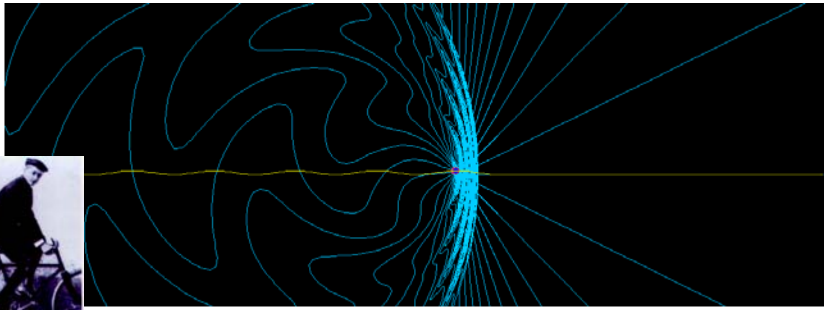
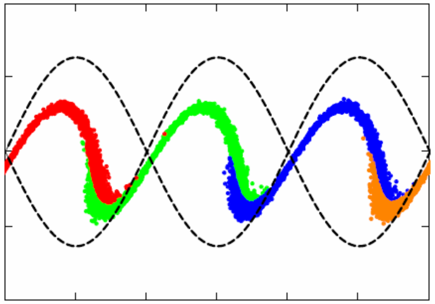
Free-Electron Laser

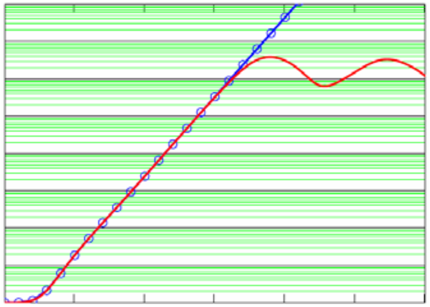
A) Motivation and Introduction



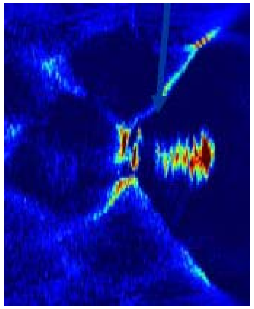


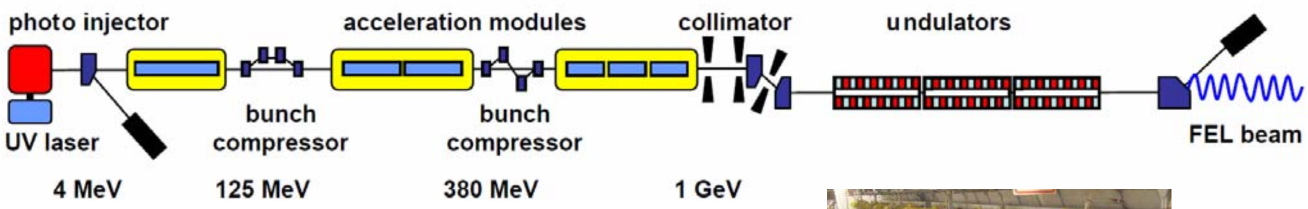
B) Theoretical Approach





C) Experimental Realization / Challenges





Free-Electron Laser

appendix for “experts”

Theoretical Approach: FEL

1 Introduction

2 Effects (decoupled considerations)

- 2.1 Particle Motion – Trajectory in Undulator
- 2.2 Continuous Source Distribution
- 2.3 Electromagnetic Fields
- 2.4 Particle Motion - Energy

3 Coupled Equations

- 3.1 Numerical Solution
- 3.2 Effects (now coupled)
- 3.3 Analytic Solution (Vlasov etc.)

A) Motivation and Introduction

Need for Short Wavelengths

Why FELs?

Free Electron ↔ Wave Interaction

Micro-Bunching

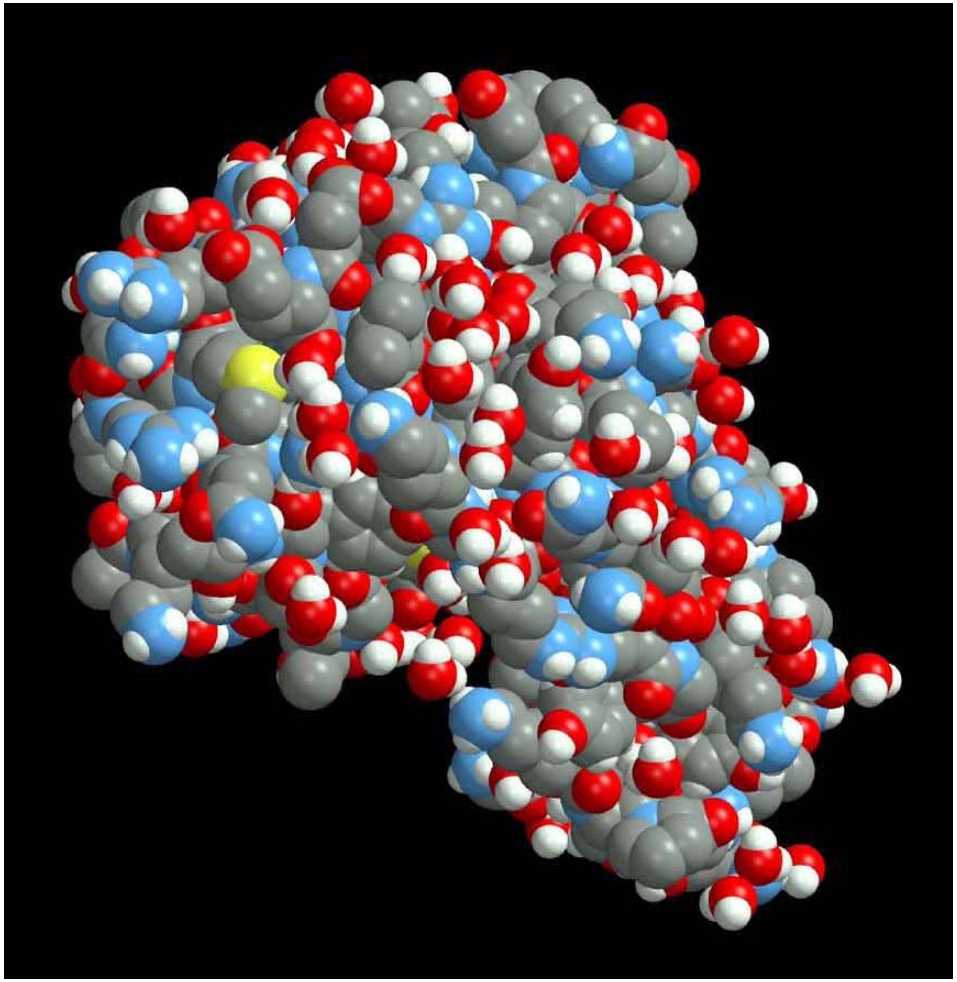
Amplifier and Oscillator

Self Amplifying Spontaneous Emission (SASE)

Why SASE?

Coherent Radiation

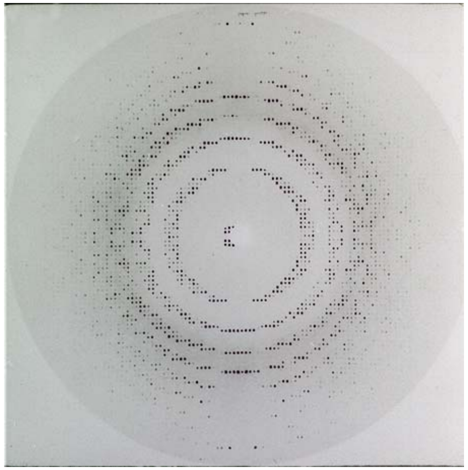
Need for Short Wavelengths



LYSOZYME MW=19,806

state of the art:
structure of biological macromolecule

reconstructed from diffraction pattern of protein crystal:



needs $\approx 10^{15}$ samples

crystallized → not in life environment

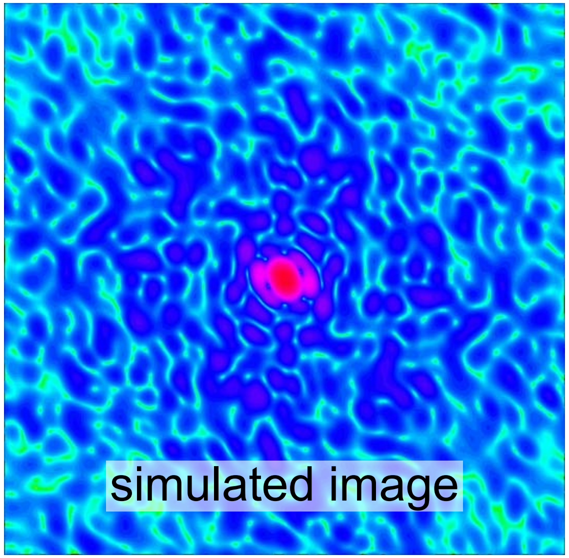
the crystal lattice imposes restrictions on molecular motion

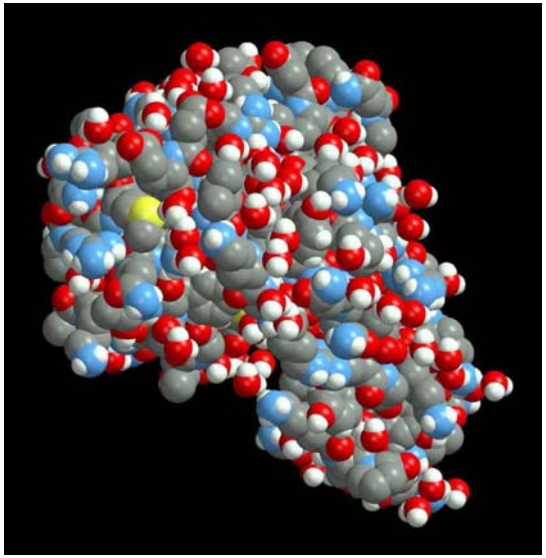
images courtesy Janos Hajdu, slide from Jörg Rossbach

Need for Short Wavelengths - 2

SINGLE
MACROMOLECULE

courtesy Janos Hajdu





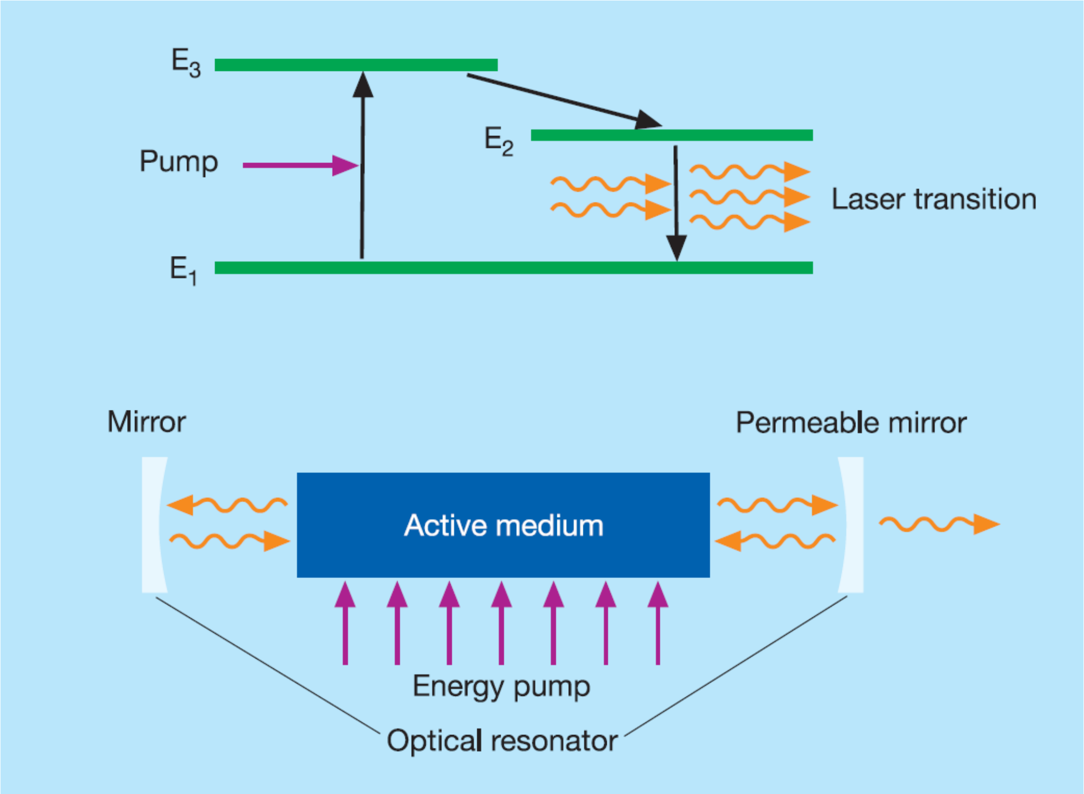
resolution does not depend on sample quality
needs very high radiation power @ λ ≈1Å
can see dynamics if pulse length < 100 fs

we need a radiation source with

- very high peak and average power
- wavelengths down to atomic scale λ ~ 1Å
- spatially coherent
- monochromatic
- fast tunability in wavelength & timing
- sub-picosecond pulse length

Why FELs?

principle of a quantum laser



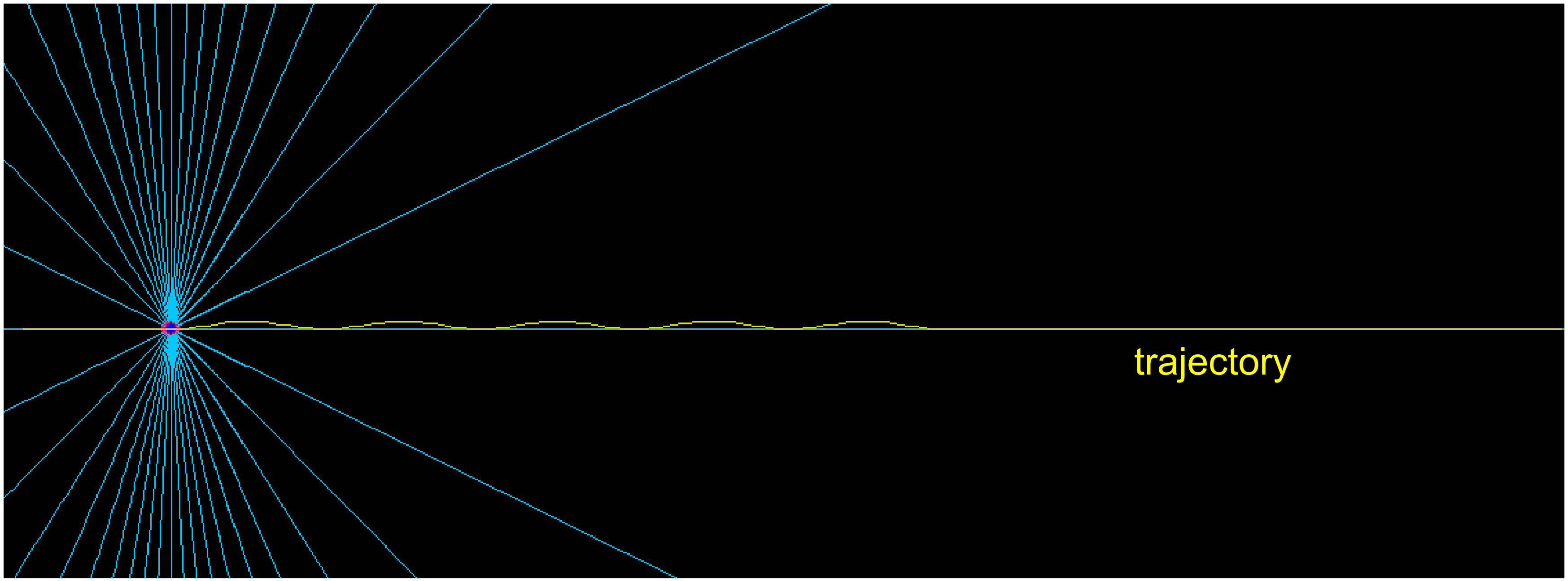
problem & solution:

