Resummation identities in the boundary-to-bound (B2B) correspondence

Marco Ambrosini, Scuola Normale Superiore

Supervisors: Rafael A. Porto, Gregor Kälin, Zhengwen Liu

September 7, 2022

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Abstract

The extraction of the information embedded inside Gravitational Waves (GW) depends on our ability to make precise theoretical predictions. One option is to consider the bound 2-body problem and see expansions in the velocity in the non relativistic limit. A more recent approach is to consider instead relativistic scattering, from which we can obtain quantities as power series in the coupling G, and then relate these to the bound motion invariants via the boundary-to-bound (B2B) map. This map, to all orders in velocity, between gauge invariant quantities, illustrates how physical information is encoded in the scattering data and how to translate it into observables for bound states.

We introduce the basics of the B2B map and how tidal effects can be included in the EFT approach. We then discuss our progress in the search for partial resummation identities for the scattering angle and obtain in the no-recoil limit (that is the test particle limit) new compact expressions for the tidal corrections to all orders in the velocity.

Introduction

The detection of gravitational waves (GW) provides us with a brand new source of information, which can be extracted via an interplay of numerical and analytical tools. When studying a binary inspiralling motion, the majority of the cycles occur where the perturbative approximation of the Einstein's equation is valid. Thus we can use, for example, the Post-Newtonian (PN) expansion in velocity to reconstruct the signal and allow to reduce the problem to a computation of Feynman diagram in the corresponding EFT. The PN expansion is truncated at order in velocities depending to the power of G appearing in each contribution. This happens because for bound states the virial theorem relates $GM/r \sim v^2$, so that only terms scaling as $G^l v^{2(n-l)}$ (with $l \leq n$) are needed at nPN. We see now that the usual field theoretical expansion in powers of the coupling G (here the Post-Minkowskian PM expansion) requires the inclusion of special relativistic effects to all orders. This can be obtained in principle by resumming all velocity correction of the non-relativistic EFT, but is ultimately unnecessary in the framework that we are going to present (for a more detailed discussion [7]).

Our approach will be to consider the study of relativistic scattering, in which the PM framework is natural. This problem is unrelated only in appearance to the bound motion. Indeed we will show that it is possible to build a Boundary-to-Bound (B2B) map, to relate the computed quantities to the bound motion's dynamical invariants of interest.

In the next section we will introduce the basics of the B2B map and discuss the no-recoil limit and the inclusion of tidal effects. Then we will explore partial resummation for the scattering angle, and obtain new compact all-loop results for tidal contributions in the no-recoil limit.

B2B correspondence

2.1 Review of basic relations

The scope of this section is introducing a dictionary between gravitational observables for scattering processes (measured at the boundary) and adiabatic invariants of bound orbits (in the bulk), to all PM orders. For example, an advantage of this map is that it allows to bypass the Hamiltonian in the computations, relating directly gauge invariant quantities. We follow [5], [6] and [2], so for more details we refer you to these articles. Here and in the next sections we will refer to the following quantities in the 2-body problem for m_1 and m_2: the total mass $M = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/M$, the symmetric mass ratio $\nu = m_1 m_2/M^2$, E the energy of the system, J the conserved angular momentum, $j \equiv J/(GM\mu)$, $p_\infty \equiv p(r \to \infty)$ and $\mathcal{E} \equiv (E - M)/\nu$.

In classical GR we are interested in computing observables (for example the scattering angle $\chi(J, E)$) from the Hamiltonian for the 2-body problem $H(r, \mathbf{p}^2) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(r, \mathbf{p}^2)$ (which allows us to solve for $\mathbf{p}^2(r, H = E)$):

$$\chi(J, E) = -\pi + 2J \int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,E) - J^2/r^2}} \ . \tag{2.1}$$

In a Post-Minkowskian (PM) expansion, for impact parameter $b \gg GM$ we can compute χ as a series in GM/b, with $\chi_j^{(n)} = p_\infty^n \chi_b^{(n)}$ and $\hat{p}_\infty = p(r \to \infty)/\mu$:

$$\frac{1}{2}\chi(b, E) = \sum_n \chi_b^{(n)}(E) \left(\frac{GM}{b}\right)^n = \sum_n \chi_b^{(n)}(E) \frac{1}{j^n} \ . \tag{2.2}$$

