Strebel differential for holography

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Abstract

The Strebel differential is introduced as a mathematical tool to parameterise the moduli space $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$, also providing a combinatorial structure derived from the orbifold of ribbon graphs.

We examined its role in holographic theories, with a focus on the case of AdS_3/CFT_2.

An analysis of particularly tractable cases of planar 4-point functions is made. We first study the UV behaviour of a generic solution, then we proceed to the explicit calculation of the differential relative to the Square (and the Whale) critical graph and finally, we make a perturbative expansion around the latter.

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1. Introduction

The purpose of this report is to provide an introduction to the Strebel differential and some of its applications. Defined indirectly by a theorem by Strebel, its usefulness lies in the fact that it allows you to parameterise the *moduli space of marked Riemann surfaces* (actually $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$) and associate each point in that space with a *ribbon graph*. For this reason, it is often employed in holographic theories to explicitly reconstruct the integrand of the string theory from the correlation functions of the corresponding (putative) field theory. There are other numerous applications in maths (*dessins d'enfant*, Witten's conjecture, etc...) not treated here.
This work is divided as follows. In section 2 we introduce all the necessary theoretical background. In 2.1 we define the object that is the focus of the work (following [1]) and go on to state the theorem that makes it so interesting, also providing a sketch of how one can actually construct a surface from a graph.

In the following subsection 2.2, we provide a quick review of the work that led to the derivation of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality [2, 3, 4, 5], thus attempting to give a physical motivation behind the study of the mathematical tool. We first explain how it is possible to define a sigma model with target space a symmetric orbifold. In order to calculate the correlation functions of this theory, it is useful to introduce the so-called twist operators that implement the group action via monodromy: Feynman diagrams are constructed for these operators and the covering map method is introduced (examples of which are presented). This map is then related to the spectral curve of a Penner-like matrix model and that to a Strebel differential. Finally, it is explained how the covering map can be interpreted as the string worldsheet.
Section 3 is entirely devoted to the calculations performed during the summer project: starting with the simplest case, i.e. a planar four-point function, we studied its UV behaviour in 3.1 following [6]. We moved on to an even simpler case, finding the exact solution relative to the Square (or Whale) graph in 3.2. We finally perturbed the solution by splitting one of the two double roots into two simple roots (separated by a small epsilon): at the level of the graph, this is equivalent to adding a (small) diagonal to the square [7].
In future work, we hope to extend the analysis to surfaces of higher genus, as well as generalise some aspects of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality.

2. Theoretical preliminaries

2.1. Strebel differential

Given Σ_g a compact Riemann surface, a *(meromorphic) quadratic differential* on Σ_g is the local data of $q = f(z)(dz)^2$, where (f is a meromorphic function) which transform under change of chart as:

$$z = z(w), \quad q = f(z)(dz)^2 = g(w)(dw)^2 \;\Rightarrow\; f(z(w)) \left(\frac{dz}{dw}\right)^2 = g(w) \; . \tag{1}$$

Given a quadratic differential q, a parametric curve γ is a *horizontal (veritcal) leaf* if:

$$f(\gamma(t)) \left(\frac{d\gamma(t)}{dt}\right)^2 > (<) 0 \; .$$

Example 1. Let $q = (dz)^2$ then the horizontal and vertical leaves are (ref. figure 1a):

$$\alpha(t) = t + ic, \quad \beta(t) = c + it, \quad t \in \mathbb{R} \; .$$

If a quadratic differential $q = f(z)(dz)^2$ is holomorphic and non-zero at $z = z_0$, then on a neighbourhood of z_0 we can introduce a canonical coordinate $w(z) = \int_{z_0}^{z} \sqrt{f(z)}dz$. It follows from eq. 1 that in w-coordinate the quadratic differential is given $q = (dw)^2$. Therefore, the leaves of q near z_0 look exactly as in example 1 (hence the name).

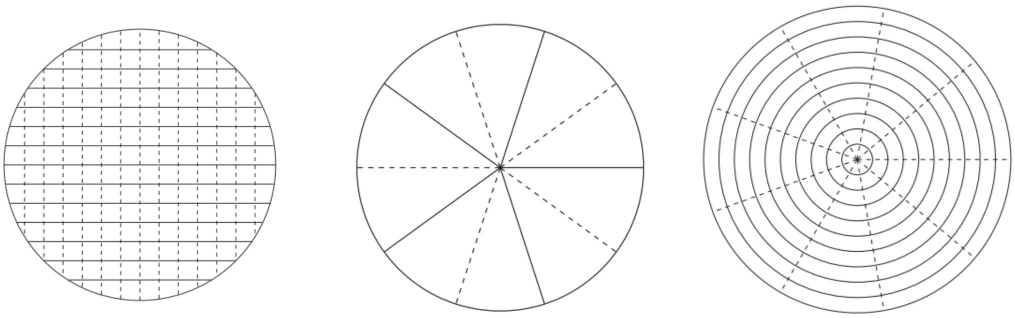
An important consequence of the definition of horizontal leaf is that the quantity $\int \sqrt{q}$ calculated on a segment of γ horizontal is always real (positive up to a choice of branch) and can thus be used to define lengths.
While for a generic point $p \in \Sigma_g$ there is a unique horizontal and vertical leaf passing through it intersecting at right angle, the foliation behaves differently around zeroes and poles of q:

- Let $q = z^m (dz)^2$, then both horizontal and vertical trajectories (α_k and β_k respectively) are given by $(m+2)$ half rays departing from $z = 0$ (ref. figure 1b):

$$\alpha_k(t) = t \cdot \exp \frac{2\pi i k}{m+2} \quad \beta_k(t) = t \cdot \exp \frac{\pi i(2k+1)}{m+2} \ .$$

- The foliation becomes quite wild at singularities of q. However, the situation is milder around a quadratic pole with a negative real coefficient. Let $q = -\left(\frac{dz}{z}\right)^2$, then the horizontal leaves are given by concentric circles around the origin (and in particular are compact curves), while the vertical ones are half-rays (ref. figure 1c):

$$\alpha_r(t) = re^{it} \quad \beta_\theta(t) = te^{i\theta} \ .$$



(a) (b) (c)

Figure 1: Frome the left to the right: foliations defined by $(dz)^2$, $z^3(dz)^2$, $-\left(\frac{dz}{z}\right)^2$

Thus, the set of horizontal (vertical) lines gives a foliation of the surface minus the poles $\Sigma_g \setminus \{z_1, \ldots, z_n\}$[1].
Our interest in this mathematical device is given by the following important theorem by Strebel:

Theorem 1. *Given a Riemann surface Σ_g with n marked points $\{z_1, \ldots, z_n\}$ (i.e. an element of $\mathcal{M}_{g,n}$ and an ordered n-tuple of positive real numbers $(p_1, \ldots, p_n) \in \mathbb{R}^n_+$, there is a unique quadratic differential q (called Strebel differential) such that:*

[1]Although the choice of notation is unfortunate, with z_i we indicate points in the set Σ_g and not their (chart dependent) coordinates

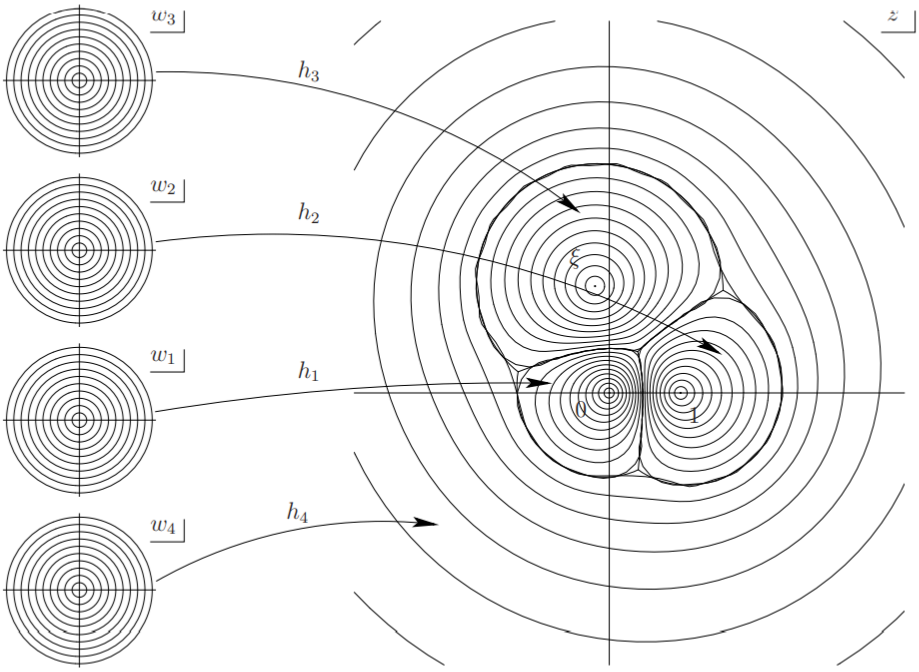


Figure 2: The horizontal curves for a four-punctured sphere. Each domain is conformally a disk. The critical graph is easily seen.

- *q is holomorphic on $\Sigma_g \setminus \{z_1, \ldots, z_n\}$*
- *q has double poles on each z_j*
- *Every compact horizontal leaf α is a simple loop circling around one of the poles, say z_j, and it satisfies*

$$p_j = \oint_\alpha \sqrt{q}\ ,$$

where the branch of the square root is chosen so that the integral has a positive value with respect to the positive orientation of α that is determined by the complex structure of Σ_g.