active medium → free electron – EM wave interaction

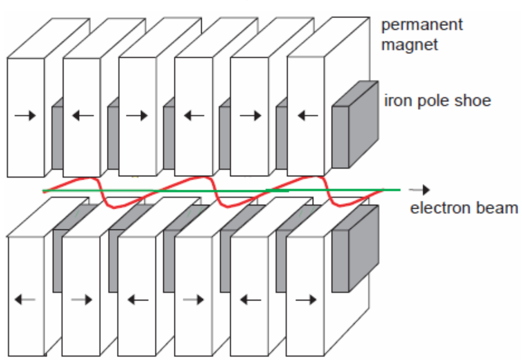
Free Electron → Wave Interaction

free electron in uniform motion + electric field lines

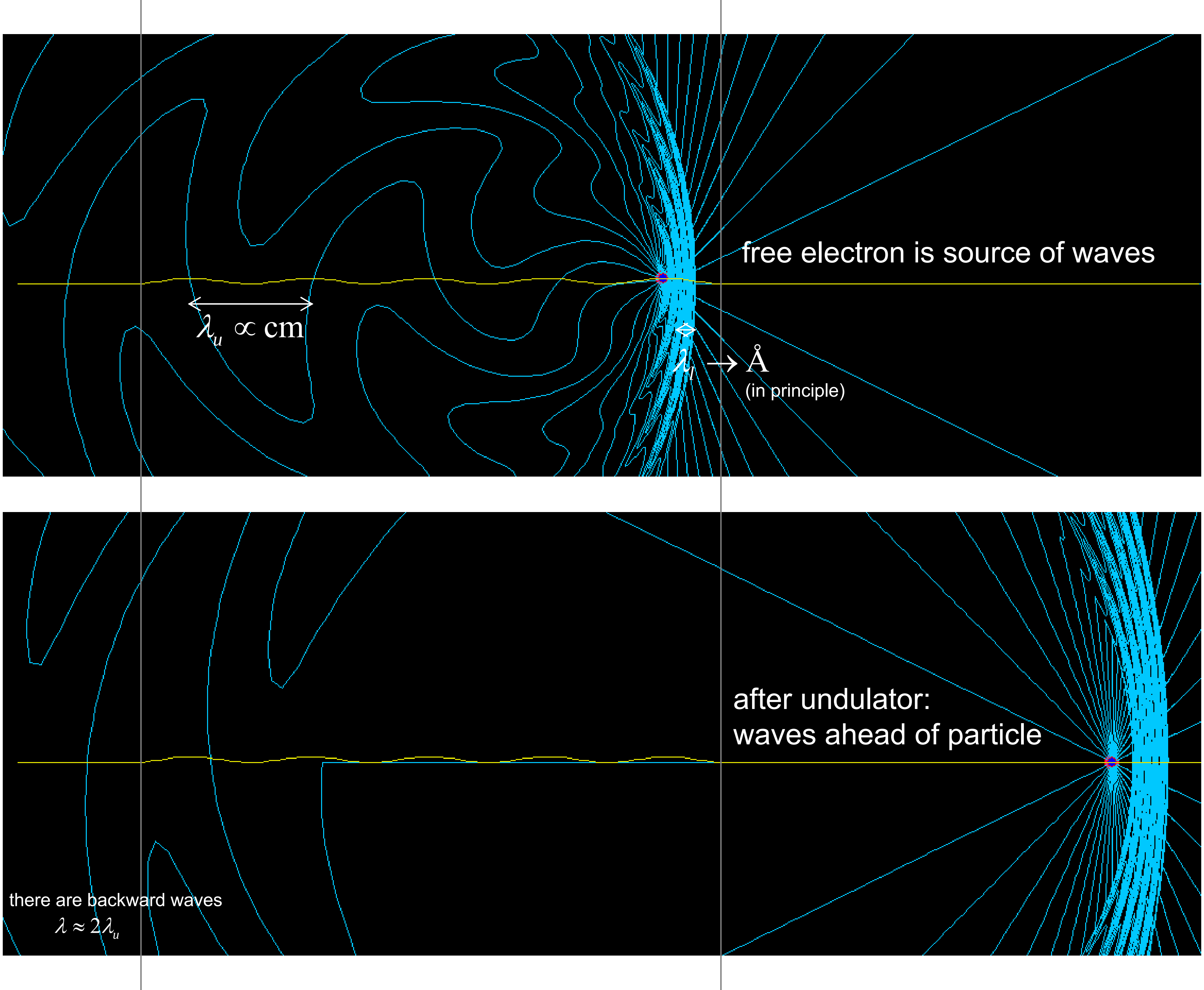
(before undulator)



courtesy T. Shintake
http://www.shintakelab.com/en/enEducationalSoft.htm



undulator



free electron is source of waves
$\lambda_u \propto$ cm
$\lambda_l \to$ Å
(in principle)
after undulator:
waves ahead of particle
there are backward waves
$\lambda \approx 2\lambda_u$

Free Electron $\leftrightarrow$ Wave Interaction

- electrons $\rightarrow$ wave

  Maxwell theory …

  z = position in undulator

$$\frac{d}{dz}\hat{E} = -\frac{\mu c}{2}\hat{J}$$

amplitude of bunched current

amplitude of EM wave

- wave $\rightarrow$ electron

  equation of motion

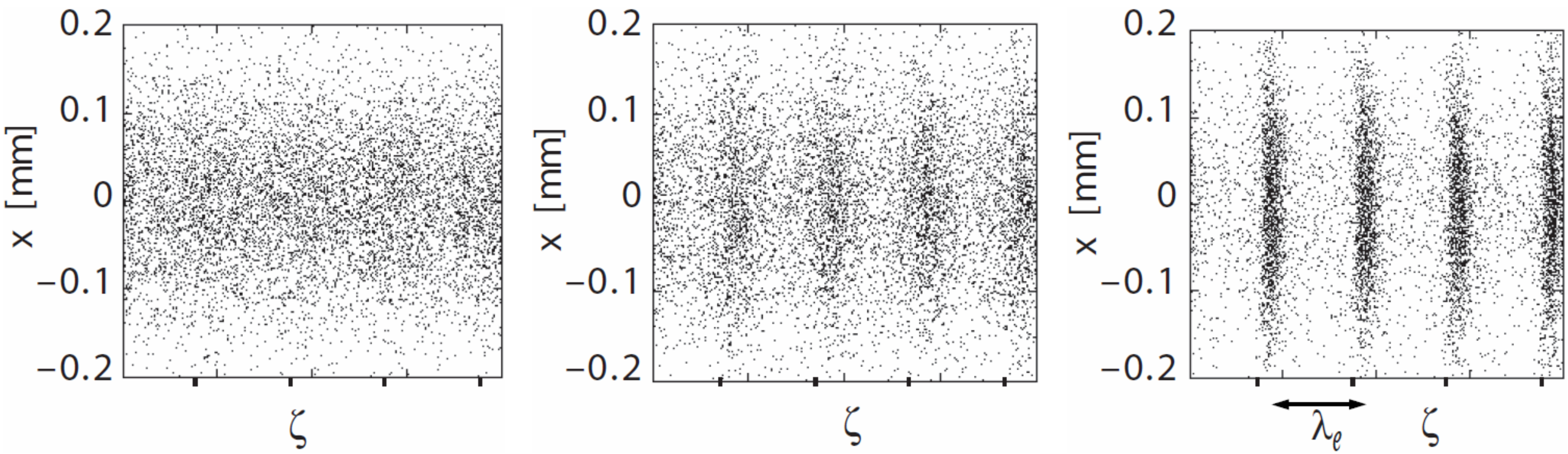
  change of kinetic energy
  of particle

$$\frac{dW}{dt} = -e\mathbf{v}(t)\cdot\mathbf{E}(\mathbf{r}(t),t)$$

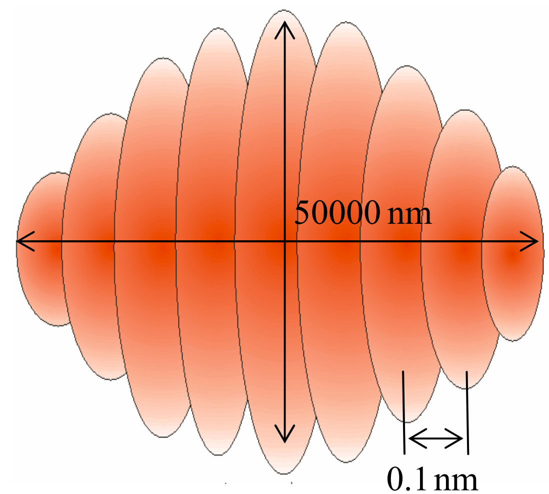
change of averaged longitudinal velocity

- change of longitudinal micro structure “micro bunching”

Micro-Bunching



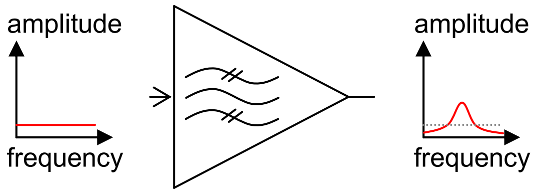
- longitudinal motion to 1st order is trivial, **but**
- micro-bunching is a 2nd order effect
  → coupled theory of particle motion and
  wave generation
- transverse bunch structure is much larger
  than longitudinal sub-structure
  → 1d theory with plane waves

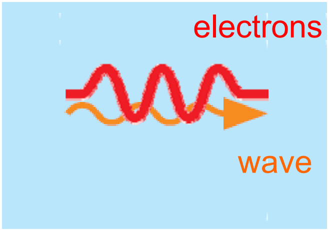


Amplifier and Oscillator

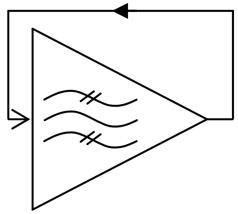
in principle | FEL

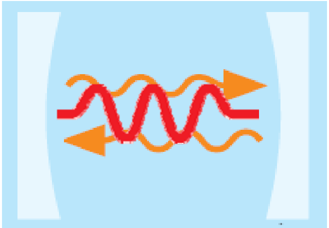
- amplifier:





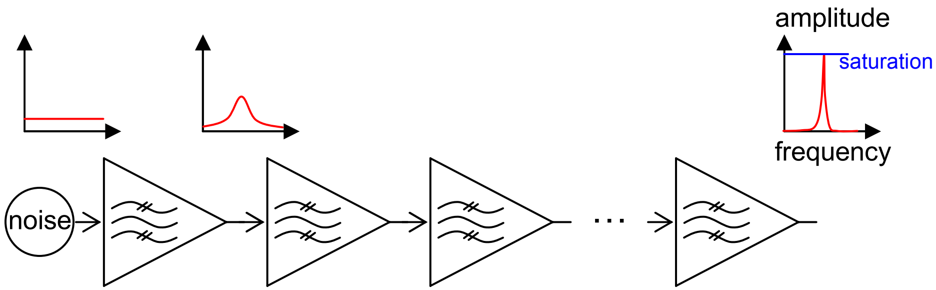
- oscillator:

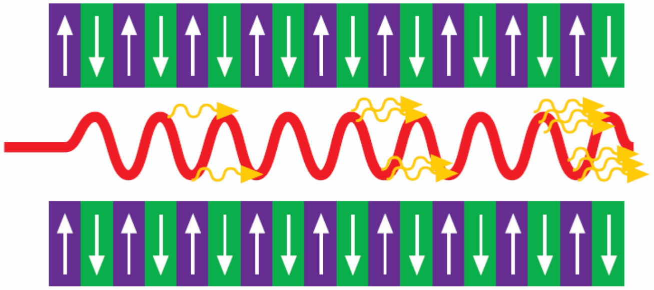




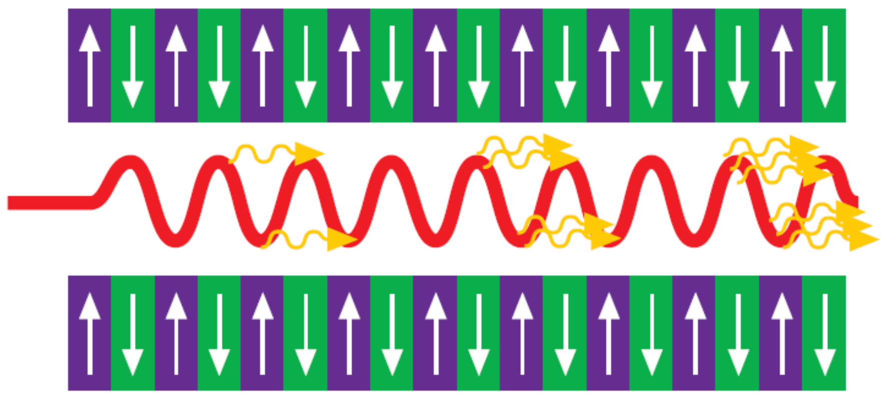
instability, driven by noise, growth until amplifier saturates

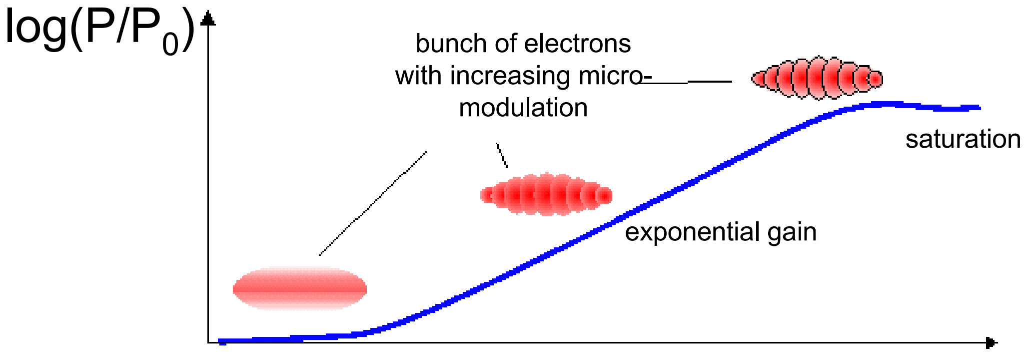
- amplified noise:





Self Amplifying Spontaneous Emission (SASE)



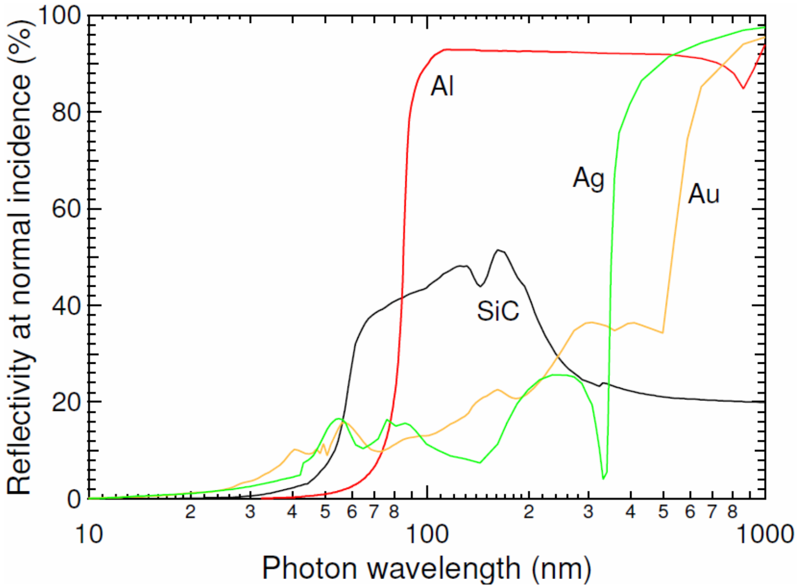


- uniform **random** distribution of particles at entrance
- **incoherent** emission of EM waves (noise, wide bandwidth)
- amplification ($\rightarrow$ resonant wavelength, micro-bunching)
- saturation, full **micro modulation**, **coherent** radiation

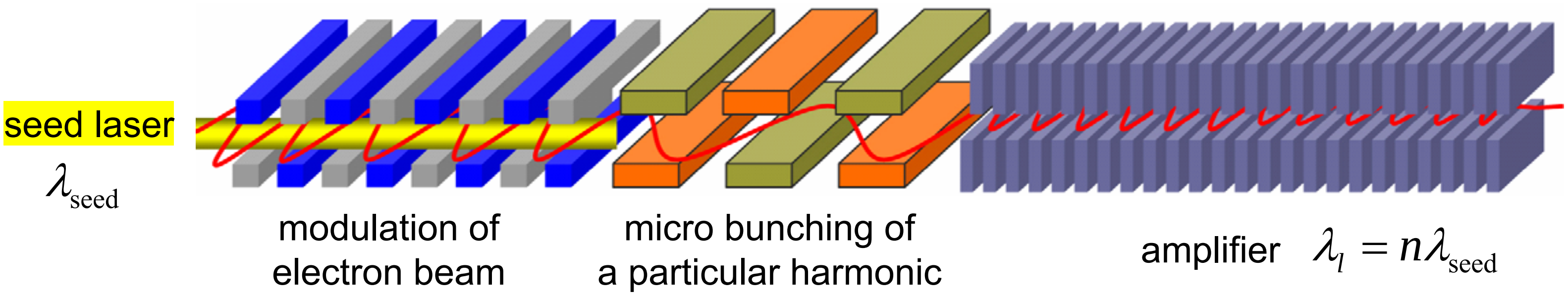
Why SASE?