We can also proceed with the inverse problem, determining $\mathbf{p}^2$ (and hence the Hamiltonian) from the scattering angle. The Firsov formula (in which $\bar{\mathbf{p}} = \mathbf{p}/p_\infty$):

$$\bar{\mathbf{p}}^2(r, E) = exp\left(\frac{2}{\pi}\int_{r|\bar{p}(r,E)|}^{\infty} \frac{\chi_b(\tilde{b}, E)d\tilde{b}}{\sqrt{\tilde{b}^2 - r^2\bar{p}^2(r,E)}}\right) . \tag{2.3}$$

By inserting the PM expansion (2.2) for χ and Taylor expanding we obtain:

$$\bar{\mathbf{p}}^2(r, E) = p_\infty^2(E)\left(1 + \sum_i f_i(E)\left(\frac{GM}{r}\right)^i\right) , \tag{2.4}$$

where the f_n's are connected to the $\chi_b^{(n)}$'s via:

$$f_n = \sum_{\sigma\in\mathcal{P}(n)} \frac{2(2-n)^{\Sigma_l - 1}}{\prod_l (2\sigma^l)!!}\left(\prod_l \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{\sigma_l}{2})}{\Gamma(\frac{\sigma_l+1}{2})} \chi_b^{(\sigma_l)}\right)^{\sigma^l} . \tag{2.5}$$

Here we are summing over the integers partition σ of n and σ_l are unique integers appearing σ^l times in the partition, so that $n = \sigma_l \sigma^l$ and $\Sigma^l = \sum_l \sigma^l$

These equations provide a perturbative map between χ and $\bar{\mathbf{p}}^2$. Now we can for example invert (2.5) and obtain:

$$\chi_b^{(n)} = \frac{\sqrt{\pi}}{2}\Gamma(\frac{n+1}{2}) \sum_{\sigma\in\mathcal{P}(n)} \frac{1}{\Gamma(1 + n/2 - \Sigma^l)} \prod_l \frac{f_{\sigma_l}^{\sigma^l}}{\sigma^l!} \ . \tag{2.6}$$

Now, in principle, the knowledge of all the f_i's, allows to read off an infinite serie of PM terms χ_b^n's for the deflection angle.

2.2 B2B relations for circular orbits

For circular orbits we can pursue a more geometrical approach: using Firsov, we can extract information about hyperbolic motion and then analytically continue to elliptic ones. Then, for circular orbits, we can use the condition of vanishing eccentricity to determine observables.
We observe that the point of closest approach $r_{\min}$ satisfies:

$$r^2\left(1 + \sum f_i(\mathcal{E})\left(\frac{GM}{r}\right)^i\right) = b^2 \ . \tag{2.7}$$

It is possible to transform the boundary data of the scattering problem (hyperbolic motion) into 2 real positive $r_\pm(\mathcal{E}, J)$ (elliptic) via an analytic continuation of the impact parameter.
For the hyperbolic motion, described by $r = \tilde{a}(\tilde{e}\cosh u - 1)$ (where u is the eccentric anomaly, and $\tilde{a}$, $\tilde{e}$ the orbital elements) one of the solutions $\tilde{r_\pm}$ of (2.7) is negative. We can find $\tilde{r}_-(J, E) = r_{\min}(ib, i\beta)$ of (2.7) (where $\beta \equiv \cosh\gamma^{-1}$ is the rapidity), and $\tilde{r}_- = r_{\min}$, while for the other we use the map $b \longrightarrow -b : r_+(b > 0) = r_-(-b)$.

An evident drawback of this method is the difficulty of finding the roots for a theory with many non-null f_i's, but we can construct a generalized map via the scattering angle that takes care of this issue. The representation of $\tilde{r_-}$ in terms of the scattering angle is known:

$$\tilde{r_-} = b \prod_{n=1}^{\infty} e^{-\frac{(GM)^n \chi_b^{(n)}(\beta)\Gamma(n/2)}{b^n \sqrt{\pi}\Gamma(\frac{n+1}{2})}} . \tag{2.8}$$

In principle we could try to resum at the exponent and then obtain $r_\pm$ by analytic continuation. At this point for the circular orbits we can use the condition $r_- = r_+$ (vanishing eccentricity):

$$-2\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{\pi}}\left(\frac{GM}{z}\right)^{2n+1}\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)\chi_b^{2n+1}}\right) = i\pi + 2\pi i N , \tag{2.9}$$

where z is b promoted to a complex. We can solve for b, and then find $j(\mathcal{E})$ as a function of the f_i's bypassing the computation of the radial action. Now let us call a f_n theory a theory which has only the first n coefficient f_i's non-null. For f_1 and f_2 one can directly solve (2.8), so this last described approach seems useless. For f_3 is still possible to find the general roots, but for higher theories it is much more cumbersome than this resummation scheme.