Remark. Around every marked point z_j there is a foliated disk of compact horizontal leaves with length equal to the prescribed value p_j . As the loop becomes larger in size (but not in length, because it is a constant), it hits zeroes of q and the shape becomes a polygon and we have a relation between the residue and the length of the edges ${\gamma_i}^2$:

[2] At first glance there seems to be an ambiguity of signs, all quantities involved being positive by definition and in general $|\int_{\cup\gamma_i} f| \neq \sum |\int_{\gamma_i} f|$. However, the integrand is real (by definition of horizontal leaf) and of constant sign since by choosing a small enough path we can avoid zeros. This property remains valid even when the path is extended to the edge of the polygon (precisely because we continue to encounter no zeros, vertices of the polygon itself)

$$p_j = L(\alpha) = L(\gamma_1) + \cdots + L(\gamma_m).$$

The theorem gives us a way to associate to each point of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ a *ribbon graph* (i.e. fatgraph with an assigned length for every edge, we will denote the space of such graphs as Γ_{met}) given by the noncompact horizontal leaves together with their lengths (i.e. the *critical graph*). However, the relationship is actually invertible! Let's see how we can cunstruct a surface from a given graph in Γ_{met}[3]:

- Given an edge E of length L we assign the strip of infinite length and width L.

  $$U_E = \{z \in \mathbb{C} | 0 < \Re(z) < L\} \ .$$

  The strip has a complex structure defined by the coordinate z, and a holomorphic quadratic differential $(dz)^2$ on it. Every horizontal leaf of the foliation defined by this quadratic differential is a horizontal line of length L.

- Given a vertex V of degree m. Let us place the vertex V at the origin of the w-plane. For every edge E_j coming out of V, we glue a neighbourhood of the boundary point $z_j = 0$ of each of the strips U_{E_j} together on the w-plane by:

  $$w = z_j^{2/m} \cdot \exp \frac{2\pi i(j-1)}{m} \quad \Rightarrow \quad (dz_j)^2 = \frac{m^2}{4} w^{m-2} (dw)^2 := q \ . \tag{2}$$

  It is important to note that the quadratic differential (2) is independent of j and thus is well defined in a neighbourhood U_V of $z_j = 0$. Note also that q has a zero of degree $m-2$ at $w = 0$ and, at least locally on U_V, the horizontal leaves of the foliation defined by q coincide with the image of the edges E_j via (2).

- For every boundary component B (formed by edges $E_1, \ldots, E_h$ of length L_i) we glue together the upper halves of U_{E_j} with ($a_B := \sum_{i=0}^{h} L_i$):

  $$u = \exp \left[\frac{2\pi i}{a_B} \left(\sum_{i=0}^{j-1} L_i + z_j \right) \right] \quad \Rightarrow \quad (dz_j)^2 = -\left(\frac{a_B}{2\pi} \frac{du}{u} \right)^2 := q \ . \tag{3}$$

  As before, the differential is independent of j and thus well-defined on U_B: it has a pole of order 2 at $u = 0$ with a negative real coefficient. The horizontal leaves of the foliation defined by q are concentric circles that are centered at $u = 0$, which correspond to the horizontal lines on U_{E_j} through (3). Note that the length of a compact horizontal leaf around $u = 0$ is always a_B.

[3]There are many details missing: U_E stripes depend on the orientation of the sides, U_V's depend on the cyclic orientation of vertices (inherent in the definition of ribbon graphs), these observations propagate over into the construction of U_B, ...

2.2. AdS$_3$/CFT$_2$

We will now present a brief review of some aspects of recent developments in the explicit derivation of a AdS$_3$/CFT$_2$ duality: the aim is to (hopefully) show in practice an exciting and recent application of the Strebel differential in physics. In particular, we will explain how this duality is realised at the level of correlation functions (the study of the spectrum, although interesting, is beyond the scope of this project).

$$\left\langle \mathcal{O}^{(w_1)}_{h_1}(x_1)\dots\mathcal{O}^{(w_n)}_{h_n}(x_n)\right\rangle_{\mathrm{S}^d}\Big|_g = \int_{\mathcal{M}_{g,n}} d\mu \left\langle \mathcal{V}^{w_1}_{h_1}(x_1;z_1)\dots\mathcal{V}^{w_n}_{h_n}(x_n;z_n)\right\rangle_{\Sigma_{g,n}} .$$

2.2.1. Simmetryc orbifold

The field theory side is given by a symmetric product orbifold $(\mathbb{T}^4)^{\otimes K}/S_K$. The orbifold theory can be described by the same field of the original theory but obeying twisted boundary conditions[4][8] (and, by definition, points related by the group action are identified):

$$X\left(e^{2\pi i}z\right) = (g\cdot X)(z) = (h\cdot X)\left(e^{2\pi i}z\right) = \left(hgh^{-1}\cdot(h\cdot X)\right)(z) .$$

It is evident now that, the Hilbert space is divided into subspaces labelled by the conjugacy classes of G (labelled $[g]$).
A useful way to deal with the twisting is to introduce *twist fields*, defined by the monodromy of other fields around them [5]:

$$X\left(e^{2\pi i}z+\zeta\right)\sigma_g(\zeta) = (g\cdot X)(z+\zeta)\sigma_g(\zeta) .$$

Well defined (gauge invariant) operators can thus be obtained by summing over the orbit of the group. Thus for the twist fields, they are given by:

$$\sigma_{[g]}(z) \equiv \mathcal{N}_{[g]}\sum_{h\in G}\sigma_{hgh^{-1}}(z) \qquad \mathcal{N}_{[g]} = \frac{1}{\sqrt{|\operatorname{Stab}([g])||G|}} = \frac{1}{\sqrt{(N-w)!N!w}}$$

(the normalization factor ensures a properly $\sigma_[g](z)$ normalized two-point function, assuming that $\sigma_g(z)$ has one).
In our specific case, conjugacy classes are labelled by the lengths of the cycles in the decomposition of an element. We will consider only single cycle classes $w := [(1,\dots,w)]$: multi-cycles can be obtained by multiplication of non intersecting single cycles. Moreover, in this case $|G| = N!$ and $|\operatorname{Stab}([g])| = (N-w)!w$ (the first factor counts the permutation of the index not involved in g, the *inactive colours*, while the second counts the cyclic permutation of the *active* ones).

[4] We only deal with the bosonic case. The fermionic case is similar, but switching to the covering, the lifted fields differ from the original ones by a factor $\partial\Gamma^{\frac{1}{2}} \sim t^{\frac{w-1}{2}}$. Because of this, they may not be well defined, but have monodromy around the preimage of an insertion point of a twist field $\psi_t(t) \to (-1)^{w-1}\psi_t(t)$

[5] Actually, the twist fields have themselves monodromy around each other. This can be derived to act as a conjugation by the element $g := g_2g_1$ (when g_2 runs around g_1)

Note that for $z_2 \sim z_2$, $\sigma_{g_1}(z_1)\sigma_{g_2}(z_2)$ and $\sigma_{g_1g2}(z_1)$ implement the same boundary conditions on fields. In other words, states in the conjugacy classes of g_1 and g_2 fuse to states in the conjugacy class of $g_1 g_2$ [6]:

$$[g_1] \times [g_2] = \sum_{[g_3]} N_{[g_1][g_2][g_3^{-1}]} [g_3] \ , \quad N_{[g_1][g_2][g_3]} = |\{(g_1, g_2, g_3) \mid g_1 g_2 g_3 = \mathbb{1}\} / \sim| \ ,$$

where the equivalence is related to an overall relabelling of the elements (conjugacy): $(g_1, g_2, g_3) \sim (g g_1 g^{-1}, g g_2 g^{-1}, g g_3 g^{-1})$.
We are interested in correlation functions between gauge invariant quantities: they are defined in terms of gauge dependent fields

$$\left\langle \sigma_{[g_1]}(z_1) \dots \sigma_{[g_m]}(z_m) \right\rangle = \prod_{j=1}^{m} \mathcal{N}_{[g_j]} \sum_{h_1, \dots, h_m \in G} \left\langle \sigma_{h_1 g_1 h_1^{-1}}(z_1) \dots \sigma_{h_m g_m h_m^{-1}}(z_m) \right\rangle \tag{4}$$

and, by taking successive OPEs, we see that these are only non-vanishing if $g_1 g_2 \cdots = \mathbb{1}$. As will become apparent, it is useful to introduce a genus for the individual terms in the sum of (4) by defining $g := 1 - n + 1/2 \sum (w_j - 1)$ (where w_j are the lengths of the involved cycles and n is the number of active colours). Fixed the genus, there are multiple factors that give the same contributions: for each equivalence class, there are $|\mathrm{Stab}((1, \dots, w_i))| = (N - w_j)! w_j$ trivial choices that do not influence g_j, $\binom{N}{n}$ ways to choose the active colours and $n!$ ways to relabel them (equivalent elements obtained by a global gauge give the same contribution). Now, each genus g contributes as $N^{1-g-\frac{m}{2}} = N^{-\frac{1}{2}\chi(g,m)}$ (applying Stirling's formula).