- oscillator needs resonator

  but there are no mirrors for wavelengths < 100 nm



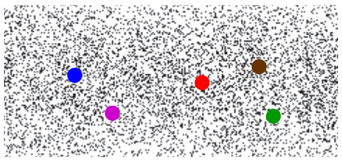
- alternative: seed laser + harmonic generation + amplifier



Coherent Radiation

electron in undulator $\rightarrow$ plane wave in far field

incoherent superposition of plane waves:



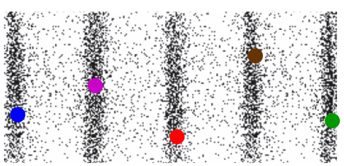
φ_1 φ_2 φ_3

$$E = \sum_{\nu=1}^{N} \cos\left(k_l\left(z - ct\right) + \psi_\nu\right) = E_\Sigma \cos\left(k_l\left(z - ct\right) + \psi_\Sigma\right)$$

field amplitude: $E_\Sigma = \left|\sum_{\nu=1}^{N} \exp\left(i\varphi_\nu\right)\right| \propto \sqrt{N}$

radiated power: $P_\Sigma \propto E_\Sigma^2 \propto N$

coherent superposition of plane waves:

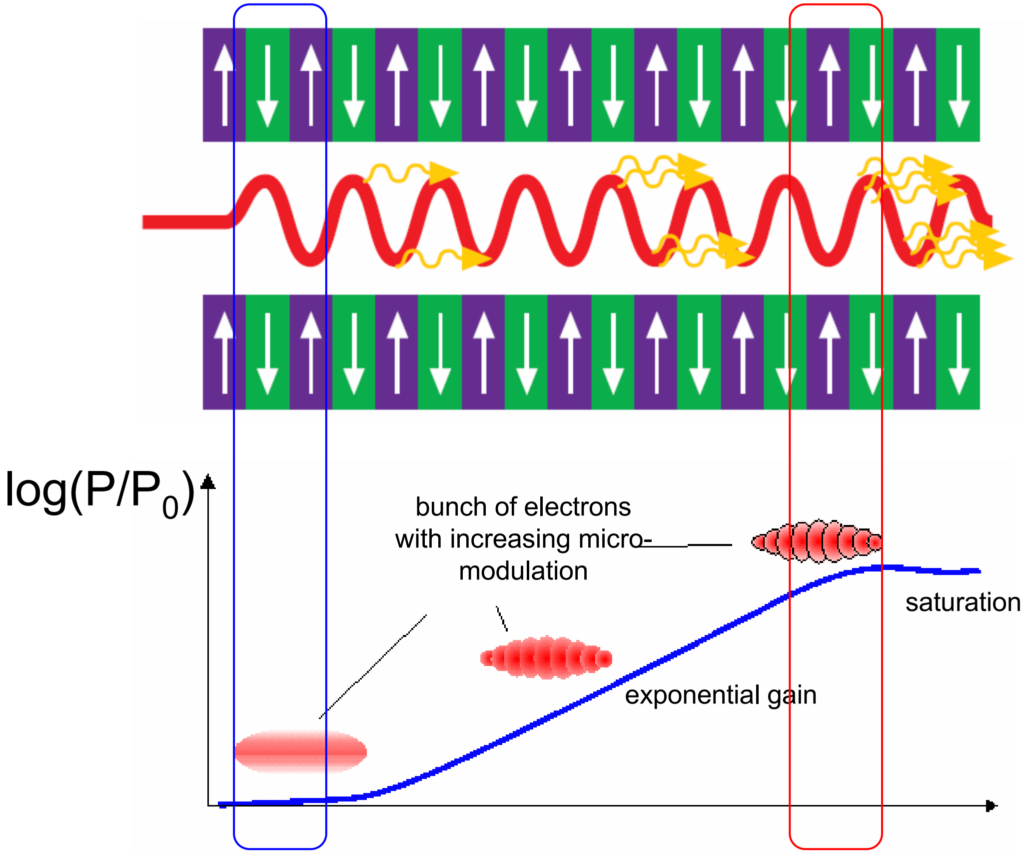


φ_1 φ_2 φ_3

field amplitude: $E_\Sigma = \left|\sum_{\nu=1}^{N} \exp\left(i\varphi_\nu\right)\right| \approx N$

radiated power: $P_\Sigma \propto E_\Sigma^2 \approx N^2$

Coherent Radiation - 2



no micro-bunching
(only shot noise)
$P_0 \propto N$

saturation:
full micro-bunching
$P \propto N^2$

with N = particles per λ_l

B) Theoretical Approach

Wave $\rightarrow$ Free Electron interaction

Resonance Condition

Particle Energy and Ponderomotive Phase

Longitudinal Equation of Motion

Low Gain Theory

FEL Gain

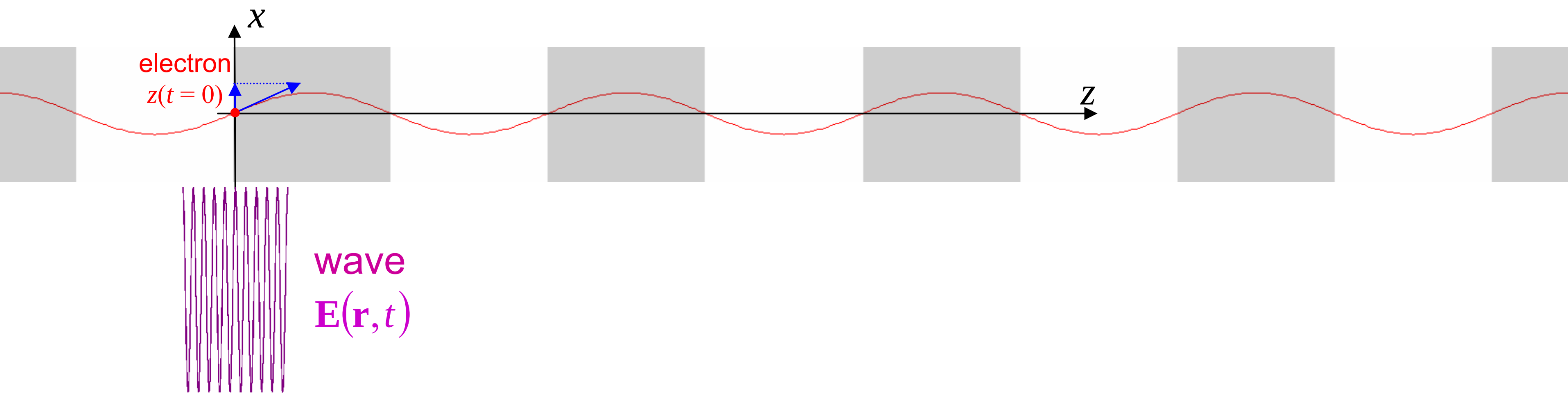
Micro-Bunching

High Gain Theory

Continuous Phase Space

Gain Length

Wave $\rightarrow$ Free Electron Interaction



undulator period λ_u $\quad$ $$k_u = \frac{2\pi}{\lambda_u}$$

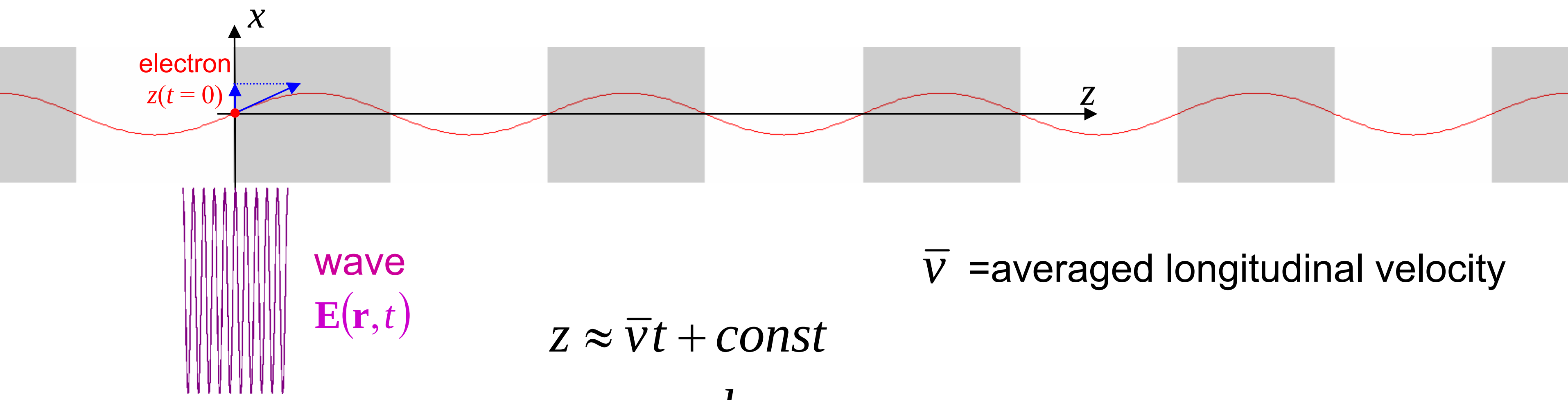
wave period period λ_l $\quad$ $$k_l = \frac{2\pi}{\lambda_l} = \frac{\omega_l}{c}$$

wave number and angular frequency

change of electron energy $\quad$ $$\frac{dW}{dt} = -e\mathbf{v}(t) \cdot \mathbf{E}(\mathbf{r}(t), t)$$

undulator trajectory

Free Electron ← Wave Interaction - 2



$\bar{v}$ =averaged longitudinal velocity

$$z \approx \bar{v}t + const$$

$$v_x \sim \cos k_u z$$

$$E_x \sim \cos\{k_l(z - ct) + \varphi\}$$

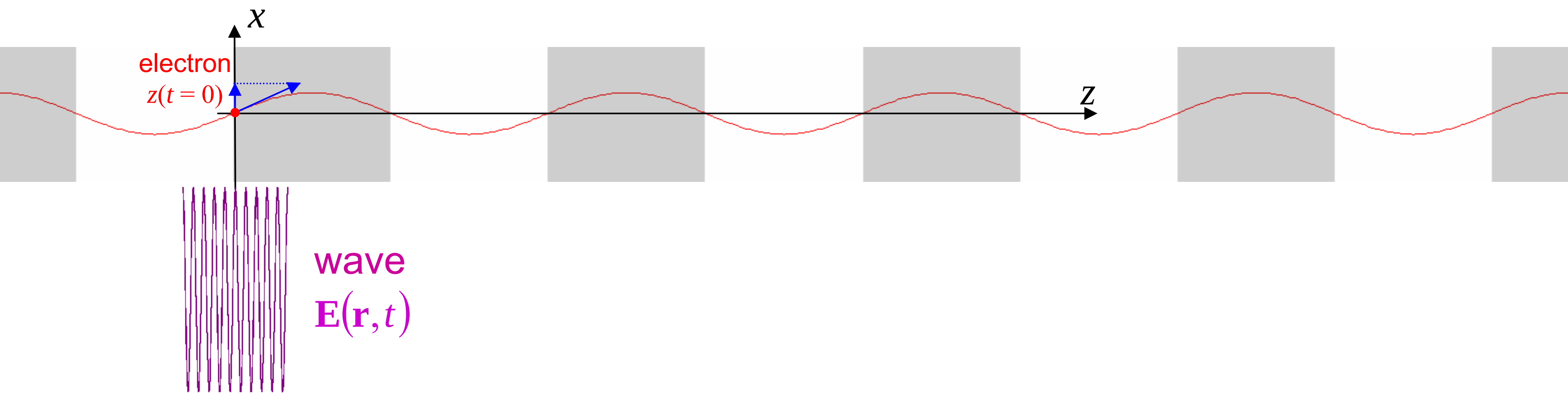
$$\frac{dW}{dt} \sim \cos k_u z \;\; \cos\{k_l(z-ct)+\varphi\} = \frac{1}{2}\cos\{k_u z + k_l(z-ct)+\varphi\} + \frac{1}{2}\cos\{k_u z - k_l(z-ct)-\varphi\}$$

$$\cos\{k_u z + k_l(z-ct)+\varphi\} = \cos\{\underbrace{(k_u + k_l)\bar{v}t - k_l ct}_{0} + \underbrace{const}_{\psi}\}$$

slippage condition: 0

$$k_u\bar{v} = k_l(c - \bar{v})$$

Free Electron ← Wave Interaction - 3



with slippage condition:

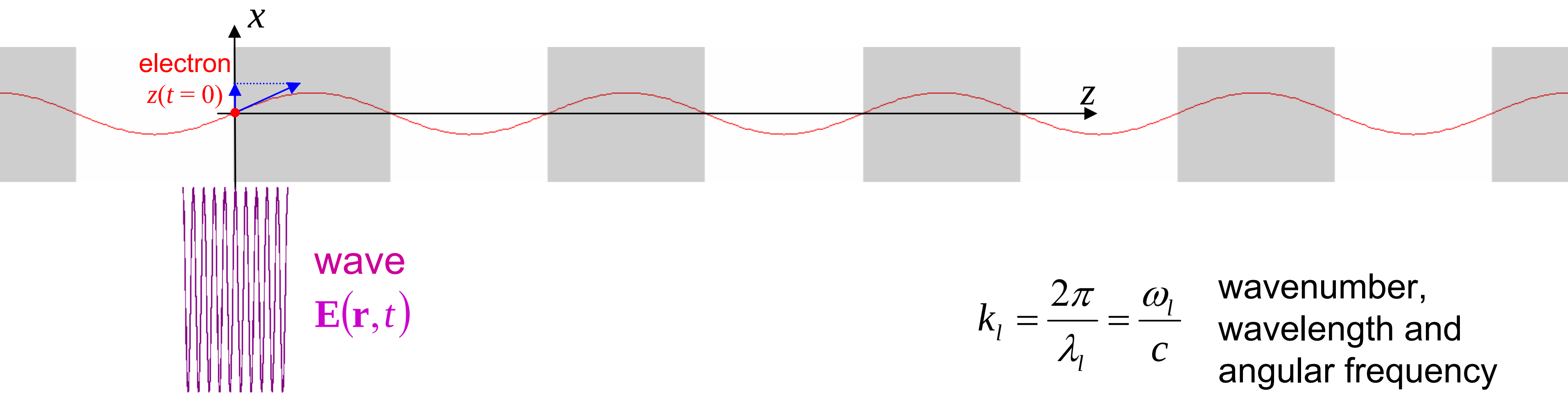
$$\frac{dW}{dt} \sim \frac{1}{2}\cos\psi + \frac{1}{2}\cos\{2k_u z + \xi\}$$

averaged versus one undulator period:

$$\frac{dW}{dt} \sim \frac{1}{2}\cos\psi$$

systematic gain or loss of particle energy

Resonance Condition



$$k_l = \frac{2\pi}{\lambda_l} = \frac{\omega_l}{c}$$ wavenumber, wavelength and angular frequency

- systematic gain or loss of particle energy if

$$k_u \bar{v} = k_l (c - \bar{v})$$

- equation of motion in undulator (without wave) $\rightarrow$

$$x(z) \approx \frac{K_p}{k_u \gamma} \sin k_u z \qquad z(t) \approx \bar{v} t$$

with $K_p = \dfrac{q_0 |B_0|}{m_0 k_u c}$ undulator parameter (~ 1)

and $\dfrac{\bar{v}}{c} \approx 1 - \dfrac{1}{2\gamma^2}\left(1 + \dfrac{K_p^2}{2}\right)$ γ =Lorentz factor (~ energy)

- resonance condition

$$\lambda_l = (c - \bar{v}) \frac{\lambda_u}{c} \approx \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K_p^2}{2}\right)$$

Particle Energy and Ponderomotive Phase

- resonance condition $\rightarrow$ resonant energy

$$\lambda_l = \frac{\lambda_u}{2\gamma_{\text{res}}^2}\left(1+\frac{K_p^2}{2}\right)$$

- if resonance condition fulfilled: $\left\langle \frac{dW}{dt} \right\rangle \propto -\hat{E}\cos\psi$ with ψ = const

ponderomotive phase

$\psi = 0 \rightarrow$ kinetic energy transfer EM $\rightarrow$ wave "laser"
$\psi = \pi \rightarrow$ transfer EM wave $\rightarrow$ kinetic energy "accelerator"