2.3 Example resummations for f_1 and f_2 theories

Our objective is to obtain example resummation identities in the framework described in the previous sections. In the f_1 theory, by summing in (2.6), we find:

$$\chi_b^n(f_1) = \frac{(-1)^n}{2n+1}\left(\frac{f_1}{2}\right)^{2n+1} \implies \chi[f_1] = 2\sum_n \frac{(-1)^n}{2n+1}\left(\frac{f_1}{2}\right)^{2n+1}\left(\frac{GM}{b}\right)^n = 2\arctan(y/2) , \tag{2.10}$$

where $y = GMf_1/b$. If we resum in the $\tilde{r}_-$ exponent we obtain:

$$\tilde{r_-} = be^{\sinh^{-1}\left(-\frac{GMf_1}{2b}\right)} , \tag{2.11}$$

so that:

$$r_-(\tilde{b},\beta) = \tilde{r_-}(ib, i\beta) = -\frac{GMf_1}{2} + \mathrm{sgn}(b)\sqrt{\frac{(GMf_1)^2}{4} - b^2} = r_-(b>0,\beta) . \tag{2.12}$$

Of course this procedure can be repeated analogously also for more general theories. Let us see as an example the condition (2.9) for the f_2 theory. Starting from (2.6), we can find two separate sequences for odd and even χ_b^n:

$$\chi_b^{2n}[f_{1,2}] = \frac{\sqrt{\pi} f_2^n \Gamma(n+1/2)}{2\Gamma(n+1)} \qquad n = 1,\ 2... \tag{2.13}$$

$$\chi_b^{2n+1}[f_{1,2}] = \frac{1}{2} f_1 f_2^n \, {}_2F_1\left(\frac{1}{2}, -n; \frac{3}{2}; \frac{f_1^2}{4f_2}\right) \qquad n = 0,\ 1... \tag{2.14}$$

By performing the sum we get the total scattering angle:

$$\frac{\chi[f_{1,2}] + \pi}{2} = \frac{1}{\sqrt{1-\mathcal{F}_2 y^2}}\left(\frac{\pi}{2} + \arctan\left(\frac{y}{2\sqrt{1-\mathcal{F}_2 y^2}}\right)\right) , \tag{2.15}$$

where $\mathcal{F}_2 \equiv f_2/f_1^2$. We can perform the sum inside (2.9), which becomes:

$$\sinh^{-1}\sqrt{\frac{(GMf_1)^2}{4(z^2-(GM)^2 f_2)}} = i\frac{\pi}{2} + i\pi N . \tag{2.16}$$

As a consistence check, this is precisely equivalent to the condition of degenerate roots for (2.8). Again for this theory we can attempt a resummation of the $\tilde{r}_-$ exponent obtaining:

$$r_-(b,\beta) = ib\prod_{n=1}^{\infty} e^{-\frac{(GM)^n \chi_b^{(n)}(i\beta)\Gamma(n/2)}{(ib)^n\sqrt{\pi}\Gamma((n+1)/2)}} . \tag{2.17}$$

2.4 No recoil approximation and inclusion of tidal effects

Now we consider the no recoil approximation, which allows for easier partial resummation, that is to leading order in the symmetric mass ratio ν. Indeed we will see in the next section that the simplifying power of this limit allows us to find new compact expressions for the tidal corrections to the scattering angle to all orders in the velocity.
The no recoil limit coincides with the test particle limit $\nu \to 0$. Thus if we assume $m_1 \gg m_2$, we have $m_2 \to \mu$, $m_1 \to M$, $E = M(1 + \mathcal{O}(\nu))$, and we can ignore self force effects.
In this approximation $\mathbf{p}^2$ is a known function of r and E, computed on a Schwarzschild background. The no recoil approximation successfully reproduces the 2PM order exactly (see [9]), but fails at 3PM due to presence of unaccounted self force effects. Nonetheless some terms at higher PM contribute to the ν independent part of the no recoil quantities.