2.2.2. Diagrams

In order to use the Strebel differential, we would like to have some diagrams. Here's a prescription[3]: for each active colour we draw a "fatgraph" loop, writing the corresponding index inside the inner circle. The two sides of the fatgraph are inequivalent the inner circle is drawn with a solid line and the outer circle with a dashed line. We will refer to the solid line as the "*color line*". We mark the external (dashed) line of each fatgraph with the labels of the twist fields that contain the corresponding colour. Finally, we glue the non-colour loops together at the positions of the twist fields, in such a way that the order of the loops at each vertex (circling the vertex counter clockwise) corresponds to the cycle structure of the corresponding twist field.
A term is said to be *reducible* if the group elements g_i can be split into two sets so that the elements in each set act trivially by conjugation on the elements of the other set[7]. A reducible term factorizes into irreducible components. If a term is reducible, all the terms in the same class are reducible, so we may speak of reducible and irreducible classes.

[6] And this fusion rule is compatible with the mondromy of twist fields because $g_1' g_2' = g_2 g_1$

[7] Another way to state this condition is to say that the group elements g_j of an irreducible term generate a transitive subgroup of S_N

It is clear that our procedure associates irreducible classes to connected diagrams, and reducible classes to disconnected diagrams. The usual combinatorial arguments apply: the generating functional of all diagrams is the exponential of the generating functional of irreducible diagrams.
An w-cycle twist field corresponds to a vertex with $2w$ fatgraph propagators emanating from it: w solid ("colour") and w dashed oriented lines, in alternating order.

Remark. The genus of each diagram is easily computed: for s-point function involving n active colours we have $v = s$, $f = 2n$, $e = \sum_{j=1}^{s} w_j$,

$$g = \frac{1}{2}(e - v - f + 2) = \frac{1}{2}\sum_{j=1}^{s}(w_j - 1) - n + 1 \ .$$

This justifies the definition of the genus given earlier.

Remark. There seems to be a strong similarity between these diagrams and the one arising from a gauge theory. However, while in both cases the correlator organizes itself as sums of contributions of different fatgraphs weighted by the genus, here the basic vertex is quartic, associated to a twist-two field, and not cubic (and there are many other subtle constraints, further discussed in [3]). These differences disappear when passing to the skeleton graph where homotopically equivalent edges are fused together and the graph becomes metric.

Let us summarize what we have found. The fundamental correlators one has to compute are classified by equivalence classes of tuples $(g_1, \ldots, g_m)$ satisfying:

- **Conjugacy cass**: each g_j is a syngle cycle of length w_j
- **Connectedness:** $\langle g_1, \ldots, g_m \rangle \subset S_n$ is a transitive subgroup
- **Nontriviality (fusion rules)**: $g_1 g_2 \cdots = \mathbb{1}$
- $(g_1, g_2, g_3) \sim (g g_1 g^{-1}, g g_2 g^{-1}, g g_3 g^{-1})$

One theorem by Hurwitz asserts that this data characterises precisely the possible inequivalent covering maps of the Riemann sphere by a genus g surface with ramification indices w_j at the respective insertion points.
This suggests that this expansion can be identified with the string worldsheet genus expansion under the AdS/CFT correspondence, with the identification $g^2_{\text{string}} \sim 1/N$ and the worldsheet being identified with the covering map itself.

2.2.3. Covering map

This map is the one that unwind the twists: take n copy of the base space at the insertion point of σ_j merge w_j of them in accordance with the data of the permutation (i.e. such that if we assign indexes to the copies, if we turn around the insertion point starting in the sheet i we end up in $\sigma_j(i)$. We claim is there a way to assign one of the original n fields $X^j(z)$, mixed by these twists, so that on the overlay there is a single, well-defined

field $X(t)$. Finally, in order to have the same physics, we need to define the action of the new theory as $S = S_{CFT}|_{\text{patch}}$.
The partition function has to be rescaled by a factor due to conformal anomaly: the metric on Σ is given by the pullback of the metric on the z-plane (that can be chosen to be the flat one except at infinity): $ds^2 = \left|\frac{d\Gamma}{dt}\right|^2 dt\, d\bar{t}$.
If we fix a fiduciary metric on Σ (e.g. for $\Sigma = \mathbb{CP}^1$ we can use the same as the z plane) the partition function is the same of the fiducial one up to the Louville term

$$S_L = \frac{c}{96\pi} \int d^2t \sqrt{-g^{(\hat{s})}} \left[\partial_\mu \phi \partial_\nu \phi g^{(\hat{s})\mu\nu} + 2R^{(\hat{s})} \phi\right] \ ,$$

where c is the central charge of the seed theory and $\phi = \log \partial\Gamma + \log \overline{\partial\Gamma}$.
As we have seen before, a theorem by Hurwitz tells us that the *number* of covering maps is the same as the *number* of diagrams. Let's present an explicit correspondence. Given a covering map, we draw a closed loop without self-crossings on the base sphere, touching the positions of the twist fields and enclosing infinity. The closed loop divides the base sphere into two regions: after choosing an orientation for the loop, by convention the "colour" region is to the left of the loop and the "non-colour" region to the right. The inverse image of this loop on the covering space defines the diagram. For future purposes, it is important to note that preimages of infinity can be now interpreted as the centre of the coloured faces.

2.2.4. Correspondence

What we have shown so far is pretty suggestive but actually finding the covering map is not an easy task. Let's look at the $g = 0$ case. The covering map takes the form of a rational function of degree N[8]:

$$\Gamma(z) = x_i + a_i^\Gamma (z - z_i)^{w_i} + \cdots \qquad \text{for } z \sim z_i \ , \qquad \Gamma(z) = \frac{p_N(z)}{q_N(z)} = \frac{p_N(z)}{\prod_{a=1}^N (z - \lambda_a)}$$

where both $p_N(z)$ and $q_N(z)$ are polynomials of degree N. We have chosen the latter, without loss of generality, to be a monic polynomial with N distinct zeroes corresponding to the poles λ_a of $\Gamma(z)$ (we have chosen infinity to be a general point, $x_i \neq \infty$).
We now observe that the poles and zeros of $\partial\Gamma(z)$ are fix occur at $z = z_i$ with order $(w_i - 1)$. On the other hand, the only poles appear at $z = \lambda_a$, and they are all double poles with residue zero (because there's no logarithmic term in Γ). This residue constraint (is not an additional constraint, but just an observation) leads us to a series of scattering equations:

$$\sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a}^{N} \frac{2}{\lambda_a - \lambda_b} \ . \qquad (5)$$

[8]**Achtung:** from now on, we are only interested on the covering space, thus we chose z to be its coordinate (hopefully not causing too much confusion)

In theory, one could solve this system for the zeroes λ_a[9], but it is still a difficult task. However, the equation (5) looks a lot like the stationary equation of a Penner-like matrix model (i.e. with potential $W(z) = \sum_{i=1}^{n-1} \alpha_i \log(z - z_i)$ with $\alpha_i := (w_i - 1)/N$).
We can thus use standard matrix model techniques (reviewed in Appendix A). A particular useful one is the spectral curve which, rewritten in term of our problem, becomes:

$$y(z) = \frac{1}{N} \partial \log \partial \Gamma \ .$$

2.2.5. Spectral curve and covering map

While actually solving the problem we need a strong change of prospective: in the matrix model, the z_j are given *ab initio* (defining the potential) and we solve for the spectral curve finding the eigenvalue density (including its support) in terms of the input data (α_i, z_i, ν_i).
In our context it is natural, instead, to specify the $2(n-3)$ independent periods of the spectral curve (over both the A- and B-cycles). This comes from the relationship between the spectral curve and the (skeleton graph of the) diagram of the covering map: after gluing together homotopically equivalent edges we are left with:

- the number of edges between pairs of vertices (i, j) in the original double-line diagram ($n_{ij} = n_{ji}$). These integers are constrained by their sum around a fixed vertex i to be $2w_i$. In a generic graph (3 edges from each vertex) there are $(3n-6)$ edges so $2n-6$ independent n_{ij}
- in the original double-line diagram we had one pole λ_a for each of the N coloured faces, but in the skeleton graph only $(2n-4)$ (generically triangular) faces remain. This implies that, at large N, most of the N poles are associated with the two-edged faces formed from homotopic Wick contraction. These poles coalesce in the large N limit into a system of cuts $\mathcal{C}$, which are transverse to the original edges, and are now seen to build up the edges of the dual skeleton graph (approximately n_{ij} for the cut transverse to the segment (ij)).

We can tie everything together in a nice way: starting from the spectral curve[10], we identify the cut system $\mathcal{C}$ with the dual skeleton graph $\mathcal{G}^D$. The discontinuity of $y_0(z)$ across a cut counts the fraction of λ_a poles associated to this cut, and thus:

$$\frac{1}{4\pi i} \oint_{A_l} y_0(z) dz \equiv \nu_l = \frac{n^{(l)}}{2N}, \quad \frac{1}{4\pi i} \oint_{B_l} y_0(z) dz \equiv \mu_l = \frac{\tilde{n}^{(l)}}{2N} \ .$$

We can then use these period integrals to determine the $(n-3)$ independent parameters of $\tilde{R}_{n-3}(z)$ together with the $(n-3)$ cross ratios of the z_i.