- if resonance condition is not fullfilled: $\gamma \neq \gamma_{\text{res}}$

particle slips in one period by $\lambda_l - \zeta = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K_p^2}{2}\right)$

change of ponderomotive phase $\Delta\psi = k_l \zeta$

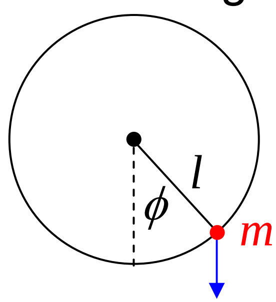
Longitudinal Equation of Motion
(in average)

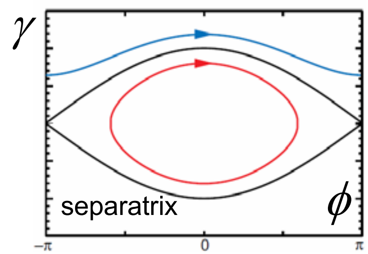
- longitudinal phase space

$$\frac{d\psi}{dt} \propto \gamma - \gamma_{\text{res}} \qquad\qquad \frac{d\gamma}{dt} \propto -\hat{E}\cos\psi$$

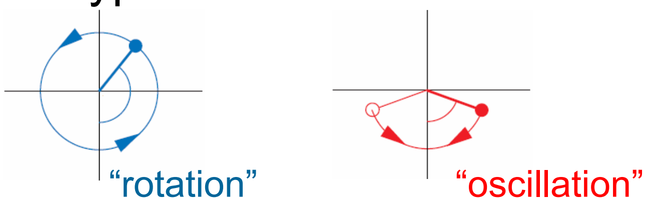
longitudinal position $\rightarrow \psi$
energy $\rightarrow \gamma$

- equations are analog to mathematical pendulum





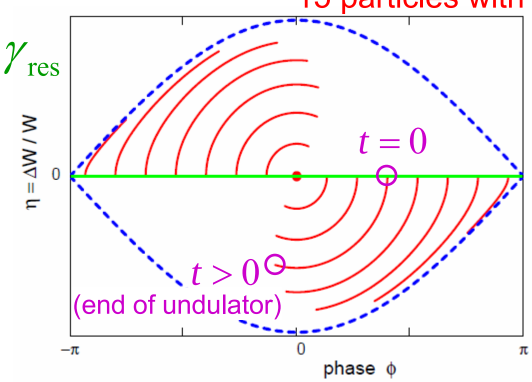
two types of solution:



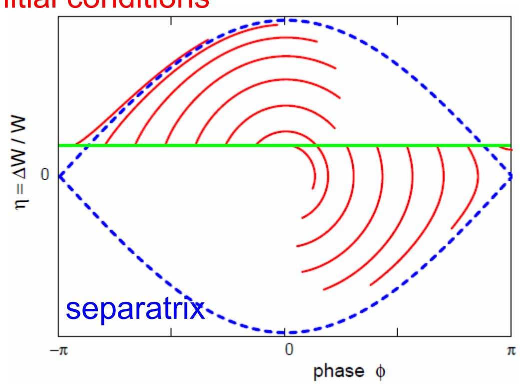
- trajectories in phase space

15 particles with different initial conditions

$\gamma_n(0) = \gamma_{\text{res}}$



$\gamma_n(0) > \gamma_{\text{res}}$



Low Gain Theory

- neglect change of field amplitude

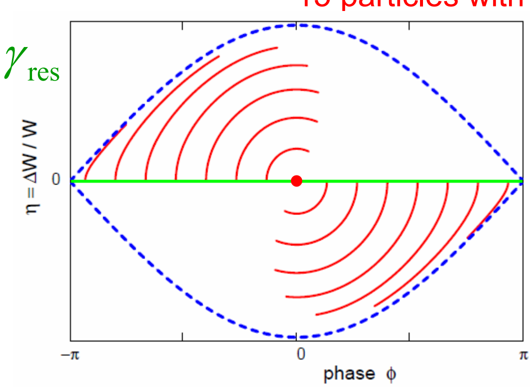
$$\frac{d\psi}{dt} \propto \gamma - \gamma_{\mathrm{res}} \qquad\qquad \frac{d\gamma}{dt} \propto -\boxed{\hat{E}}\cos\psi$$

- indirect gain calculation

$$G = \frac{\text{gain of field energy}}{\text{initial field energy}} = \frac{\text{loss of particle energy}}{\text{initial field energy}} = \frac{W_\Sigma(\text{in}) - W_\Sigma(\text{out})}{\text{initial field energy}}$$

15 particles with different initial conditions

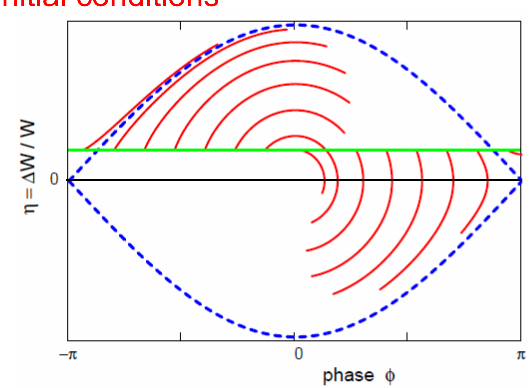
$\gamma_n(0) = \gamma_{\mathrm{res}}$



$W_\Sigma(\text{in}) = W_\Sigma(\text{out})$

$G = 0$

$\gamma_n(0) > \gamma_{\mathrm{res}}$



$W_\Sigma(\text{in}) < W_\Sigma(\text{out})$

$G > 0$

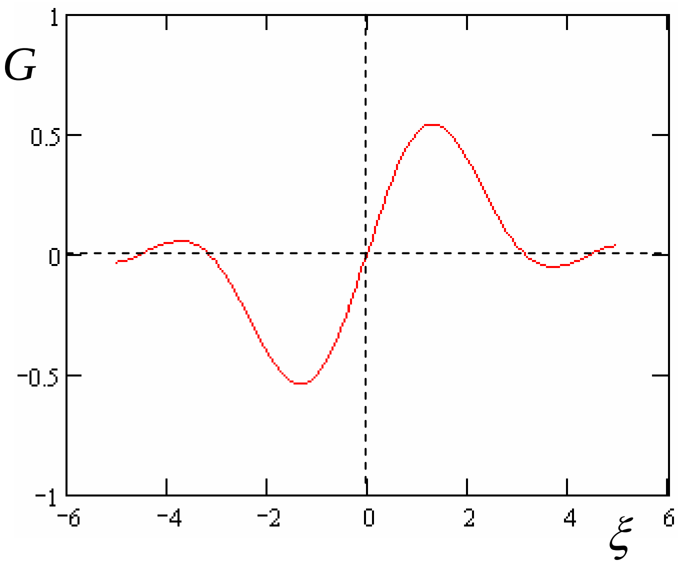
FEL Gain

- analytical analysis →

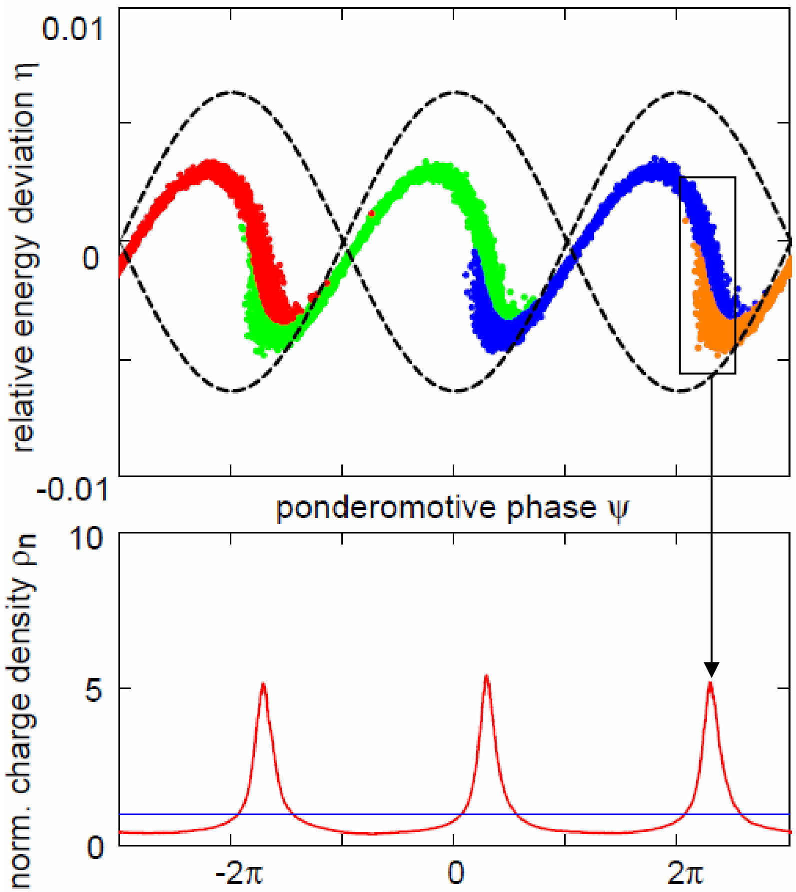
$$G \propto -\frac{d}{d\xi}\left(\frac{\sin^2 \xi}{\xi^2}\right)$$

$$\xi \propto \boxed{M} \frac{\gamma - \gamma_{\text{res}}}{\gamma_{\text{res}}}$$

periods of undulator



Micro-Bunching



- Fourier analysis of longitudinal particles positions
  → amplitude of micro modulation

$$\hat{I} \propto \sum \exp(-i\psi_{\nu})$$

(fundamental mode)

High Gain Theory

- logitudinal position in undulator $z = \bar{v}t \approx ct$
- set of equations:

particles

$$\frac{d\psi}{dz} \propto \gamma - \gamma_{\mathrm{res}} \qquad \frac{d\gamma}{dz} \propto -\mathrm{Re}\left\{\hat{E}(z)\exp(i\psi)\right\}$$

amplitude (and phase) of EM wave

bunching

$$\hat{I} \propto \sum \exp(-i\psi_\nu)$$

amplitude

$$\frac{d}{dz}\hat{E} \propto \hat{I}$$

(from Maxwell equations)

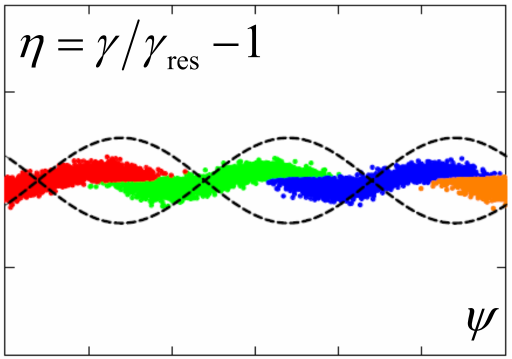
this set of equations + field equations can be solved numerically

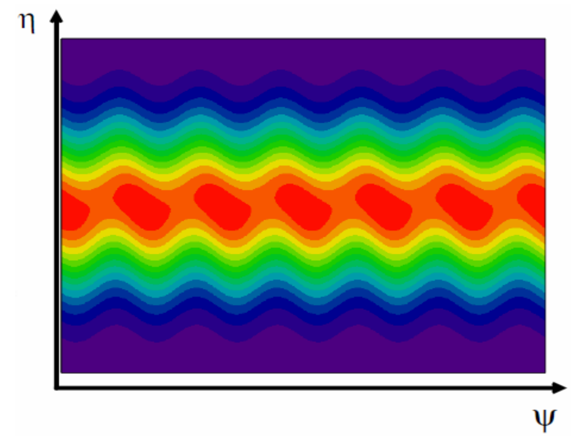
- FEL codes include transverse motion and 3D EM field calculation

Continuous Phase Space

- phase space distribution

  many point particles ψ_n, γ_n $\rightarrow$ continuous density distribution $F(z,\psi,\eta)$





charge density $\lambda(z,\psi) = \int d\eta \times F(z,\psi,\eta)$

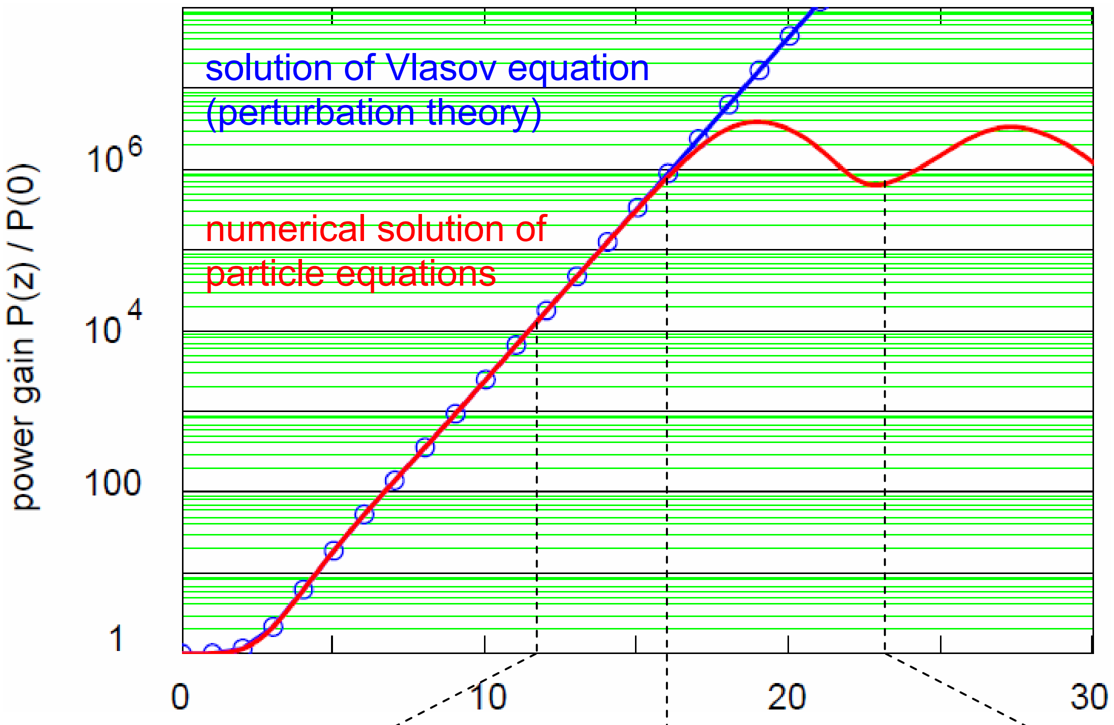
bunching $\hat{I} \propto \int d\psi \times \lambda(z,\psi) e^{-i\psi}$

- Vlasov equation

$$\frac{dF}{dz} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dz} + \frac{\partial F}{\partial \eta}\frac{d\eta}{dz} = 0$$

$$\frac{d\psi}{dz} \propto \gamma - \gamma_{\text{res}} \qquad \frac{d\eta}{dz} \propto -\text{Re}\left\{\hat{E}(z)\exp(i\psi)\right\}$$