Now we want to understand how in the above limit we can introduce tidal effects (see [3]). We start from the non tidal case of a test particle m (not extended). The on-shell condition:

$$g_{\mu\nu} p^\mu p^\nu - m^2 = 0 \; , \tag{2.18}$$

where:

$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \; , \tag{2.19}$$

with $M \gg m$, $\mu \sim m$, $\nu \ll 1$. We take the momentum of the test particle for the non-spinning case (motion bound on the plane $\theta = \pi/2$) as $p_\mu = (m\mathcal{E}_0, p_r, 0, p_\phi)$. So now from the on-shell condition I can find p_r as a function of $J = p_\phi = jGMm$ and $u = GM/r$. At this point the scattering angle can be obtained as:

$$\chi + \pi = \int_{-\infty}^{\infty} dr \frac{\partial p_r(J, u, \mathcal{E})}{\partial J} \; . \tag{2.20}$$

By substituting for the variable $y = J/(mr)$ we eventually find (originally obtained by Z. Liu in work to be published):

$$\chi = \sum_{n=1}^{\infty} \frac{(-2)^n}{j^n(\gamma^2-1)^{n/2}} \frac{\Gamma(\frac{1}{2}-n)\Gamma(n+\frac{1}{2})\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma(n+1)} \, {}_2\tilde{F}_1(-n, \frac{n+1}{2}, 1-\frac{n}{2}, 1-\gamma^2) \; , \tag{2.21}$$

where ${}_2\tilde{F}_1$ is the regularized Gaussian renormalized hypergeometric function. As an example of the simplifying power of the no-recoil approximation we can perform the resummation for the f_n's using the above result and (2.5) obtaining:

$$f_n = \frac{2^{6-n}(n-2)(1+v^2)}{v^2} \quad \text{for} \quad n > 4 \tag{2.22}$$

$$f_4 = \frac{8}{v^2} + \frac{129}{16}, \quad f_3 = \frac{17}{2v^2} + 9, \quad f_2 = \frac{3\left(5v^2+4\right)}{2v^2}, \quad f_1 = \frac{2\left(2v^2+1\right)}{v^2} \; . \tag{2.23}$$

It would be an interesting future development understanding if there is a physical reason behind the appearance of a general closed form only after $n = 4$.

Now let us consider an extended body in an external long-wavelength gravitational field. The typical body of interests are for example neutron stars, whose tidal deformation contribution to GWs carry the imprint of the nuclear matter equation of state and may open new paths for beyond the standard model searches (for an example [1]). The center of mass will trace a world-line that corresponds approximately to a geodesic motion of a test particle. Then we can try to incorporate finite size effects in a world-line action approach (for additional details see [4]).
Often the exact effective action is unknown, but we can build it by writing all the possible local terms respecting the symmetries of the system. The extra terms (with respect to the point particle) are organized in power of derivatives, so that for the first corrections:

$$S_{pp} = \sum_{a=1,2} \int d\tau_a \left(\frac{m_a}{2} g_{\mu\nu} v_a^\mu v_a^\nu + c^{(a)}_{E^2} E_{\mu\nu} E^{\mu\nu} + c^{(a)}_{B^2} B_{\mu\nu} B^{\mu\nu} - c^{(a)}_{\tilde{E}^2} \tilde{E}_{\mu\nu\alpha} \tilde{E}^{\mu\nu\alpha} - c^{(a)}_{\tilde{B}^2} \tilde{B}_{\mu\nu\alpha} \tilde{B}^{\mu\nu\alpha} \right) \; , \tag{2.24}$$

where $E_{\alpha\beta} = R_{\mu\alpha\nu\beta}u^\mu u^\nu$, $B_{\alpha\beta} = R^*_{\mu\alpha\nu\beta}u^\mu u^\nu$, $\tilde{E}_{\mu\nu\rho} = \nabla^\perp{}_{\{\mu}R_{\nu\alpha\rho\}\beta}u^\alpha u^\beta$, $\tilde{B}_{\mu\nu\rho} = \nabla^\perp{}_{\{\mu}R^*_{\nu\alpha\rho\}\beta}u^\alpha u^\beta$ and $R^*_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}_{\alpha\beta}/2$, $(\nabla^\perp)^\mu_a = (g^{\mu\nu} - u^\mu_a u^\nu_a)\nabla_\nu$ (u_1, u_2 are the velocities of the 2 bodies). Note that for the moment we have neglected 'mixed contributions'. Of course this approach can be continued by adding higher derivative (diffeomorphism invariant) terms (for more details see [8]).