[9] Actually, there are only $(N-1)$ independent equations because the sum of all of them is zero. However, we could still determined $(N-1)$ λ_a in term of λ_N. One is then left with three undetermined parameters: λ_N , M_Γ and the constant of integration in going from $\partial\Gamma(z)$ to $\Gamma(z)$. These are fixed, for instance, by requiring that $\Gamma(z_i) = x_i$ (for $i = 1, 2, n$)

[10] The converse is immediate: the lengths of the graph, when interpreted as filling fraction, completely determines the spectral curve

2.2.6. Spectral curve and Strebel differential

Another striking fact is that, if we keep the $1/N$ term in equation 13 and we express it in term of the original covering map we obtain the Schwarzian derivative of Γ:

$$y^2(z) - \frac{2}{N}y'(z) = \frac{1}{N^2}\left[\left(\frac{\Gamma''}{\Gamma'}\right)^2 - 2\frac{\Gamma'''}{\Gamma'} + 2\left(\frac{\Gamma''}{\Gamma'}\right)^2\right] - \frac{2}{N^2}S[\Gamma] \ .$$

This is actually more important than it appears: the transformation property of this derivative, implies that the spectral curve is a quadratic differential[11]:

$$S[\Gamma(f(z))] = f'(z)^2 S[\Gamma(z)] \ \Rightarrow \ 4\pi^2\phi_S(z)dz^2 \equiv -y_0^2(z)dz^2 = -\frac{\alpha_n^2}{\prod_{i=1}^{n-1}(z-z_i)^2}\prod_{k=1}^{2n-4}(z-a_k) \ .$$

Most importantly, it has poles at z_i with residue α_n which are real, and also real lengths between the zeroes (they are, following the previous discussion, the periods around the branch cuts of the original spectral curve). Finally, the Strebel graph is nothing other than $\mathcal{G}^D$. Now, the sum over all the branched covers defining the symmetric product correlator goes over, in the large twist limit, to an integral over the moduli space of the n-punctured sphere, where Strebel's theorem guarantees that we cover the moduli space exactly once.

Example 2. Let's actually compute the covering map of the simplest four-point function, i.e. $w_i = 2 \ \forall i$: $\langle\sigma_2(0)\sigma_2(1)\sigma_2(u)\sigma_2(\infty)\rangle$ (we also fix the preimages to be in $0,\ 1,\ x, \infty$). The number of preimages of a generic point is thus $N = \sum(w_i - 1)/2 + 1 = 3$, and thus there is another pole at l. The scattering system is now one single equation that can easily be solved:

$$\frac{1}{\ell} + \frac{1}{\ell - 1} + \frac{1}{\ell - x} = 0 \Rightarrow \ell = \frac{1}{3}\left(1 + x \pm \sqrt{1 - x + x^2}\right), \quad x = \frac{l(3l-2)}{2l-1} \ .$$

And the map is found by integrating (with the boundary conditions $\Gamma(0) = 0,\ \Gamma(1) = 1$):

$$\Gamma(t) = \frac{t^2(2\ell t - 3\ell - t + 2)}{t - \ell} \ .$$

Example 3. We now compute all relevant quantities for the simplest case: the large-N three-point function. We can fix all three of the points by $SL(2,\mathbb{C})$ invariance, as well as their images. The potential of the matrix model, and the resolvent are given:

$$W(p)' = \frac{\alpha_0}{p} + \frac{\alpha_1}{p-1} \rightsquigarrow G(p) = \frac{1}{2}\oint_{\mathcal{C}} \frac{dz}{2\pi i}\frac{\frac{\alpha_0}{z} + \frac{\alpha_1}{z-1}}{p - z}\sqrt{\frac{(p-a)(p-b)}{(z-a)(z-b)}} \ .$$

[11] Another reason is that it presents the wanted invariance under post-composition with an element of $SL(2;\mathbb{C})$

There is only one cut, whose extremes are found imposing the constraints (11) (the integrals reduce to sum of residues). They can be seen to be solutions of a specific polynomial:

$$\delta_{\ell s} = \frac{1}{2}\oint_{\mathcal{C}} \frac{dz}{2\pi i}\left(\frac{\alpha_0}{z} + \frac{\alpha_1}{z-1}\right)\frac{z^\ell}{\sqrt{(z-a)(z-b)}}, \quad \ell = 0,1 \ ,$$
$$z^2 - z\left(1 + \frac{\alpha_o^2 - \alpha_1^2}{(2-\alpha_0-\alpha_1)^2}\right) + \frac{\alpha_0^2}{(2-\alpha_0-\alpha_1)^2}$$

We can thus explicitly compute the resolvent, which in turn gives us the density of the poles and the spectral curve:

$$y(p) = -(2-\alpha_0+\alpha_1)\frac{\sqrt{(p-a)(p-b)}}{p(p-1)} = 2\pi i \rho(p)$$

Where the last equality is true only on the support of $\mathcal{C}$. It is easy to see that its square is a well-defined quadratic differential with the right poles and residues!

2.2.7. String on AdS_3

In the following we will give a (very) brief review of the other side of the correspondence (for a much more for a more detailed discussion please refer to [5]).
Treating AdS_3 as the coset space $SL(2,\mathbb{C})/SU(2)$ and choosing the parameterisation:

$$h = \begin{pmatrix} e^{-\phi} + \gamma\bar{\gamma}e^{\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}$$

In other words, ϕ is the radial coordinate and γ and $\overline{\gamma}$ are the
coordinates of the boundary sphere. The boundary of AdS_3 is located at $\phi \to \infty$
The Nambu-Goto action becomes:

$$S_{\mathrm{AdS}_3} = \frac{k}{4\pi}\int \mathrm{d}^2 z\sqrt{g}\left(4\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \mathrm{e}^{-2\Phi}\beta\bar{\beta}\right)$$

Moreover, in order to guarantee conformal invariance, one should also add a WZW term of factor k. From this, one can find the classical solutions being parametrized by three holomorphic ($\rho(z)$, $b(z)$ and $a(z)$) and three anti-holomorphic counterparts.

$$\Phi(z,\bar{z}) = \rho(z) + \bar{\rho}(\bar{z}) + \log(1 + b(z)\bar{b}(\bar{z}))$$
$$\gamma(z,\bar{z}) = a(z) + \frac{\mathrm{e}^{-2\rho(z)}\bar{b}(\bar{z})}{1 + b(z)\bar{b}(\bar{z})}$$
$$\bar{\gamma}(z,\bar{z}) = \bar{a}(\bar{z}) + \frac{\mathrm{e}^{-2\bar{\rho}(\bar{z})}b(z)}{1 + b(z)\bar{b}(\bar{z})}$$

Taking now the limit:

$$b(z) = b_0 + \epsilon c(z), \quad \rho(z) = -\frac{1}{2}\log\epsilon + \sigma(z) \quad \epsilon \to 0$$

We find:

$$\begin{aligned}
\Phi(z,\bar{z}) &= -\log\epsilon + \log\left(1 + b_0\bar{b}_0\right) + \sigma(z) + \bar{\sigma}(\bar{z}) \\
\beta(z) &= \mathrm{e}^{2\sigma(z)}\partial c(z) \\
\gamma(z) &= a(z)
\end{aligned}$$

This solution reflects the fact that ϕ has an infinite additive constant $(-\log\epsilon)$ which essentially places the worldsheet at the boundary.
On the other hand, by studying the Ward identities for the correlation functions of vertex operators one find:

$$\left\langle \prod_{i=1}^{n} V_{h_i}^{w_i}\left(x_i; z_i\right)\right\rangle = \sum_{\Gamma}\prod_{i=1}^{n}\left(a_i^{\Gamma}\right)^{-h_i}\prod_{i=4}^{n}\delta\left(x_i - \Gamma\left(z_i\right)\right) W_{\Gamma}\left(z_4, \ldots, z_n\right)$$

They exactly match our hope: they localize (thanks to the δs) where the covering map previously discussed exists with the interpretation that the vertex operators are inserted in the preimages of the points where the twist operators are situated. This geometric argument can be made even more specific: it can be showed that the same correlators lead to the identifications:

$$\begin{aligned}
\gamma(z) &= \Gamma(z) \\
\partial\Phi(z) &= -\frac{\partial^2\Gamma(z)}{2\partial\Gamma(z)} \Rightarrow \Phi = -\frac{1}{2}\log(\partial\Gamma) - \frac{1}{2}\log(\bar{\partial}\bar{\Gamma}) + \text{ constant}
\end{aligned}$$

3. Limiting Strebel differential

Although the theorems are very fascinating, actually computing the Strebel differential for a given data (z_i, p_i) is a complicated task, mainly due to the transcendentality of the reality condition (which involves elliptic integrals).

$$\Im\int_{z_i}^{z_j}\sqrt{\phi(z)}dz = 0$$

For this project, we focused on a few particularly tractable cases.

3.1. The easiest 4-point function

To begin with, we analysed the simplest non-trivial case: the sphere with four punctures, in the limiting cases where one or two of the Strebel lengths are much larger than the others.