Gain Length

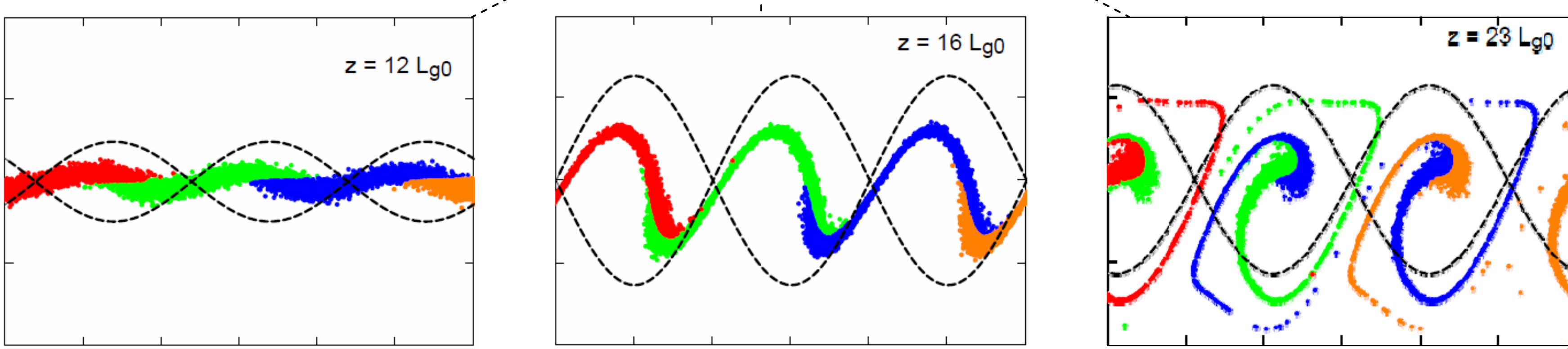


} saturation

$$P \propto \exp\left(z/L_g\right)$$

$$L_g \propto \gamma \left(\frac{\sigma_r^2}{I} \right)^{1/3}$$

beam current



C) Experimental Realization / Challenges

Linac Coherent Light Source - LCLS

Scales

Challenges

RF Gun

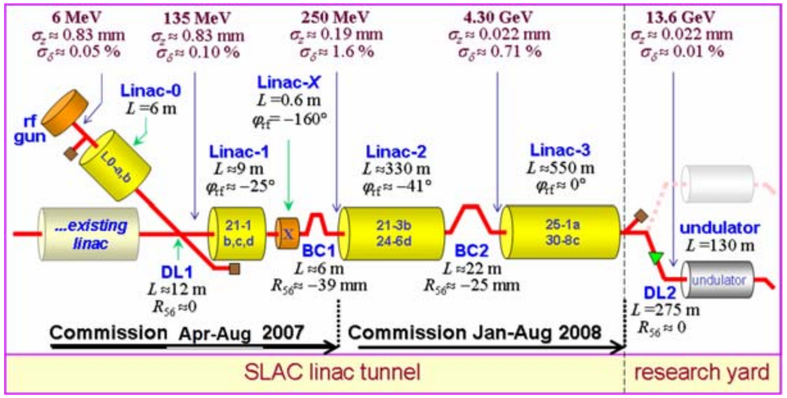
Bunch Compression

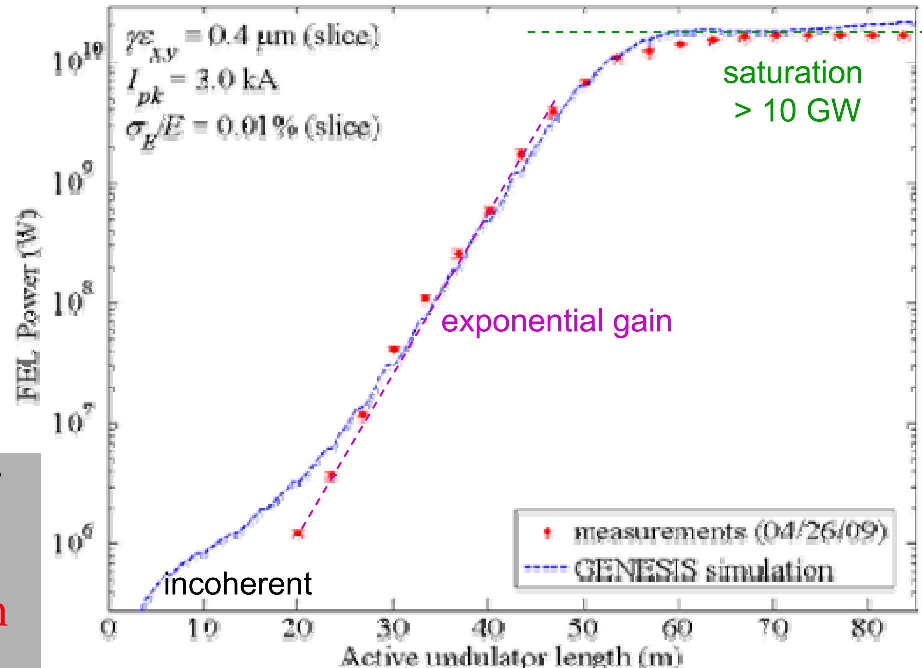
European X-FEL

Table Top FEL

Linac Coherent Light Source- LCLS

SLAC mid-April 2009 – first lasing at 1.5 Å





injector	linac & bunch compression	undulator
	$L \propto 10^3\,\mathrm{m}$	$L \propto 100\,\mathrm{m}$
	$E = 13.6\,\mathrm{GeV}$	$\lambda_u \approx 3\,\mathrm{cm}$
$q \approx 0.25\,\mathrm{nC}$	$I = 3\,\mathrm{kA}$	$L_g \approx 3.3\,\mathrm{m}$
		$\lambda_l = 1.5\,\text{Å}$

particles per λ_l

$$N = \frac{I\lambda_l}{ec} \approx 10^4$$

Scales

photon wavelength	$\lambda_l \propto 10^{-10}\,\mathrm{m} \quad \propto \lambda_u / \gamma^2$	
cooperation length	$L_l \propto 10^{-8}\,\mathrm{m}$	
transverse oscillation	$\hat{x} \propto 10^{-6}\,\mathrm{m}$	(undulator trajectory)
bunch length	$L_b \propto 10^{-5}\,\mathrm{m}$	
bunch width	$\sigma_w^{\mathrm{bunch}} \propto 10^{-5}\,\mathrm{m}$	width of photon beam $\sigma_w^{\mathrm{wave}} \propto \sqrt{\lambda_l L_R}$
undulator period	$\lambda_u \propto 10^{-2}\,\mathrm{m}$	
power gain length	$L_g \approx 1..10\,\mathrm{m}$	
Rayleigh length	L_R	(scale of widening of photon beam)
saturation length	$L_s \approx 10L_g..20L_g < L_u$	
undulator length	$L_u \propto 100\,\mathrm{m}$	
length with linac	$L \propto 10^3\,\mathrm{m}$	

Scales

photon wavelength	$\lambda_l \propto 10^{-10}\,\mathrm{m} \quad \propto \lambda_u / \gamma^2$	
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Rayleigh length	L_R	(scale of widening of photon beam)
saturation length	$L_s \approx 10 L_g\,..\,20 L_g < L_u$	
undulator length	$L_u \propto 100\,\mathrm{m}$	
total length	$L \propto 10^3\,\mathrm{m}$	

overlap of particle beam
with photon beam

Challenges

$$\lambda_l = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$$L_g = \frac{1}{\sqrt{3}}\left(\frac{4mc}{\mu e}\frac{\gamma^3\lambda_u}{K^2}\boxed{\frac{\sigma_r^2}{I}}\right)^{1/3}$$

$$\sigma_r^2 \propto \lambda_l L_g$$

(undulator parameter K $\propto$ 1)

- $\lambda_l \rightarrow$ Å
- Energy $\rightarrow$ 10 .. 20 GeV
- gain length L_g <~ 10 m
- high peak current >~ kA
- transverse beam size $\sigma_r \propto 10\ \mu\text{m}$
- energy spread
- overlap electron-photon beam

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transverse: generate low emittance beam
preservation of emittance

longitudinal: compression

acceleration

diagnostic and steering

undulator alignment

$$\boxed{\frac{\sigma_r^2}{I}} = \frac{\boxed{\sigma_r^2 L_b}}{\boxed{q}c}$$

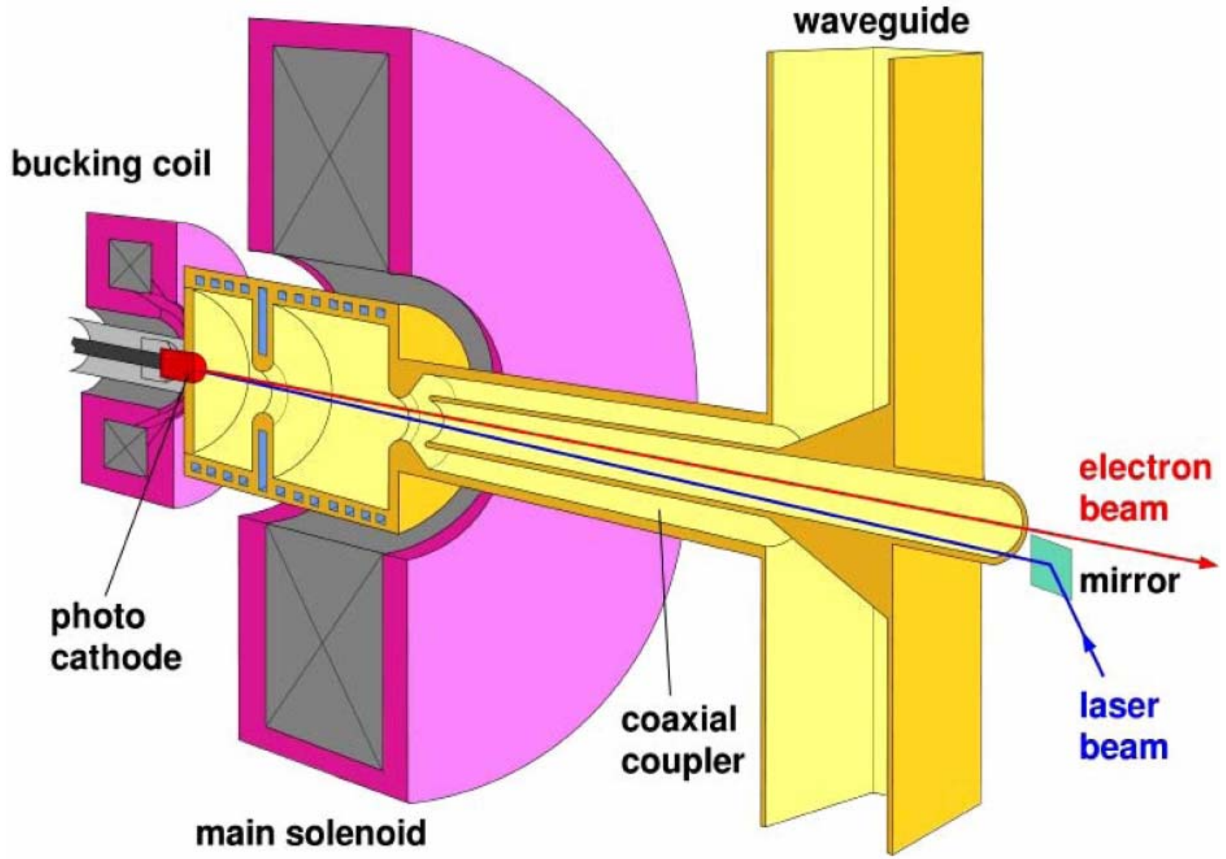
volume

bunch charge

space charge forces:

$$E_{sq} \propto \frac{1}{\gamma^2}\frac{q}{\sigma_r^2}$$

RF Gun



typical parameters of FLASH & European XFEL:

$$q \propto 1\,\mathrm{nC} \qquad E \propto 5\,\mathrm{MeV} \qquad I \propto 50\,\mathrm{A}$$

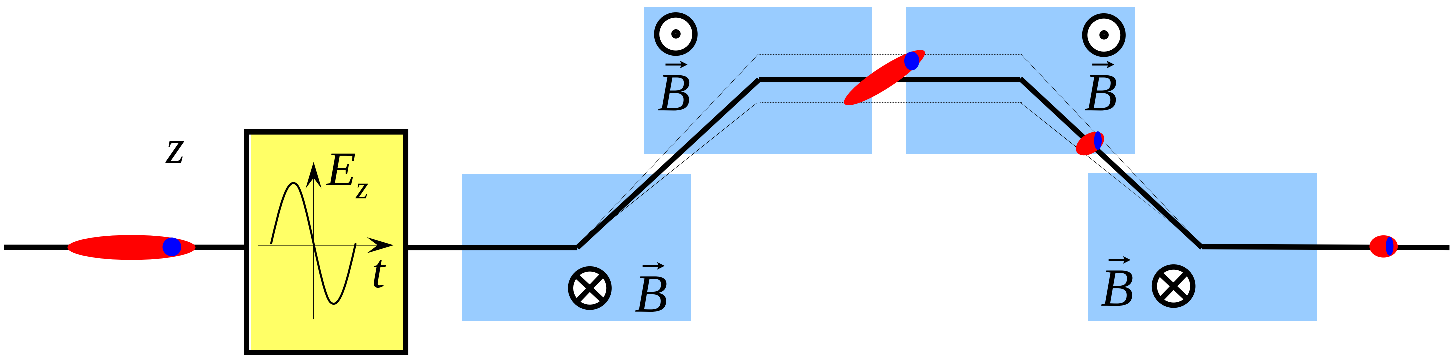
$$\gamma \propto 10$$

longitudinal compression 1 $\rightarrow$ 0.01 needed !

Magnetic Bunch Compression

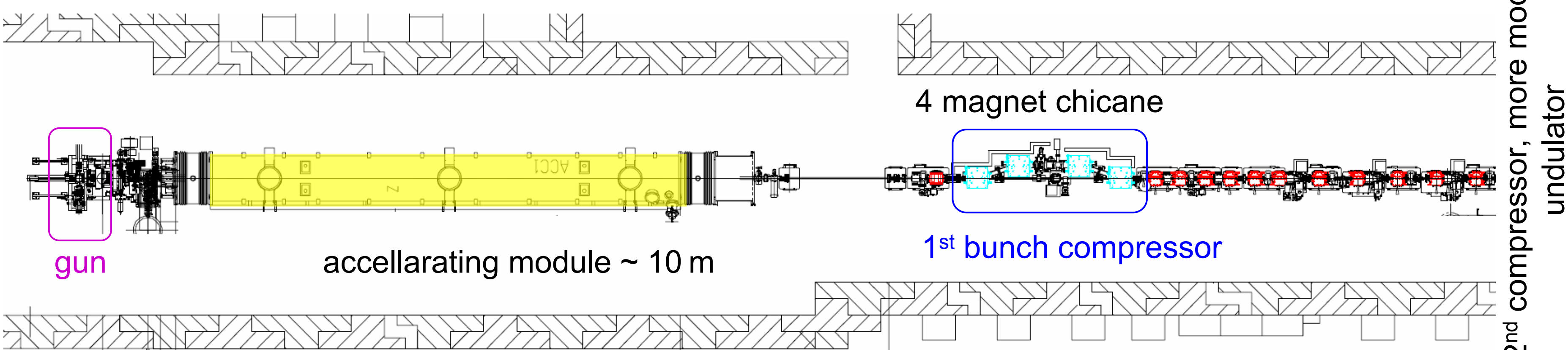
$\gamma >> 1 \rightarrow$ velocity differences are too small for effective compression

magnetic compression: path length depends on energy



accelleration “off crest” →
head particle with less energy than tail

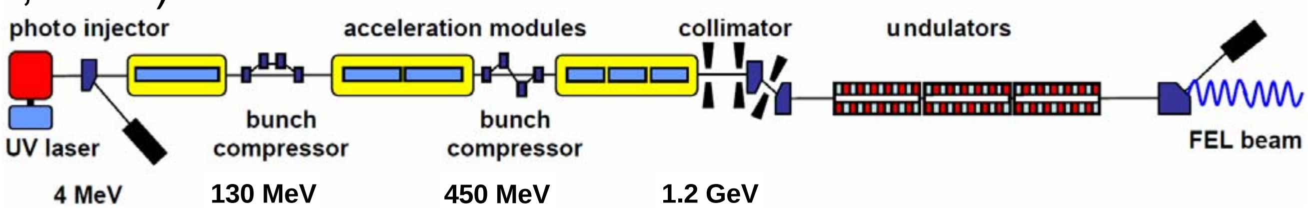
FLASH:



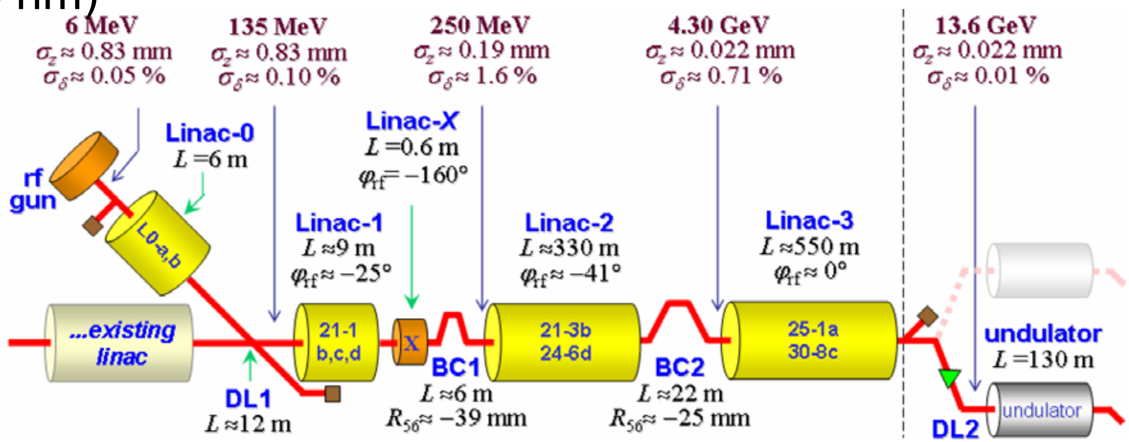
beam dynamics with space charge and CSR effects

Magnetic Bunch Compression - 2

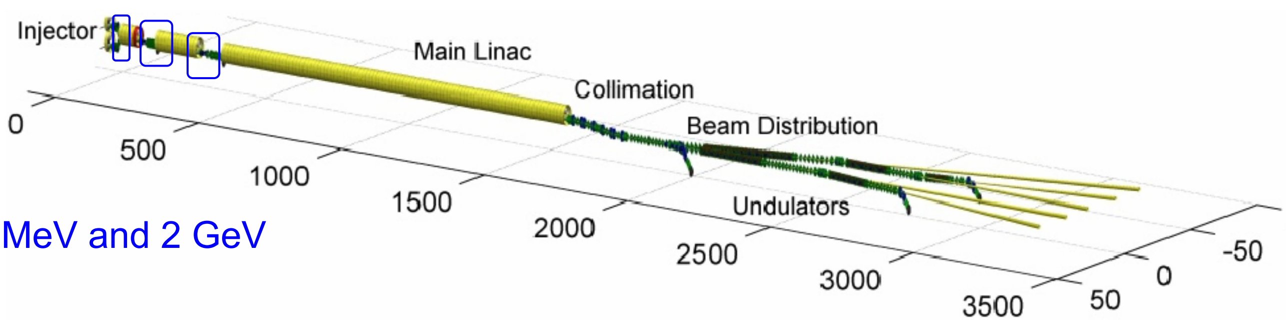
FLASH (1.2 GeV, 4 nm)



LCLS (14 GeV, 0.15 nm)

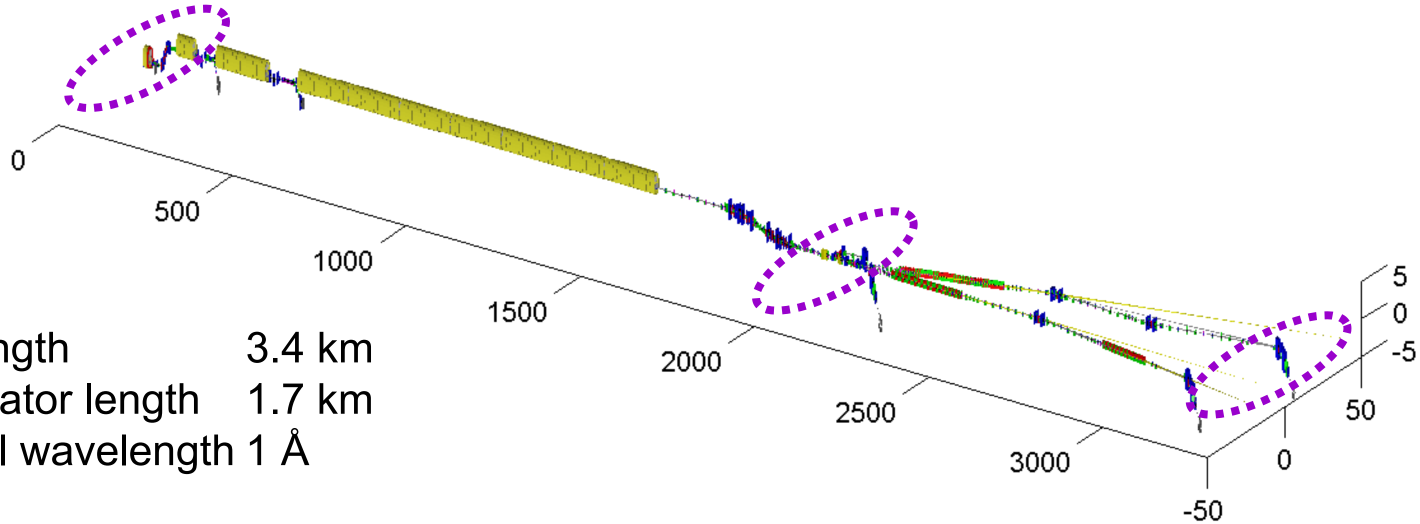


European XFEL (0.1 nm)



BCs at 130 MeV, 500 MeV and 2 GeV

European XFEL



total length 3.4 km
accelerator length 1.7 km
minimal wavelength 1 Å

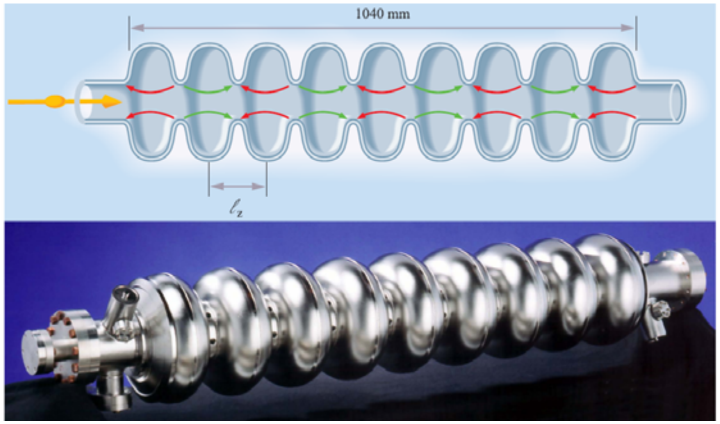
webcams (Aug 2009)

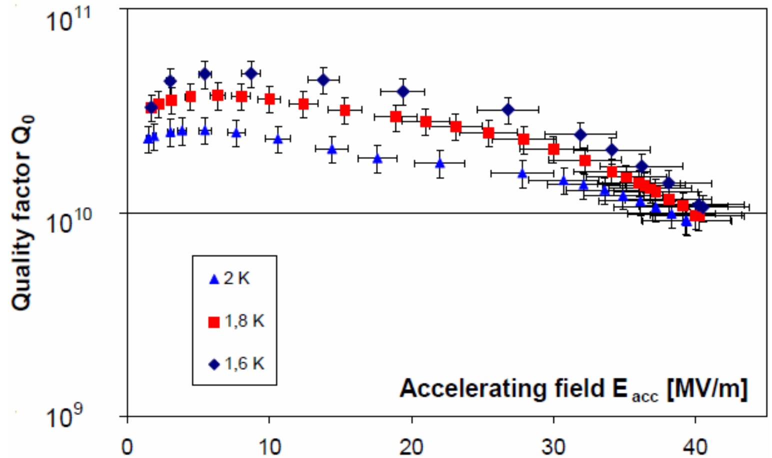
June 2010: start of tunnel construction

webcams (Jul 2010)

European X-FEL - 2

superconducting cavity, 1.3 GHz $E_{acc} \rightarrow$ 40 MeV/m
22.5 MeV/m are needed





FLASH tunnel: cryo module

undulator

European X-FEL - 3

	LCLS	SCSS	European XFEL
Abbreviation for	Linac Coherent Light Source	Spring-8 Compact SASE Source	European X-Ray Free-Electron Laser
Location	California, USA	Japan	Germany
Start of commissioning	2009	2010	2014
Accelerator technology	normal conducting	normal conducting	superconducting
Number of light flashes per second	120	60	30 000
Minimum wavelength of the laser light	0.15 nanometres	0.1 nanometres	0.1 nanometres
Maximum electron energy	14.3 billion electron volts (14.3 GeV)	6-8 billion electron volts (6-8 GeV)	17.5 billion electron volts (17.5 GeV)
Length of the facility	3 Kilometer	750 Meter	3.4 Kilometer
Number of undulators (magnet structures for light generation)	1	3	5
Number of experiment stations	3-5	4	10
Peak brilliance [photons / s / mm^2 / mrad^2/ 0.1% bandwidth]	$8.5 \cdot 10^{32}$	$5 \cdot 10^{33}$	$5 \cdot 10^{33}$

multi bunch operation

beamlines

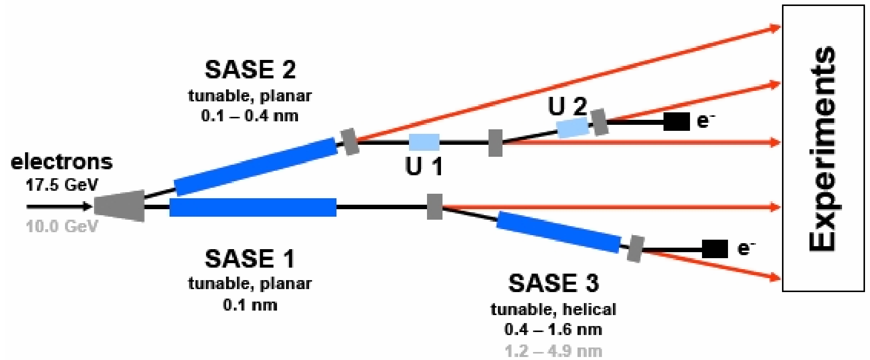
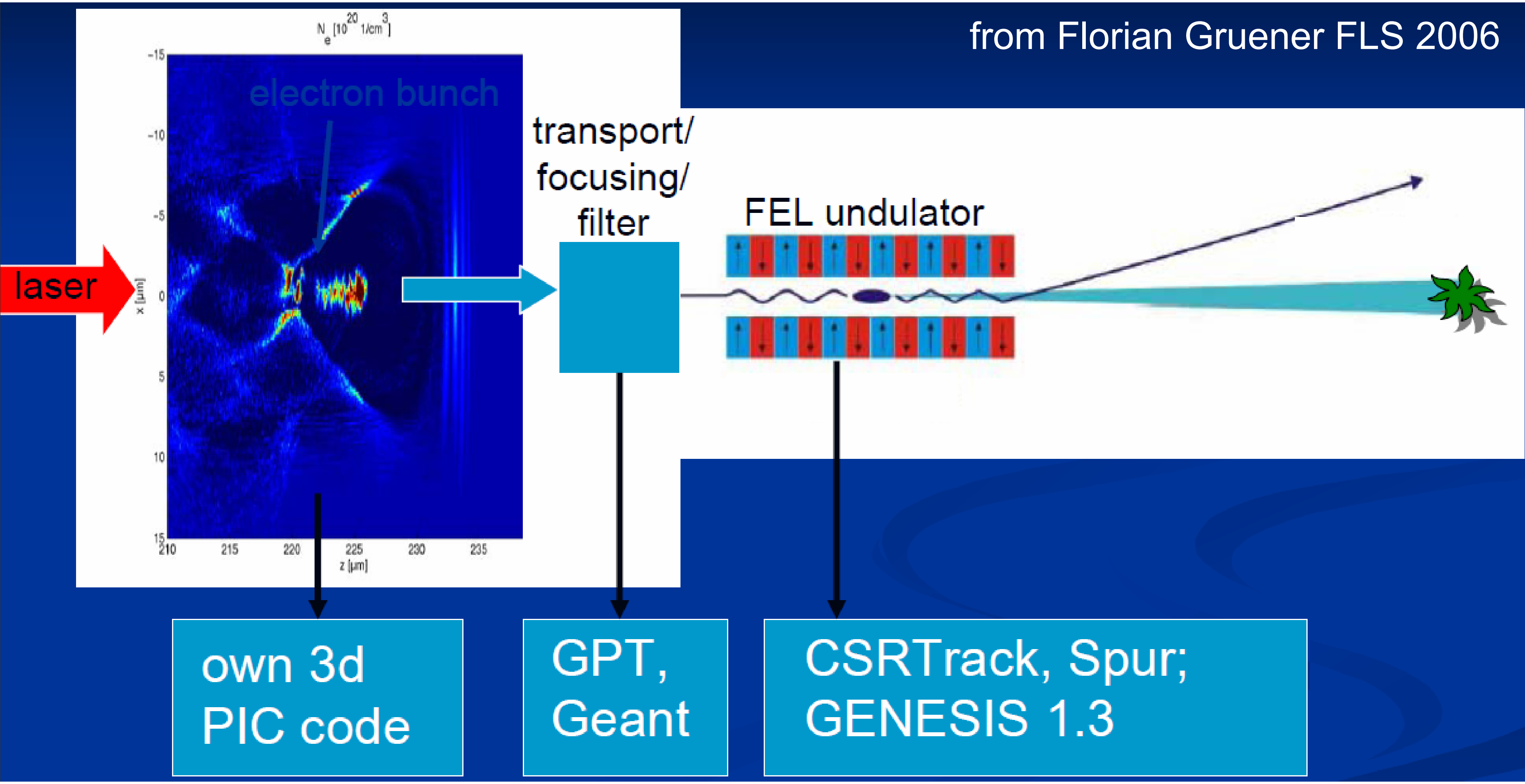


Table Top FEL

Laser-Pasma accelerators: “bubble acceleration”



$q \approx 1\ \mathrm{nC},\ \sigma_r \approx 1\ \mu\mathrm{m}$
$I \sim 100\ \mathrm{kA},\ \gamma < 10^4$

all types of effects; usual approximations fail
a challenge in experiment and theory !

Appendix

Theoretical Approach: FEL

1 Introduction

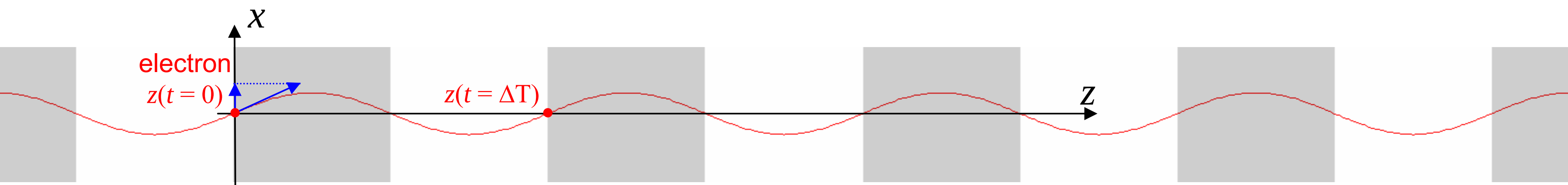
2 Effects (decoupled considerations)

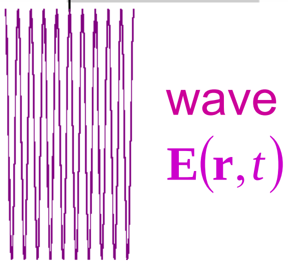
- 2.1 Particle Motion – Trajectory in Undulator
- 2.2 Continuous Source Distribution
- 2.3 Electromagnetic Fields
- 2.4 Particle Motion - Energy

3 Coupled Equations

- 3.1 Numerical Solution
- 3.2 Effects (now coupled)
- 3.3 Analytic Solution (Vlasov etc.)

2.1 Particle Motion – Trajectory in Undulator





$\bar{v} = \dfrac{\lambda_u}{\Delta T}$ averaged longitudinal velocity of a particle with nominal energy $\mathcal{E} = \gamma\, m_0 c^2$

three important periods

undulator (property of device)	λ_u	$T_u = \dfrac{\lambda_u}{c}$	$k_u = \dfrac{2\pi}{\lambda_u}$	undulator
EM wavelength (property of seed laser or nominal value)	λ_p	$T_p = \dfrac{\lambda_p}{c}$	$k_p = \dfrac{2\pi}{\lambda_p}$	photon
slippage length (period of micro modulation)	$\lambda_m = \lambda_u - \bar{v} T_u$		$k_m = \dfrac{2\pi}{\lambda_m}$	micro modulation

slippage or resonance condition: $\dfrac{\lambda_m}{\bar{v}} = \dfrac{\lambda_p}{c}$ or $\lambda_u \left(\dfrac{1}{\bar{v}} - \dfrac{1}{c} \right) = \dfrac{\lambda_p}{c}$

2.1 Particle Motion – Trajectory in **Helical** Undulator

field of helical undulator:

$$\mathbf{B}_u = B_0\left(\mathbf{e}_x \cos k_u z \mp \mathbf{e}_y \sin k_u z\right)$$

simplification: we neglect dependency on transverse offset

in equation of motion

$$\gamma_\nu m_0 \ddot{\mathbf{r}}_\nu = -q_0 \dot{\mathbf{r}}_\nu \times \mathbf{B}_u$$

solution

$$\mathbf{r}_\nu = \frac{K_h}{k_u \gamma_\nu}\left(\mathbf{e}_x \sin k_u z_\nu \pm \mathbf{e}_y \cos k_u z_\nu\right) + \mathbf{e}_z z_\nu$$

$$\dot{\mathbf{r}}_\nu = \frac{K_h}{\gamma_\nu}\left(\mathbf{e}_x \cos k_u z_\nu \mp \mathbf{e}_y \sin k_u z_\nu\right)\overline{v}_\nu + \mathbf{e}_z \overline{v}_\nu$$

$$\ddot{\mathbf{r}}_\nu = -\frac{K_h}{\gamma_\nu} k_u \left(\mathbf{e}_x \sin k_u z_\nu \pm \mathbf{e}_y \cos k_u z_\nu\right)\overline{v}_\nu^2$$

not the general solution!