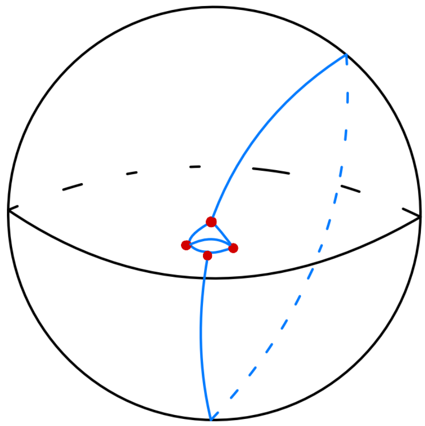


Figure 3: Critical graph in the limiting cases where one of the Strebel lengths are much larger than the others

Motivations The physical reasons behind this choice can be found in the work of Gopakumar ([6] and earlier). In these papers the differential was used in the context of the duality between $\mathcal{N} = 4$ SYM and strings on AdS_5. In this case, the operators involved in the correlation functions are of the trace type $\mathrm{Tr}\,\Phi^{J}$[12]. These correlation function can be computed through the fatgraph introduced by 't Hooft.
In this framework, Gopakumar realised in [9] that by writing the propagator in the Schwinger representation,

$$\frac{1}{p^2+m^2} = \int_0^\infty d\tau \exp\left\{-\tau\left(p^2+m^2\right)\right\} \tag{6}$$

the correlation function can be rewritten in terms of only quantity related to the skeleton graph (assuming the Schwinger parameters behave as resistors):

$$\frac{1}{\tau_r^{eff}} = \sum_{\mu_r=1}^{m_r} \frac{1}{\tau_{r\mu_r}} \rightsquigarrow G_g^{\{J_i\}}(k_1,\ldots k_n) = \sum_{\substack{\text{skel.}\\ \text{graphs}}} \int_0^\infty \frac{\prod_r d\tau_r f^{\{J_i\}}(\tau)}{\Delta_{skel}(\tau)^{\frac{d}{2}}} e^{-P_{skel}(\tau,k)} \ .$$

The Strebel differential is obtained after the identification $l = 1/\tau$[13].
Now having one (or two) length(s) bigger than the others means that the corresponding τ is small or equivalently (looking at (6)) we are selecting $p \sim 1/\tau^2$ big: the UV limit! From this, for example, we can get the OPE of trace operator.

[12] These are the simplest gauge invariant operators that can be created with the three complex scalars (in the adjoint representation) of the theory. They have many interesting features (e.g. they are primary half BPS) and play a fundamental role in the integrability of the theory (they can be mapped to particular states of a $\mathfrak{su}(2,2|4)$ spin chain, then resolved via for example the BA)

[13] This dictionary is more problematic than the one seen in the previous sections. For example, it does not seem to preserve all symmetries, breaking the special conformal one. The way to restore the space time symmetries is, indeed, to avoid direct coupling between the space time variables and the length parameters of the metric graphs

Computation The general quadratic differential for a four-point function is of the form:

$$\phi(z)dz^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{\prod_{i=1}^4 (z - z_i)}{(z-1)^2(z^2-t^2)^2} dz^2$$

where we used the $SL(2,\mathbb{C})$ (reparameterization) invariance to place the punctures at $(1, \pm t, \infty)$. The numerator is fixed to be a polynomial of degree four because of the requirement of a double pole at ∞.
We now want to obtain $z_i = z_i(t, p_i)$ and $a = a(t, p_i)$ (actually the residue at ∞ we immediately see $a = p_\infty$).
Let start with the first case: one of the Strebel lengths is very large compared to the others, or equivalently (after an overall rescaling [14]), when all but one length is scaling to zero, with the relative ratios of these finite in the limit.
Before starting, we ponder on the geometrical picture: the requirements on the lengths are achieved by the configuration where all the zeroes converge to a single point. If we fix the finite length to be $l_{1\infty}$ then the two punctures $\pm t$ are enclosed by "small circles" so it makes sense to make the following ansatz:

$$|t| \to 0, \quad z_i(t) = |t|^{\alpha_i} \tilde{z}_i(t)$$

with $\tilde{z}_i(t) \to \tilde{z}_i$ finite, in the limit $|t| \to 0$. Now the differential becomes:

$$\phi(z)dz^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{\prod_{i=1}^4 (z - |t|^{\alpha_i} \tilde{z}_i)}{(z-1)^2(z^2-t^2)^2} dz^2$$

However, it is important to emphasise that at this point there is no definitive proof that the condition $|t| \to 0$ is necessary, as the reasoning on lengths is done with the Strebel measure, and thus not clearly translatable into constraints on Euclidean distances (or similarly on the t coordinate). However, if we arrive to a consistent solution, the ansatz is justified *a posteriori* due to the uniqueness of the Strebel differential.
In fact, we will prove that a consistent solution is given by $\alpha_i = 1\ \forall i$. The proof follows a few steps:

- Looking at the residue at $z = 1$ and requiring it to be finite we get $\alpha_i \geq 0$.
- Next, we looked at the region $z \sim 0$ where two poles and possibly some zeroes are converging. In order to so, we make the change of variable $w = |t|/z$ (and defined $w_i = 1/\tilde{z}_i$):

$$\phi(w)dw^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{|t|^{\sum_{i=1}^4 \alpha_i \; -2}}{e^{4i\theta}} \frac{\prod_{i=1}^4 (w - |t|^{1-\alpha_i} w_i)}{w^2(w-|t|)^2(w^2 - e^{-2i\theta})^2} dw^2$$

  This, in fact, ensure that the interesting punctures have now a fix coordinate $\pm e^{i\theta}$ (while the other punctures will converge to 0).

[14]It is important to note that such a rescaling doesn't affect the topology of the critical graph, consequently it doesn't change anything in the $\mathcal{M}_{g,n}$ part of the moduli space

The residue $p_\pm$ are now easily seen to scale as $p_\pm^2 \sim |t|^{\sum_{i=0}^4 \min(\alpha_i,1)\ -2}$. Requiring that they go to zero we get:

$$\sum_{i=0}^{4} \min(\alpha_i, 1)\ - 2 \geq 0 \quad \xrightarrow{\alpha_i \geq \min(\alpha_i,1)} \quad \sum_{i=0}^{4} \alpha_i \geq 2 \tag{7}$$

- From the formulas shown so far, there would appear to be a symmetry between the four zeros. We would then be tempted to do a further ansatz $\alpha_i = \alpha\ \forall i$, but this would seem to lead to the same scaling for the lengths of all segments, contradicting the initial hypothesis (one finite, the others infinitesimal).

  However, the length $l_{1\infty}$ is, by definition, forced to be calculated along a curve passing in between the poles at 1 and ∞: this part of the segment gives a finite contribution.

  This becomes obvious after the change of variables $w = |t|^{1-\alpha}\tilde{w}$ (believing in ansatz, we just showed that is a coherent belief). In fact, the differential becomes:

$$\phi(w)dw^2 = -\left(\frac{a}{2\pi}\right)^2 \frac{|t|^{6\alpha-4}}{e^{4i\theta}} \frac{\prod_{i=1}^4(\tilde{w} - w_i)}{\tilde{w}^2(\tilde{w} - |t|^\alpha)^2(\tilde{w}^2 - |t|^{2(\alpha-1)}e^{-2i\theta})^2} d\tilde{w}^2 \tag{8}$$

  Now the zeroes are in fixed position and it can always be chosen a path which avoids the punctures. The integrand is now regular apart from a scaling factor of $l_{ij} \sim |t|^{3\alpha-2-\min(\alpha-2,0)}$ apart from the one forced to path between the singularities at $\tilde{w} = 0, |t|^\alpha$ (in this case, condition (7) becomes $\alpha > 1/2$).

Requiring that lengths and periods scale in the same way (since all these Strebel lengths are supposed to go uniformly to zero) fixes $\boxed{\alpha = 1}$.
As we were saying at the begining, having found a consistent solution, we have now *a posteriori* confirmation of the validity of the various assumptions.
Let's consider the second case. Requiring that the length of the segment separating $w = \pm e^{-2i\theta}$ also stay finite in the limit $|t| \to 0$ doesn't change much of the reasoning: between each pair of zeros there is always a segment of infinitesimal length and therefore the ansatz $|t| \to 0$ is still reasonable; the length argument also holds, with an observation on $l_\pm$ similar to what was said for $l_{1\infty}$[15].
The first difference is that now the residue $p_\pm$ must be finite:

$$4\alpha - 2 = 0 \quad \Rightarrow \quad \boxed{\alpha = \frac{1}{2}}$$

We observe that the lengths (not affected by topological constraints) scale like $l \sim |t|^{3\alpha-2-\min(-1,0)} = |t|^{\frac{1}{2}}$, and we therefore have a consistent solution.

[15]In fact, plugging the value of α we'll later find into (8), it can be seen that the poles at $\pm t$ both go to ∞ and, in the same way as before, in order to calculate the length $l_\pm$ it must be chosen a path through them

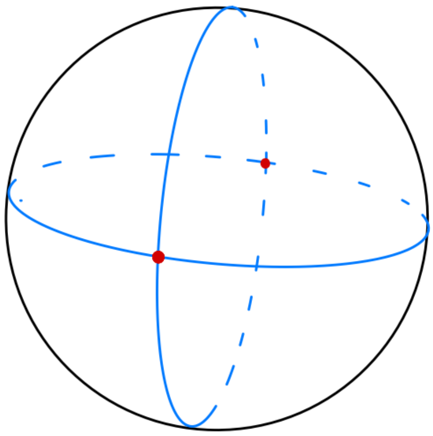


Figure 4: Dual of the Square graph

3.2. Box and Whales

Another case of interest is when (at the complete opposite of before) two opposite segments has infinitesimal lengths. More specifically, we are interested in the case where the sides of finite length are arranged in a square.
In this case, the physical motivation is provided by the hexagonisation programme. With this idea of integrability, it is possible to calculate finite coupling correlation functions by dressing diagrams with particular form factors. The diagram corresponding to the differential we are going to study is of particular interest in that the long sides of the square correspond in operators with a high number of Wick contractions between them. For this reason, the correlation function factorizes as the product of two (identical) contributions from the regions outlined by the square, making the application of the hexagon programme particularly easy.