with helical undulator parameter (~1, dimensionless)

$$K_h = \frac{q_0 \left|B_0\right|}{m_0 k_u c}$$

2.1 Particle Motion – Trajectory in Helical Undulator

nominal particle (and energy)

$$\mathbf{r} = \frac{K_h}{k_u \gamma}\left(\mathbf{e}_x \sin k_u z \pm \mathbf{e}_y \cos k_u z\right) + \mathbf{e}_z z$$

$$\dot{\mathbf{r}} = \frac{K_h}{\gamma_v}\left(\mathbf{e}_x \cos k_u z \mp \mathbf{e}_y \sin k_u z\right)\bar{v} + \mathbf{e}_z \bar{v}$$

averaged longitudinal velocity

$$\left|\dot{\mathbf{r}}_v\right| = \bar{v}_v \sqrt{1 + \frac{K_h^2}{\gamma_v^2}} = c\sqrt{1 - \frac{1}{\gamma_v^2}}$$

$$\frac{\bar{v}_v}{c} = \frac{\sqrt{1 - \frac{1}{\gamma_v^2}}}{\sqrt{1 + \frac{K_h^2}{\gamma_v^2}}} \approx 1 - \frac{1}{2\gamma_v^2} - \frac{K_h^2}{2\gamma_v^2}$$

$$\frac{\bar{v}_v}{c} \approx 1 - \frac{1}{2\gamma_v^2}\left(1 + K_h^2\right)$$

$$\frac{\bar{v}}{c} \approx 1 - \frac{1}{2\gamma^2}\left(1 + K_h^2\right)$$

in slippage or resonance condition: $\lambda_u\left(\frac{1}{\bar{v}} - \frac{1}{c}\right) = \frac{\lambda_p}{c}$

$$\boxed{\lambda_p \approx \frac{\lambda_u}{2\gamma^2}\left(1 + K_h^2\right)}$$

2.1 Particle Motion – Trajectory in Helical Undulator

ponderomotive phase

$$\psi_\nu = k_m\left(z_\nu - \bar{v}t\right)$$

describes particle position with respect to nominal particle and micro modulation

$$\frac{d\psi_\nu}{dt} = k_m\left(\bar{v}_\nu - \bar{v}\right)$$

$$\frac{\bar{v}_\nu}{c} \approx 1 - \frac{1}{2\gamma_\nu^2}\left(1 + K_h^2\right)$$

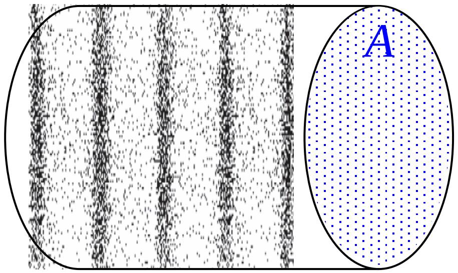
$$\frac{\bar{v}}{c} \approx 1 - \frac{1}{2\gamma^2}\left(1 + K_h^2\right)$$

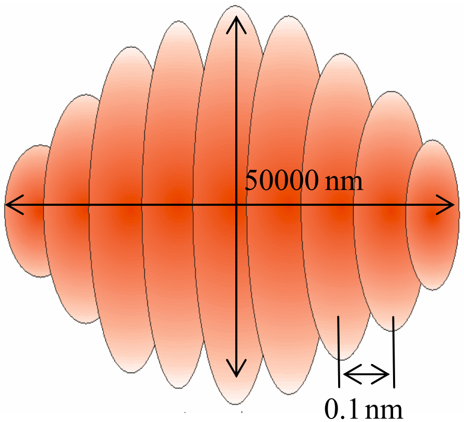
$$\boxed{\frac{d\psi_\nu}{dt} \approx 2\bar{v}k_u \frac{\mathcal{E}_\nu - \mathcal{E}}{\mathcal{E}}}$$

2.2 Continuous Source Distribution

1D approach: neglect transverse dimensions (very large transverse dimension)

simplification: uniform in cross section A





charge density:

$$\rho(z,t) \leftarrow -q_0 \sum \delta(\mathbf{r} - \mathbf{r}_\nu(t))$$

current density:

$$\mathbf{J}(z,t) \leftarrow -q_0 \sum \dot{\mathbf{r}}_\nu \delta(\mathbf{r} - \mathbf{r}_\nu(t))$$

$$\mathbf{J}(z,t) \approx\leftarrow -q_0 \dot{\mathbf{r}} \sum \delta(\mathbf{r} - \mathbf{r}_\nu(t)) \quad \text{with} \quad \dot{\mathbf{r}} = \frac{K_h}{\gamma}\left(\mathbf{e}_x \cos k_u z \mp \mathbf{e}_y \sin k_u z\right)\bar{v} + \mathbf{e}_z \bar{v}$$

in particular:

$$J_x(z,t) = -q_0 \bar{v} \frac{K_h}{\gamma A} \sum \delta(z - z_\nu(t)) \cos k_u z$$

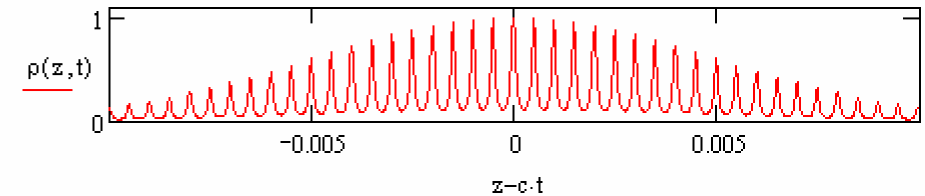
2.2 Continuous Source Distribution

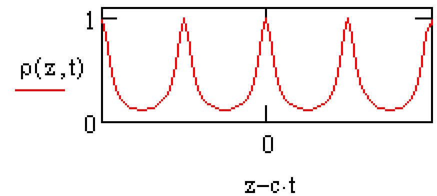
example:

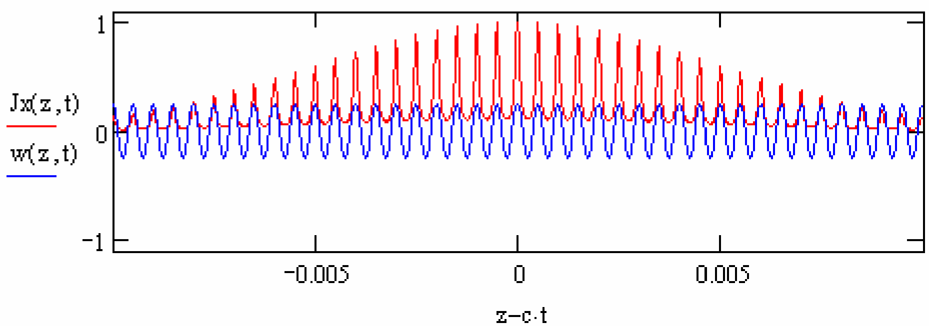
$$\rho(z,t) := \rho m(z - v\cdot t) \qquad Jx(z,t) := \rho(z,t)\cos\left(\frac{2\cdot\pi}{\lambda u}\cdot z\right) \qquad w(z,t) := 0.25\cdot\cos\left[\frac{2\cdot\pi}{\lambda p}(z - c\cdot t)\right]$$

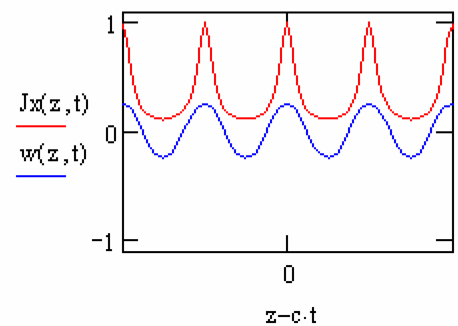
$$Tu := \frac{\lambda u}{c}$$

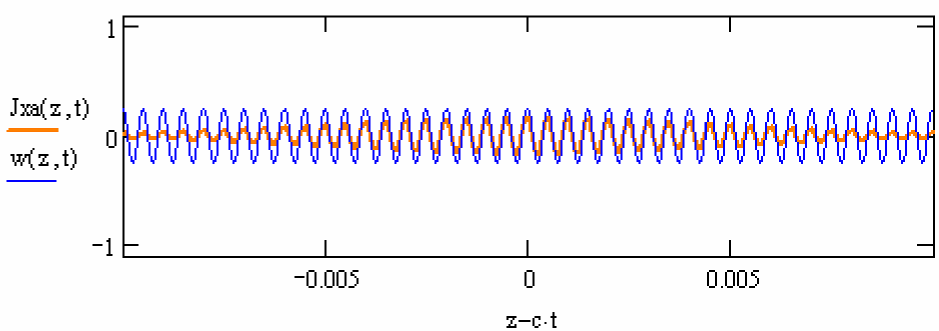
$$t := 0\cdot Tu \qquad z := -20\cdot\lambda p + c\cdot t, -19.95\cdot\lambda p + c\cdot t .. 20\cdot\lambda p + c\cdot t$$

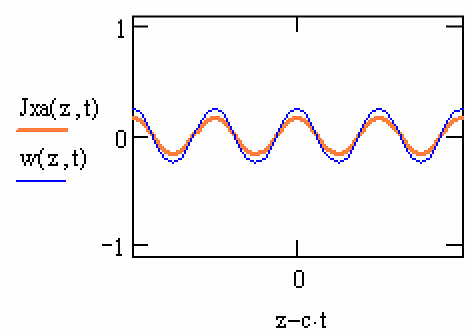






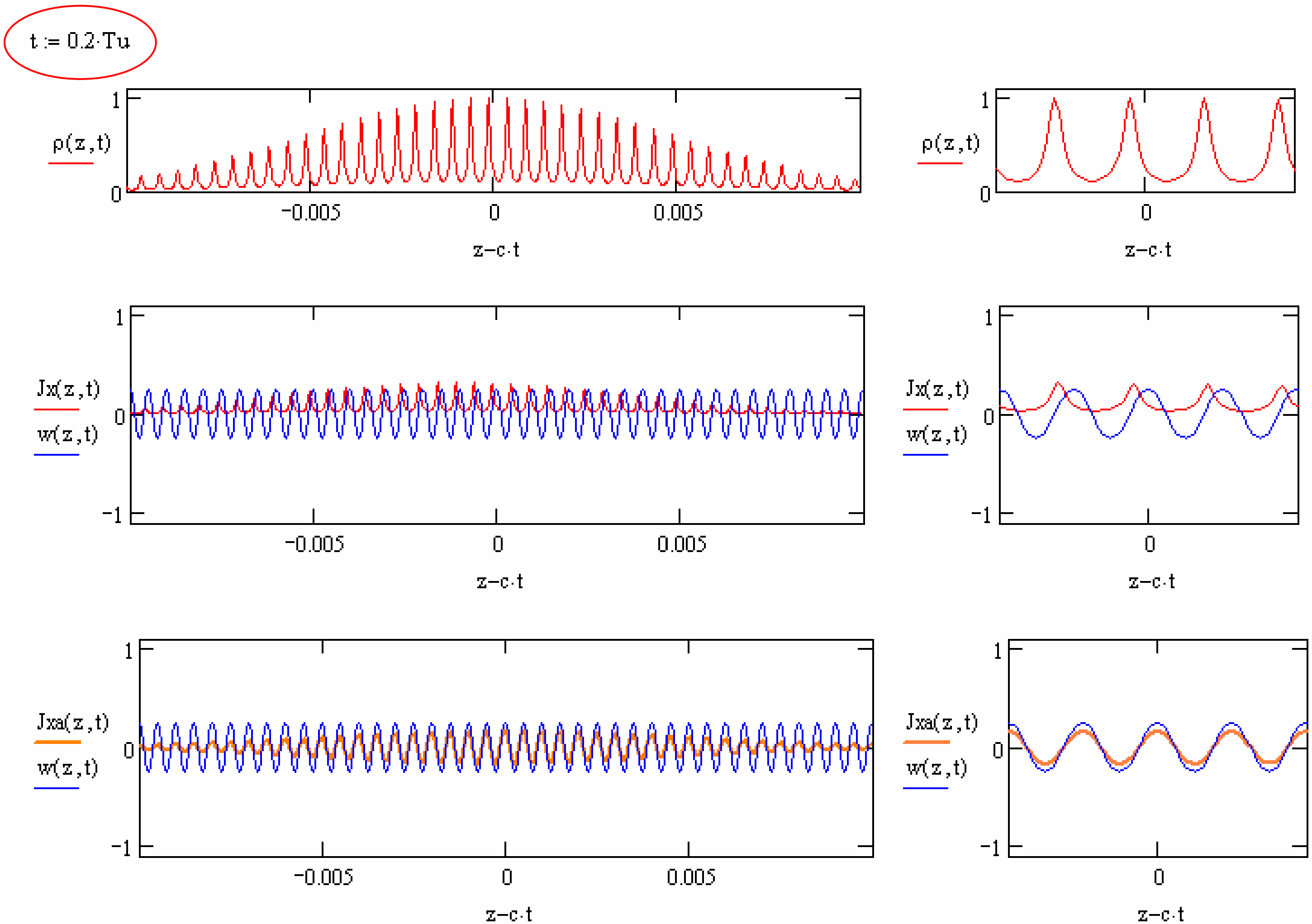






2.2 Continuous Source Distribution

example:



2.2 Continuous Source Distribution

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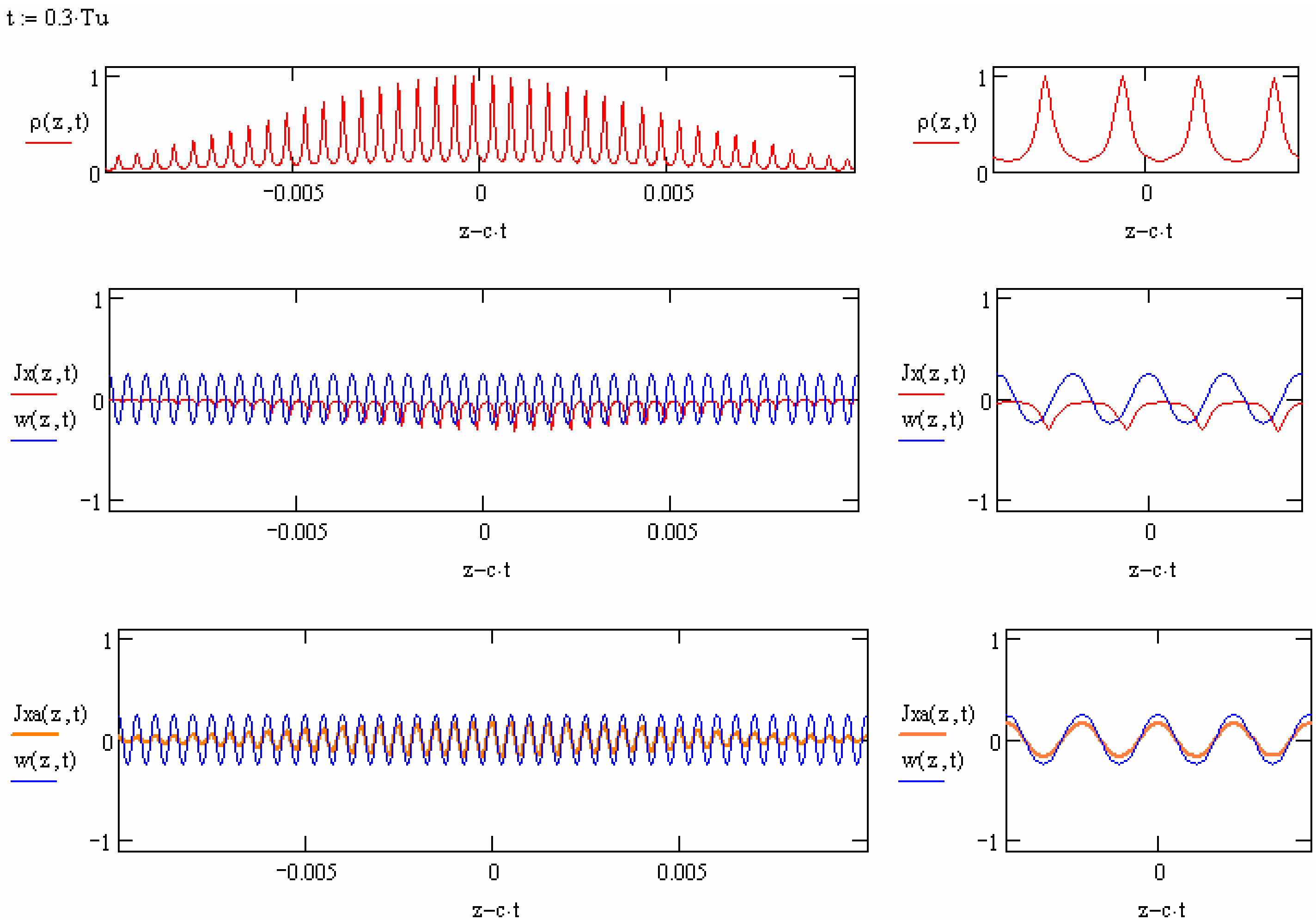
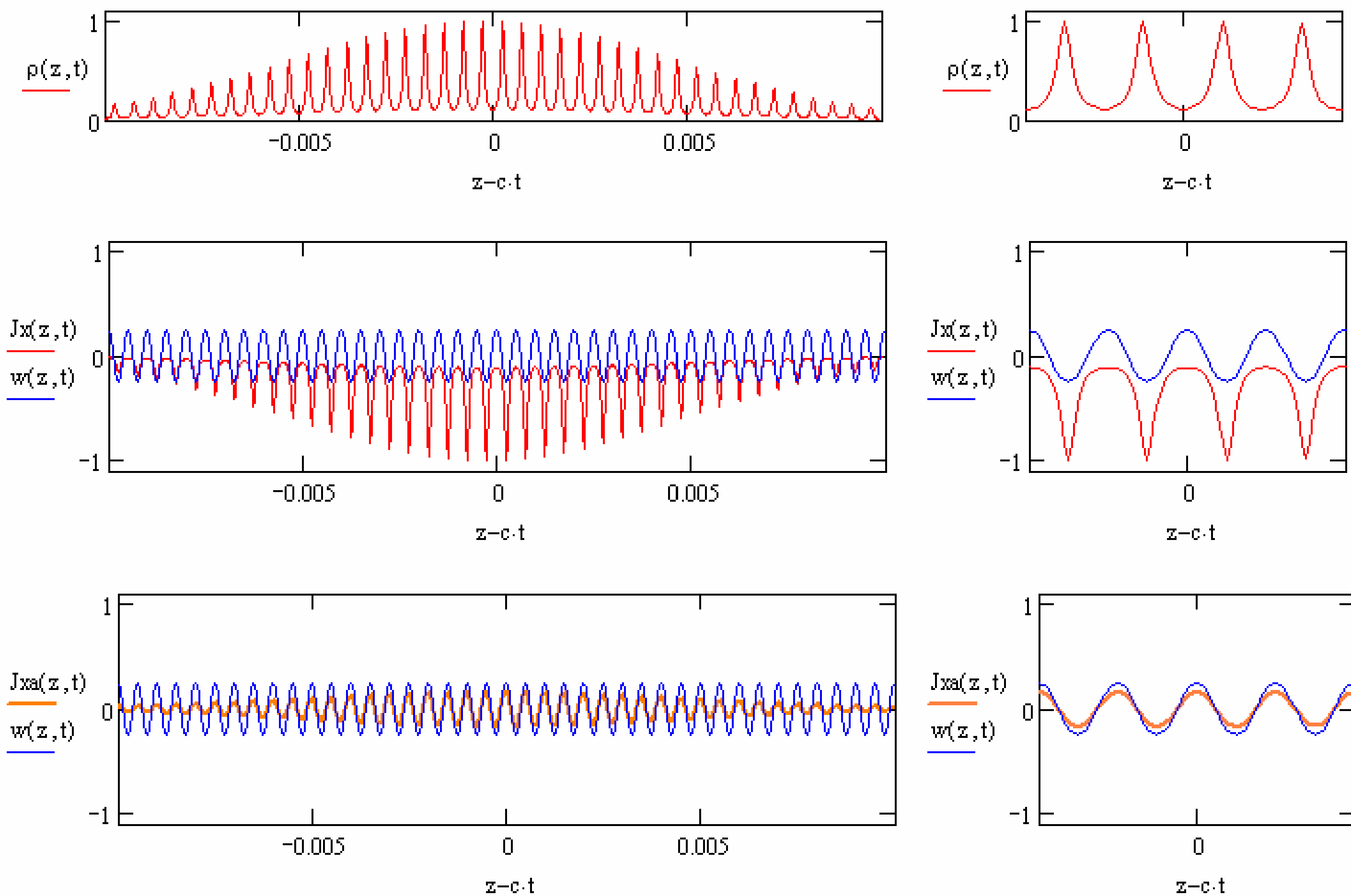


2.2 Continuous Source Distribution

example:

$$Jxa(z,t) := \frac{1}{Tu} \cdot \int_{-0.5 \cdot Tu}^{0.5 \cdot Tu} Jx(z - c \cdot \tau, t - \tau)\, d\tau$$

$t := 0.5 \cdot Tu$



2.2 Continuous Source Distribution

averaged transverse current (as “seen from the wave”):
vs. one undulator period

$$J_{xa}(z,t) = \frac{1}{T_u}\int_{-T_u/2}^{T_u/2} J_x(z - c\tau, t - \tau)\,d\tau$$

$$J_{xa}(z,t) = -q_0\bar{v}\,\frac{K_h}{\gamma A}\,\frac{1}{T_u}\int_{-T_u/2}^{T_u/2}\sum \delta\Big(\underbrace{z - c\tau - \underbrace{z_\nu(t-\tau)}_{\approx z_\nu(t) - \bar{v}\tau}}_{=0\,\to\,\tau_\nu}\Big)\cos k_u(z - c\tau)\,d\tau$$

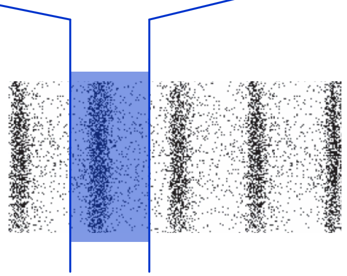
$$\{\nu\} = \left\{\frac{-T_u}{2} < \tau_\nu \le \frac{T_u}{2}\right\}$$

$$J_{xa}(z,t) = -q_0\bar{v}\,\frac{K_h}{\gamma A}\,\frac{1}{c - \bar{v}}\,\frac{1}{T_u}\sum_{\{\nu\}}\cos k_u\left(-\bar{v}\,\frac{z - ct}{c - \bar{v}} + c\,\frac{z_\nu(t) - \bar{v}t}{c - \bar{v}}\right)$$

2.2 Continuous Source Distribution

$$J_{xa}(z,t) = -q_0 \bar{v} \frac{K_h}{\gamma} \frac{1}{\lambda_p} \sum_{\{\nu\}} \cos\left(k_m\left(z_\nu(t) - \bar{v}t\right) - k_p(z - ct)\right)$$

$$z - \bar{v}t - \frac{\lambda_p}{2} < z_\nu(t) - \bar{v}t \le z - \bar{v}t + \frac{\lambda_p}{2}$$



$\nu \in$ period

$$J_{xa}(z,t) = \operatorname{Re}\left\{\hat{J}_a(z, z-\bar{v}t) \cdot \exp\left(jk_p(z-ct)\right)\right\}$$

$$\hat{J}_a(z, z-\bar{v}t) = -q_0 \bar{v} \frac{K_h}{\gamma A} \frac{1}{\lambda_p} \sum_{\{\nu\}} \exp(-j\psi_\nu)$$

$J_{ya}(z,t) = \mp \operatorname{Im}\{\cdots\}$

y component is similar

with $\psi_\nu = k_m\left(z_\nu(t) - \bar{v}t\right)$ ponderomotive particle phase
(changes slowly along many undulator periods)

2.3 Electromagnetic Fields

Maxwell $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu \mathbf{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{\partial}{\partial t}\mu \mathbf{J} + \nabla(\nabla \mathbf{E}) \qquad \nabla \mathbf{E} = \frac{\rho}{\varepsilon}$$

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{\partial}{\partial t}\mu \mathbf{J} + \frac{1}{\varepsilon}\nabla \rho}$$

inhomogeneous 1D wave equation for transverse components:

$$\boxed{\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_x = \mu \frac{\partial J_x}{\partial t}}$$

2.3 Electromagnetic Fields

solution of 1D wave equation:

$$E_x(z,t) = E_x^f(z,t) + E_x^b(z,t)$$

with $\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)E_x^f = -\frac{\mu c}{2}J_x$ forward wave

$\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)E_x^b = \frac{\mu c}{2}J_x$ backward wave

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(E_x^f + E_x^b\right) = -\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\frac{\mu c}{2}J_x + \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\frac{\mu c}{2}J_x$$

$$= \mu\frac{\partial J_x}{\partial t}$$

solution: $E_x^{f/b}(z,t) = -\frac{1}{2\varepsilon}\int_0^\infty J_x(z \mp c\tau, t-\tau)\,d\tau$

we neglect backward solution $E_x(z,t) \approx E_x^f(z,t)$ and use $\int_0^\infty J_x(\cdots)\,d\tau \approx \int_0^\infty J_{xa}(\cdots)\,d\tau$

$$E_x(z,t) = -\frac{1}{2\varepsilon}\int_0^\infty J_{xa}(z - c\tau, t-\tau)\,d\tau$$

2.3 Electromagnetic Fields

approach (same form as transverse current density):

$$E_x(z,t) = \operatorname{Re}\{\hat{E}_a(\underbrace{z, z-\bar{v}t}_{s}) \cdot \exp(jk_p(z-ct))\}$$

bunch coordinate

$$E_y(z,t) = \mp \operatorname{Im}\{\cdots\}$$

y component is similar

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)E_x = -\frac{\mu c}{2} J_{xa}$$ real fields

$$\downarrow$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\hat{E}_a = -\frac{\mu c}{2} \hat{J}_a$$ complex amplitudes

with bunch coordinate:

$$\boxed{\left(\frac{\partial}{\partial z} + \left(1-\frac{\bar{v}}{c}\right)\frac{\partial}{\partial s}\right)\hat{E}_a = -\frac{\mu c}{2} \hat{J}_a}$$

2.3 Electromagnetic Fields

electromagnetic field:

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{(\mathbf{e}_x \pm j\mathbf{e}_z)\hat{E}_a(z, z - \bar{v}t)\cdot \exp\left(jk_p(z - ct)\right)\right\} + \mathbf{e}_z E_z(z,t)$$

calculation of z component is straight forward

this is essentially a circular polarized EM wave:

$$\mathbf{E}_p(\mathbf{r},t) = E_0\left(\ \mathbf{e}_x \cos\{k_p(z - tc) + \psi_0\} \mp \mathbf{e}_y \sin\{k_p(z - tc) + \varphi_0\}\right)$$

$$-\frac{d}{dt}\mathbf{B}_p = \mathrm{curl}\,\mathbf{E}_p \rightarrow \quad \mathbf{B}_p(\mathbf{r},t) = \frac{E_0}{c}\left(\pm\mathbf{e}_x \sin\{k_p(z - tc) + \psi_0\} + \mathbf{e}_y \cos\{k_p(z - tc) + \varphi_0\}\right)$$

2.4 Particle Motion - Energy

change of particle energy

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \dot{\mathbf{r}}_\nu \cdot \mathbf{E}(\mathbf{r}_\nu, t)$$

simplified (with circular polarized wave):

$$\mathbf{E}_p(\mathbf{r},t) = E_0\left(\mathbf{e}_x \cos\{k_p(z-tc)+\psi_0\} \mp \mathbf{e}_y \sin\{k_p(z-tc)+\varphi_0\}\right)$$

$$\dot{\mathbf{r}}_\nu = \frac{K_h}{\gamma_\nu}\left(\mathbf{e}_x \cos k_u z_\nu \mp \mathbf{e}_y \sin k_u z_\nu\right)\bar{v}_\nu + \mathbf{e}_z \bar{v}_\nu$$

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \frac{K_h}{\gamma_\nu}\bar{v}_\nu E_0\left(\mathbf{e}_x \cos k_u z_\nu \mp \mathbf{e}_y \sin k_u z_\nu\right)\cdot\left(\mathbf{e}_x \cos\{\cdots\} \mp \mathbf{e}_y \sin\{\cdots\}\right)$$

$$= -q_0 \frac{K_h}{\gamma_\nu}\bar{v}_\nu E_0\left(\cos k_u z_\nu \cos\{\cdots\} + \sin k_u z_\nu \sin\{\cdots\}\right)$$

$$= -q_0 \frac{K_h}{\gamma_\nu}\bar{v}_\nu E_0 \cos\left[k_p(z_\nu - tc) + \psi_0 - k_u z_\nu\right]$$

2.4 Particle Motion - Energy

change of particle energy

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \dot{\mathbf{r}}_\nu \cdot \mathbf{E}(\mathbf{r}_\nu, t)$$

simplified (with circular polarized wave):

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \frac{K_h}{\gamma_\nu} \bar{v}_\nu E_0 \cos\left[k_p(z_\nu - tc) + \varphi_0 - k_u z_\nu\right]$$

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \frac{K_h}{\gamma_\nu} \bar{v}_\nu E_0 \cos\left[\underbrace{(k_p - k_u)}_{k_m} z_\nu - \underbrace{k_p c}_{k_m \bar{v}} t + \varphi_0\right]$$

$$\psi_\nu = k_m(z_\nu - \bar{v}t)$$ ponderomotive phase

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \frac{K_h}{\gamma_\nu} \bar{v}_\nu E_0 \cos[\psi_\nu + \varphi]$$

$$\boxed{\frac{d}{dt}\mathcal{E}_\nu \approx -q_0 \frac{K_h}{\gamma} \bar{v} E_0 \cos[\psi_\nu + \varphi]}$$

2.4 Particle Motion - Energy

change of particle energy

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \dot{\mathbf{r}}_\nu \cdot \mathbf{E}(\mathbf{r}_\nu, t)$$

without simplification:

electric field $\mathbf{E}(z,t) = \mathrm{Re}\left\{(\mathbf{e}_x \pm j\mathbf{e}_z)\hat{E}_a(\cdots)\cdot \exp\left(jk_p(z-ct)\right)\right\} + \mathbf{e}_z E_z(z,t)$

$$\frac{d}{dt}\mathcal{E}_\nu = -q_0 \frac{K_h}{\gamma_\nu}\bar{v}_\nu \,\mathrm{Re}\left\{\hat{E}_a(z_\nu, z_\nu - \bar{v}t)\cdot \exp(j\psi_\nu)\right\} - q_0 \bar{v} E_z(z_\nu, t)$$

z component contributes to FEL equation
for simplicity suppress this term in the following

$$\frac{d}{dt}\mathcal{E}_\nu \approx -q_0 \frac{K_h}{\gamma}\bar{v} E \cos[\psi_\nu + \varphi]$$

with $E\exp(j\varphi) = \hat{E}_a(z_\nu, z_\nu - \bar{v}t)$

3 Coupled Equations

particle equations (longitudinal phase space):

$$\frac{d\psi_\nu}{dt} \approx 2k_u \bar{v} \frac{\mathcal{E}_\nu - \mathcal{E}}{\mathcal{E}}$$

$$\frac{d}{dt}\mathcal{E}_\nu \approx -q_0 \bar{v} \frac{K_h}{\gamma} \mathrm{Re}\left\{\hat{E}_a(z, z_\nu - \bar{v}t) \cdot \exp(j\psi_\nu)\right\}$$

field equations:

$$\hat{J}_a(z, z - \bar{v}t) = -q_0 \bar{v} \frac{K_h}{\gamma A} \frac{1}{\lambda_p} \sum_{\{\nu\}} \exp(-j\psi_\nu)$$

$$\left(\frac{\partial}{\partial z} + \left(1 - \frac{\bar{v}}{c}\right)\frac{\partial}{\partial s}\right)\hat{E}_a = -\frac{\mu c}{2} \hat{J}_a$$

change variables:

$$\frac{\mathcal{E}_\nu(t) - \mathcal{E}}{\mathcal{E}} \rightarrow \eta_\nu(z = t\bar{v})$$

$$\psi_\nu(t) \rightarrow \psi_\nu(z = t\bar{v})$$

neglect bunch coordinate:

$$\hat{E}_a(z, z_\nu - \bar{v}t) \approx \hat{E}_a(z)$$

$$\hat{J}_a(z, z_\nu - \bar{v}t) \approx \hat{J}_a(z)$$

(periodic model)

3 Coupled Equations

periodic model
particle equations (longitudinal phase space):

$$\frac{d\psi_\nu}{dz} = 2k_u \eta_\nu$$

$$\frac{d}{dz}\eta_\nu = -\frac{q_0}{\mathcal{E}}\frac{K_h}{\gamma}\operatorname{Re}\left\{\hat{E}_a \cdot \exp\left(j\psi_\nu\right)\right\}$$

field equations:

$$\hat{J}_a = -q_0 \overline{v} \frac{K_h}{\gamma A}\frac{1}{\lambda_p}\sum_{\{\nu\}}\exp\left(-j\psi_\nu\right)$$

$$\frac{d}{dz}\hat{E}_a = -\frac{\mu c}{2}\hat{J}_a$$

3 Coupled Equations