3.2.1. Exact solution

Unfortunately, the techniques used above are not applicable in this case because the four zeros can no longer be connected to each other by a path of infinitesimal length or similarly no longer collapse to a single point but to two. Moreover, while before we were trying to find only the limiting behaviour, now we look for an exact solution (and then, try to perturbate it). Here, the simplicity camas from the fact that the quadratic differential in this case is actually the square of a usual one: the imaginary condition no longer has any ambiguity of branch cuts and the integral is no longer elliptic.
We are looking for a quadratic differential of the type:

$$-\left(\frac{p_\infty}{2\pi}\right)^2\left(\frac{(z-z_1)(z-z_2)}{(z-1)(z^2-t^2)}\right)^2(dz)^2=-\frac{1}{4\pi^2}\left(\frac{p_+/p_\infty}{z-t}+\frac{p_-/p_\infty}{z+t}-\frac{p_1/p_\infty}{z-1}\right)^2(dz)^2$$

where we have fixed the lengths l_i (and thus the residue p_i, subject to the constraint $\boxed{p_+ + p_- = p_1 + p_\infty}$) and we are looking for solutions for z_i, t. It is important to note that, due to this constraint, the relation between lengths and periods is not one-to-one (the residue have one less degree of freedom) and thus the additional degree of freedom of the lengths in the graph side of the picture will manifest itself in a degree of freedom in the moduli space (the graph is non generic, we are thus moving on a lower dimensional cell).

Remark. Rescaling the lengths by p_∞ has no effect on the topology of the graph nor on the moduli space of the surface. From now on $p_\infty = 1$.

Calculating the residue, we obtain a system of three equation

$$p_+ = \frac{(t-z_1)(t-z_2)}{2t(t-1)} \quad p_- = \frac{(t+z_1)(t+z_2)}{2t(t+1)} \quad -p_1 = \frac{(1-z_1)(1-z_2)}{1-t^2}$$

where in reality the third one is a consistency check for the constraint on the residue. Taking the sum and difference of the first to equation, and defining A, B as follows gives:

$$A := p_+ + p_-, \quad B := p_+ - p_-, \qquad A - tB = z_1 + z_2, \quad tA - B = \frac{t^2 + z_1 z_2}{t}$$

namely $z_{1,2}$ are the solutions of the following polynomial $z^2 - (A-tB)z + t(t(A-1)-B)$:

$$\boxed{2z_{1,2} = (A-tB) \pm \sqrt{(A-tB)^2 - 4t(t(A-1)-B)}} \tag{9}$$

Now we are left with the reality condition. An important observation that will render the computation easier is that the differential can be rewritten as:

$$\phi(z)(dz)^2 = -\frac{1}{4\pi^2}(d\log F(z))^2 \quad F(z) := (z-1)^{-p_1}(z-t)^{p_+}(z+t)^{p_-}$$

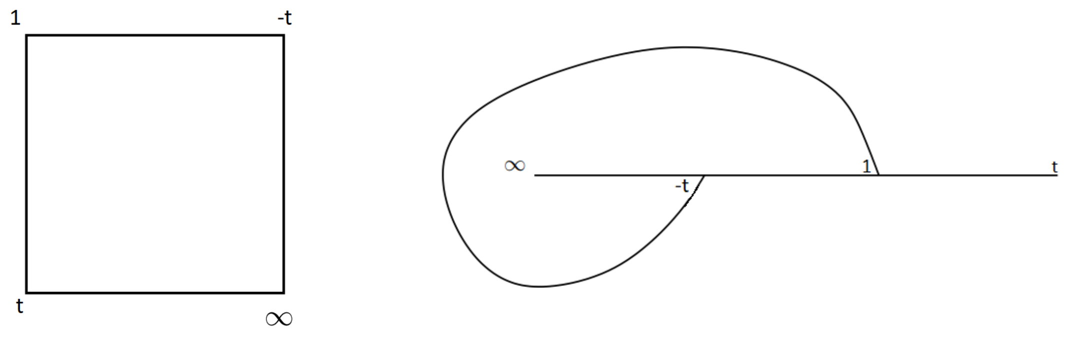
The reality condition now becomes:

$$\Im \int_{z_1}^{z_2} \frac{i}{2\pi} d\log F(z) = 0 \;\Rightarrow\; \Re\left[\log F(z)\right]_{Z_1}^{z_2} = 0 \;\Rightarrow\; F(z_1)\overline{F(z_1)} = F(z_2)\overline{F(z_2)}$$

while the horizontal leaves are characterized by:

$$\left(\frac{dz}{dt}\right)^2 > 0 \Rightarrow \Re\left[\log F(z)\right] = 0 \;\Rightarrow\; F(z)\overline{F(z)} = C$$

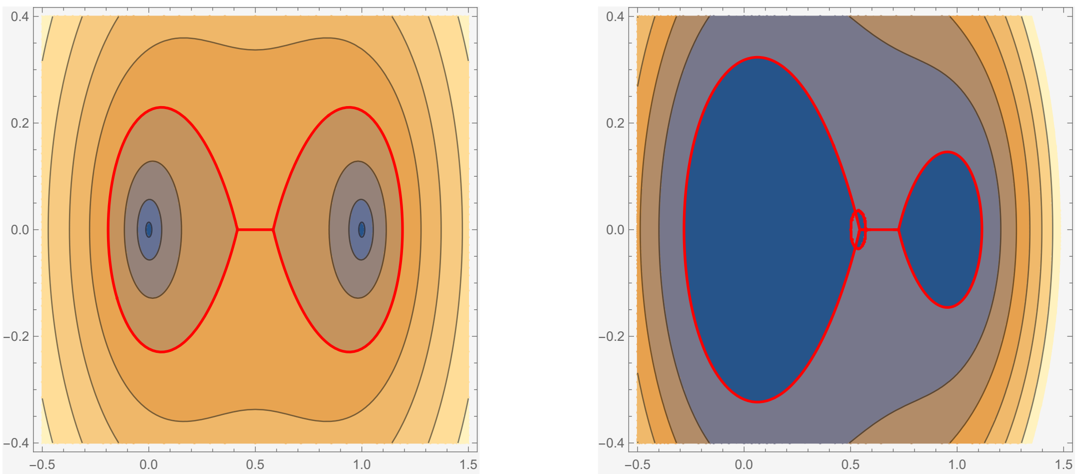
The Strebel condition now has a nice interpretation: the C parameter for both zeros has to be equal, and this is required if we want a leaf to go from one zero to another. Otherwise, either the leaves emanating from a zero will not be compact (will not end) or they will end on the same zero and the critical graph will be disconnected. Both of these cases contradict the Strebel conditions.



(a) (b)

Figure 5: Square and Strebel diagrams satisfying $p_+ + p_- = p_1 + p_\infty$

Remark. The differential found, can give rise to a critical graph dual to both the Square and the Whale. A careful analysis is needed in order to distinguish between the two cases. For the analysis is better to work with the cross ratio $\eta = (t+1)/2t$. In this way, the reality condition can be seen to single out a curve composed of a segment of the real axis, to the ends of which two bubbles are connected. We claim that the solution corresponding to the Square is the real η solution.

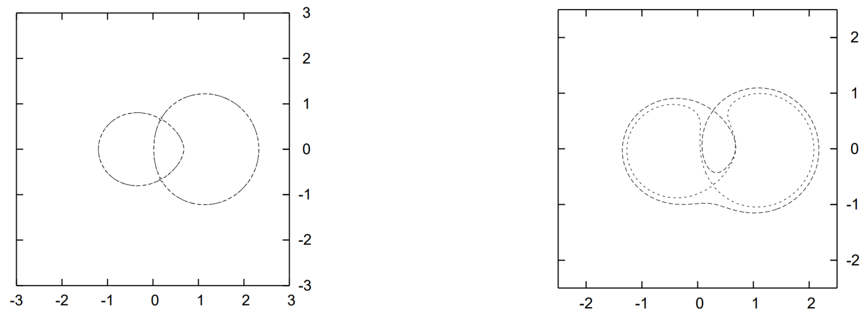


(a) (b)

Figure 6: The curve on the complex η plane for which the Strebel condition is satisfied for different fixed residue: on the left $p_{-t} = p_t = 70$ and on the right $p_{-t} = 40$, $p_t = 70$. The small circle is just an artifact due to branch cut ambiguities.

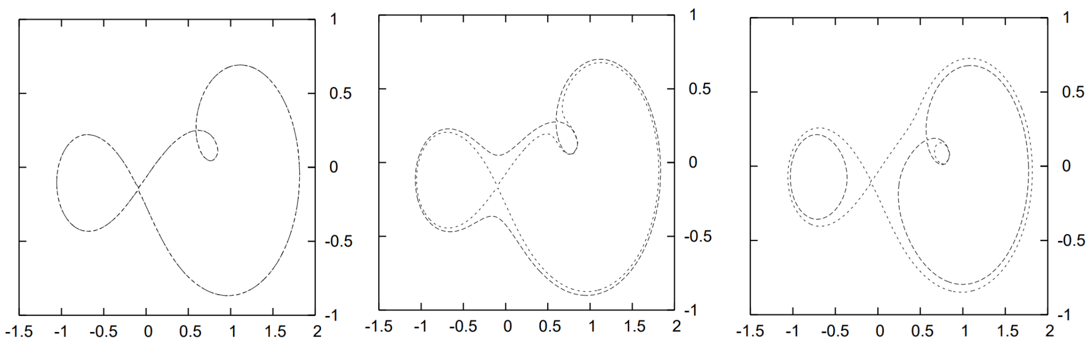
Note that at a generic point of the η plane (where the Strebel condition is not satisfied) the horizontal leaves begin and end on the same zero, and can have only three topologically distinct shapes (but only two of them can be merged to graphs that are dual to the ones satisfying the specific residue condition). We can call the two shapes shape I and shape II (they can be easily distinguished by looking at the dual graph).

Now, the real line is the boundary between two different ways to get shape I, analogous to the fact that from the square you can only reach one class of disconnected graphs. The complex solution, on the other hand, is the boundary between shape I and II as the Whale graph can be split in two topologically different ways[16].



(a) (b)

Figure 7: On the left we draw the critical graph of the Square diagram for a typical real solution. On the right we have the critical curves when slightly moving η to the complex plane.



(a) (b) (c)

Figure 8: On the left we draw the critical graph of the Whale diagram when the Strebel condition is satisfied. Then, we slightly move η away from this condition: in the middle we are in the region I, on the right we are in region II.

One important point is the intersection between the real line and the complex shapes: it represents the limiting case in which one of the segments has length 0. In both cases, fixing the residue limits our freedom in the choice of segment to cut (for example, if we are in the square case and $p_1 > p_t$ we cannot cut l_{1t} without violating the inequality) and

[16]We theorized a way to rigorously classify the different topologies reachable. It involves splitting each line of the original graph into two and then divide into two groups the boundary component of the new graph in all the *allowed* way (they will be the two new disconnected graph, after reconstructing the vertex): if two segments originating from the same one end up in the same group, the final graph is obtained by gluing them together

we arrive at the Π diagram. This is also consistent with the fact that for this diagram there is a biunivocal relationship between lengths and residues: the reality condition is no longer necessary and the periods alone fix the modulus η as the solution of a second degree polynomial [10].
Ather interesting points are the degenerate cross ratio: $\eta \in \{0, 1, \infty\}$. From the plots 6b, 6a we are led to claim that $\eta = 0, 1$ are always inside the bubbles while $\eta = \infty$ is always in region I. This behaviour is made clear by looking at how disconnected graphs arise from the Square: here you can split the insertion points in two groups in only one way, namely $\{\{p_\eta,\ p_\infty\}\ \{p_0,\ p_1\}\}$ and now, the only way to keep the graph disconnected while moving to a degenerate point is to take the limit $\eta \to \infty$ (the only puncture related to the same zero).

Example 4. The simplest case is given by all residues equal: equation (9) and the reality conditions respectively turn out to be

$$z_{1,2} = 1 \pm \sqrt{1 - t^2} \qquad t \in \mathbb{R},\ |t| > 1$$

It can be seen that in this the only solution is the real η segment. In fact, this degenerate case for the Whale diagram: since we have $p_1 = l_1,\ p_\infty = l_\infty$ but $p_{(-)t} = l_+ + l_- + l_{1(\infty)}$, having all periods equal implies having two null lengths $l_+ = l_- = 0$.
If we now fix the only degree of freedom on the lengths side, we obtain the equation:

$$2\pi l = \Im \left[\log F(z)\right]_{z_1}^{z_2} = 4\,\mathrm{Arg}(\sqrt{1 - t^2} + 1)\ \Rightarrow \mathrm{tg}(\frac{\pi}{2} l) = |\sqrt{1 - t^2}|$$

With $l \in [0, 1]$ (due to the rescaling) and $|t| > 1$. We indeed have a biunivocal relationship between the allowed ranges of the two parameters!

3.2.2. Perturbation around the exact solution

Starting from the Square solution, we want to perturb it in order to reach the physically relevant case of a four-point function with two opposite infinitesimal lengths. Following [7] we start by separating one of the double zeroes into two single ones, ϵ apart from each other.

$$\phi(z)(dz)^2 = -\left(\frac{p_\infty}{2\pi}\right)^2 \frac{(z - z_1)(z - z_1 + \epsilon)(z - z_2)^2}{(z - 1)^2(z^2 - t^2)^2}(dz)^2$$

Expanding in orders od ϵ the residues equations become:

$$\begin{aligned} p_+ &= \frac{(t - z_1)(t - z_2)}{2t(t - 1)} + \frac{\epsilon}{2}\frac{(t - z_2)}{2t(t - 1)} - \frac{\epsilon^2}{8}\frac{(t - z_2)}{2t(t - 1)(t - z_1)} + \mathcal{O}(\epsilon^3) \\ p_- &= \frac{(t + z_1)(t + z_2)}{2t(t + 1)} - \frac{\epsilon}{2}\frac{(t + z_2)}{2t(t + 1)} - \frac{\epsilon^2}{8}\frac{(t + z_2)}{2t(t + 1)(t + z_1)} + \mathcal{O}(\epsilon^3) \\ -p_1 &= \frac{(1 - z_1)(1 - z_2)}{1 - t^2} + \frac{\epsilon}{2}\frac{(1 - z_2)}{1 - t^2} - \frac{\epsilon^2}{8}\frac{(1 - z_2)}{(1 - t^2)(1 - z_1)} + \mathcal{O}(\epsilon^3) \end{aligned} \tag{10}$$

Let the (rescaled) length of the small edge of the perturbed diagram be l:

$$2l = p_+ + p_- - p_1 - 1 \overset{(10)}{\Longrightarrow} \boxed{l = -\frac{\epsilon^2}{8}\frac{z_1 - z_2}{(t^2 - z_1^2)(z_1 - 1)}}$$

Not surprisingly, the edge is non-zero only at the second order in the perturbation; the fact that the point we expand about has a two-fold degenerate zero implies that the first contribution to the additional edge vanish (just try to expand the equation for l: $\int_{z_1}^{z_1-\epsilon}\sqrt{\phi}$). The geometric meaning of this equation is clear: it is, a part from a prefactor, the residue of the two-fold zero in the Square diagram.
For any value of t and l we have two possible ways to split the zeros of the differential corresponding to Square diagram and obtain the dashed square, the difference will be in the position of the extra edge (we can choose between two diagonals of the box). Fixing the specific diagram, we want to compute uniquely fixes the branch of the square root. Going back to the system (10) and proceeding in the same way of the exact case we obtain

$$A - tB = z_1 + z_2 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8}\left(\frac{z_1 - z_2}{t^2 - z_1^2}\right) \qquad tA - B = \frac{t^2 + z_1 z_2 - \frac{\epsilon}{2}z_2 + \frac{\epsilon^2}{8}\left(\frac{m - t^2}{t^2 - z_1^2}\right)}{t}$$

$$\boxed{2z_1 = A - tB + \epsilon + \epsilon^2 \pm \sqrt{(A - tB)^2 - 4t(t(A-1) - B)}}$$

The first order perturbation can be understood geometrically: it means that the average position of the perturbed zeroes coincides with the position of the exact double one.

4. Conclusions

In this report, we introduced the Strebel differential. We presented one application to physics and we computed it in specific cases.
In future, it would be very interesting to delve into other aspects of AdS_3/CFT_2 duality:

- Moving away from the $k = 1$ case. It is observed in [11] that a perturbative approach may be viable: this consists in the insertion of k twist-two fields in the correlation function.

  We have begun to analyse how these new operators modify the differential. We first observe that in order to have $N \in \mathbb{N}$ we need $k \in 2\mathbb{N}$. In this case ($N \to N + k/2$), being $\tilde{z}_i$ their insertion points, the potential get modified in the following way:

$$W'(\lambda) = \left(1 - \frac{k}{2N}\right)\sum_{i=1}^{n}\frac{\alpha_i^{(0)}}{\lambda - z_i} + \frac{1}{N}\sum_{i=1}^{k}\frac{1}{\lambda - \tilde{z}_i}$$

  This results in a displacement of the extremes of the cuts of the spectral curve $x_j = x_j^{(0)} + \frac{1}{N}x_j^{(1)}$ which can be derived by perturbing equation (12):

$$0 = \frac{1}{2t}\oint_{\mathcal{C}} \frac{dz}{2\pi i} \frac{z^\ell}{\prod_{j=1}^{2s}\left(z - x_j^{(0)}\right)^{\frac{1}{2}}} \left(-\frac{k}{2}\sum_{i=1}^{n} \frac{\alpha_i^{(0)}}{\lambda - z_i} + \sum_{i=1}^{k} \frac{1}{\lambda - \tilde{z}_i} + \sum_{j=1}^{2s} \frac{x_j^{(1)}}{z - x_j^{(0)}} \right)$$

We would like to know how these changes affect the lengths of the critical graph.

- Although in the general discussion we have spoken of surfaces of any genus, the identification with the matrix model, the spectral curve and the Strebel differential as well as all the examples given are actually related to the $g = 0$ case. It is not entirely clear (to us) whether the discussion is directly extendable to higher genus or whether special precautions are required, and it would be nice to shed some light on this.

  For example, now the covering map is expressed in term of theta one forms (they are the equivalent of monomes $x - a$ for higher g)

$$\Gamma = c\frac{\prod_{i=1}^{N}\theta(z; Q_i)}{\prod_{i=1}^{N}\theta(z; P_i)} \qquad \sum_{i=1}^{N}\mu(Q_i) = \sum_{i=1}^{N}\mu(P_i)$$

  with new constraints due to Abel-Jacobi theorem that save the discussion of the digrees of freedom. The scattering equations

$$\sum_{i=1}^{n} \alpha_i \frac{\theta'(P_a; zi)}{\theta(P_a; z_i)} = \frac{2}{N}\sum_{b\neq a} \frac{\theta'(P_a; P_b)}{\theta(P_a; P_b)}$$

  can no longer be seen to derive from a matrix model, because the characteristic Fadeev-Popov term is absent but it is not clear if this conceptual difference renders unusable the rest of the machinery.

- The final aim is to derive a duality between AdS_5 and $\mathcal{N} = 4$ SYM (and relate it to the Hexagonisation project). An initial attempt was made in [12] but it was still limited to $AdS_3 \subset AdS_5$ subspaces. One could try to move (slightly) away from AdS_3.

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A. Appendix A: Matrix models and spectral curve

Matrix models are the simplest quantum gauge theory [13]: they are theories in 0 dimensions, with basic field a (Hermitian) $N \times N$ matrix and gauge symmetry $U(N)$ (acting by conjugation). Namely, the action is sum of trace terms:

$$\frac{1}{g_s}W(M) = \frac{1}{2g_s}\operatorname{Tr} M^2 + \frac{1}{g_s}\sum_{p\geq 3}\frac{g_p}{p}\operatorname{Tr} M^p \quad M \to UMU^\dagger$$

Fatgraphs Since we are dealing with a quantum theory of a field in the adjoint representation we can reexpress the perturbative expansions of the partition function in terms of fatgraphs, by using the double line notation due to 't Hooft.
An important result is that the contributions of the various fatgraph can be organized by the genus of the surface obtained by closing the holes of the graph $2-2g = V+h-E$: each hole give a contribution of N, each of the V_p vertices of degree p gives g_p/g_s, each propagator gives g_s. In total ($t = g_s N$):

$$g_s^{E-V} N^h \prod_p g_p^{V_p} = g_s^{2g-2+h} N^h \prod_p g_p^{V_p} = g_s^{2g-2} t^h \prod_p g_p^{V_p}$$

Quantisation, saddle point and spectral curves As any gauge theory, the matrix model can be quantised using the Fadeev-Popov method. In particular, a useful choice of gauge fixing $UMU^\dagger = D = \text{diad}(\lambda_1, \ldots, \lambda_N)$. Let $F(M) = 0$ be the gauge fixing condition and $U = e^A$ (with A anti-Hermitian), then:

$$F_{ij}\left({}^U D\right) = \left(UDU^\dagger\right)_{ij} = A_{ij}\left(\lambda_i - \lambda_j\right) + \cdots \Rightarrow \Delta^2(M) = \det \frac{\delta F\left({}^U M\right)}{\delta A}\Big|_{F=0} = \prod_{i<j}(\lambda_i - \lambda_j)^2$$

After this *reduction to eigenvalues*, the partition function can be rewritten as:

$$Z = \frac{1}{N!}\int \prod_{i=1}^N \frac{d\lambda_i}{2\pi} e^{N^2 S_{\text{eff}}(\lambda)} \qquad S_{\text{eff}}(\lambda) = -\frac{1}{tN}\sum_{i=1}^N W\left(\lambda_i\right) + \frac{2}{N^2}\sum_{i<j}\log\left|\lambda_i - \lambda_j\right|$$

We can now regard N^2 as a sort of h^{-1} in such a way that, as $N \to \infty$, the integral will be dominated by a saddle-point configuration that extremizes the effective action.

$$\frac{1}{2t}W'\left(\lambda_i\right) = \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} \tag{11}$$

In the large N limit, it is reasonable to expect that this the eigenvalues condensate along some curves $\mathcal{C} = \bigcup_{i=1}^s \mathcal{C}_i$. It is thus useful to introduce the density function:

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(\lambda - \lambda_i\right)$$

In order to solve for ρ is useful to introduce the resolvent:

$$G(p) = \int d\lambda \frac{\rho(\lambda)}{p-\lambda} \;\Rightarrow\; \rho(\lambda) = -\frac{1}{2\pi i}\left(G(\lambda + i\epsilon) - G(\lambda - i\epsilon)\right)$$

Now we are left with a Riemann-Hilbert problem for $G(p)$ which has the following solution:

$$G(p+i\epsilon) + G(p-i\epsilon) = -\frac{1}{t}W'(p) \;\Rightarrow\; G(p) = \frac{1}{2t}\oint_{\mathcal{C}}\frac{dz}{2\pi i}\frac{W'(z)}{p-z}\left(\prod_{k=1}^{2s}\frac{p-x_k}{z-x_k}\right)^{\frac{1}{2}}$$

where the x_i are the extremes of the cuts $\mathcal{C}_i = [x_{2i}, x_{2i-1}]$ and can be found by requiring the right behaviour at infinity for the resolvent:

$$G(p) \sim \frac{1}{p}\;(p\to\infty) \;\Rightarrow\; \delta_{\ell s} = \frac{1}{2t}\oint_{\mathcal{C}}\frac{dz}{2\pi i}\frac{z^{\ell}W'(z)}{\prod_{k=1}^{2s}\left(z-x_k\right)^{\frac{1}{2}}}, \quad \ell = 0, 1, \cdots, s \tag{12}$$

Another important object one can introduce is the *spectral curve*: $y(p) := W'(p) - 2G(p)$. In term of this quantity, the problem discussed in the main text can be rewritten as[17]:

$$y^2(p) - \frac{2}{N}y'(p) = \left(W'(p)\right)^2 - \frac{2}{N}W''(p) - 4R(p) \quad R(p) = \frac{1}{N}\sum_{a=1}^{N}\frac{W'\left(\lambda_a\right) - W'(p)}{\left(\lambda_a - p\right)} \tag{13}$$

Taking the large N limit of the Penner-like matrix model ($y \to y_0$, $W' \to \tilde{W}_{n-2}/S$, $R \to R_0 = \tilde{R}_{n-3}$, with S being another polynomial caused by rationalisation and the subscript signaling the degree of the numerator):

$$y_0^2(p) = \frac{\tilde{W}_{n-2}^2(p) - \prod_{i=1}^{n-1}\left(p-z_i\right)\tilde{R}_{n-3}(p)}{\prod_{i=1}^{n-1}\left(p-z_i\right)^2} \equiv \frac{Q_{2n-4}(p)}{\prod_{i=1}^{n-1}\left(p-z_i\right)^2}$$

Now $\hat{y}^2 = Q_{2n-4}$ define a genus $(n-3)$ surface on which $y_0(z)dz$ is a well-defined meromorphic differential (with poles at z_i and residue α_i). Now we are left with finding the $(n-3)$ unknown coefficient of $\tilde{R}_{n-3}$ and that can be done by specifying the same number of independent filling fractions.

[17] After some uninspiring algebra

References

[1] Michael Penkava Motohico Mulase. Combinatorial Structure of the Moduli Space of Riemann Surfaces and the KP Equations, 1997. URL: https://www.math.ucdavis.edu/~mulase/texfiles/1997moduli.pdf.

[2] Oleg Lunin and Samir D. Mathur. Correlation functions for $M^N/S(N)$ orbifolds. *Commun. Math. Phys.*, 219:399–442, 2001.

[3] Ari Pakman, Leonardo Rastelli, and Shlomo S. Razamat. Diagrams for Symmetric Product Orbifolds. *JHEP*, 10:034, 2009.

[4] Matthias R. Gaberdiel, Rajesh Gopakumar, Bob Knighton, and Pronobesh Maity. From symmetric product CFTs to AdS_3. *JHEP*, 05:073, 2021.

[5] Lorenz Eberhardt, Matthias R. Gaberdiel, and Rajesh Gopakumar. Deriving the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. *JHEP*, 02:136, 2020.

[6] Rajesh Gopakumar. From free fields to AdS: III. *Phys. Rev.*, D72:066008, 2005.

[7] Ofer Aharony, Justin R. David, Rajesh Gopakumar, Zohar Komargodski, and Shlomo S. Razamat. Comments on worldsheet theories dual to free large N gauge theories. *Phys. Rev. D*, 75:106006, 2007.

[8] Andrea Dei and Lorenz Eberhardt. Correlators of the symmetric product orbifold. *JHEP*, 01:108, 2020.

[9] Rajesh Gopakumar. From free fields to AdS. II. *Phys. Rev.*, D70:025010, 2004.

[10] Ofer Aharony, Zohar Komargodski, and Shlomo S. Razamat. On the worldsheet theories of strings dual to free large N gauge theories. *JHEP*, 05:016, 2006.

[11] Lorenz Eberhardt. A perturbative CFT dual for pure NS–NS AdS_3 strings. *J. Phys. A*, 55(6):064001, 2022.

[12] Faizan Bhat, Rajesh Gopakumar, Pronobesh Maity, and Bharathkumar Radhakrishnan. Twistor coverings and Feynman diagrams. *JHEP*, 05:150, 2022.

[13] Marcos Marino. Les Houches lectures on matrix models and topological strings. 10 2004.