This addition to the action modifies the on-shell condition to:

$$g^{\mu\nu}_{\text{Sch}}p^{(a)}_\mu p^{(a)}_\nu = m_a^2 - 2m_a(c^{(a)}_{E^2}(E^{\text{Sch}}_{\mu\nu}))^2 + c^{(a)}_{B^2}(B^{\text{Sch}}_{\mu\nu})^2 - c^{(a)}_{\tilde{E}^2}(E^{\text{Sch}}_{\mu\nu\alpha})^2 - c^{(a)}_{\tilde{B}^2}(B^{\text{Sch}}_{\mu\nu\alpha})^2)\,, \tag{2.25}$$

where all the tidal terms are computed using the Schwarzschild metric $g^{\text{Sch}}_{\mu\nu}$. After some computations it is possible to rewrite everything in terms of u, j, p_r and $E_1 = m_1\mathcal{E}$. In this way the on-shell condition becomes an equation for p_r which can be solved, order by order if needed. Once we have found p_r we can proceed with the computation of the χ^n's via (2.20). We will follow this procedure in the next chapter to find closed expressions for the tidal corrections E^2, B^2.

Resummation of the scattering angle

In the previous chapter we have already seen the usefulness of knowing a closed analytical form for the scattering angle, or at least for its PM coefficients. Moreover the knowledge of resummation identities allows for shorter numerical computation times, as in general, analytical expressions or fewer indices sums are evaluated faster.
Guided by these principles, we now address the possibility of (partial) resummations for the scattering angle in the f_3 theory and for tidal corrections in the no recoil limit.

3.1 Resumming the f_3 theory

We now address the possibility of resumming the f_3 theory. By using (2.6) we can get:

$$\begin{aligned}\chi_b &= \sum_{n_1,n_2,n_3=0}^{\infty} \left(\frac{Gm}{b}\right)^{n_1+2n_2+3n_3} \frac{\sqrt{\pi}}{2} f_1^{n_1} f_2^{n_2} f_3^{n_3} \frac{\Gamma(\frac{1+n_1+2n_2+3n_3}{2})}{n_1!\, n_2!\, n_3!\, \Gamma(1+\frac{n_3-n_1}{2})} = \\ &= \sum_{u=-\infty;n_2,n_3=0}^{\infty} \frac{\sqrt{\pi} f_1^{-u} f_2^{n_2} (f_1 f_3)^{n_3} \Gamma\left(n_2+2n_3-\frac{u}{2}+\frac{1}{2}\right) \left(\frac{Gm}{b}\right)^{2n_2+4n_3-u}}{u\Gamma(n_2+1)\Gamma(n_3+1)\Gamma\left(\frac{u}{2}\right)\Gamma(n_3-u+1)}\,,\end{aligned} \tag{3.1}$$

where we swapped the sum over n_1 for the one over $u = n_3 - n_1$. This expression cannot be rearranged in a form corresponding to a Lauricella function, so we must try to resum it index by index. Note that due to the 1/2 factor in $\Gamma(u/2)$, we must sum not only for non-negative u, but also for negative odd values. Summation over n_2 yields:

$$\chi_b = \sum_{u=-\infty,\, n_3=0}^{\infty} \frac{\sqrt{\pi} f_1^{-u} (f_1 f_3)^{n_3} \Gamma\left(\frac{1}{2}(4n_3-u+1)\right) \left(\frac{Gm}{b}\right)^{4n_3-u} \left(\frac{b^2-f_2(Gm)^2}{b^2}\right)^{-2n_3+\frac{u}{2}-\frac{1}{2}}}{u\Gamma(n_3+1)\Gamma\left(\frac{u}{2}\right)\Gamma(n_3-u+1)}\,. \tag{3.2}$$

Now we can sum over u by computing separately the contributions from non-negative and negative terms, obtaining:

$$\begin{aligned}\chi_b = \sum_{n_3=0}^{\infty} &-\frac{\left(\frac{(Gm)}{b}\right)^{4n_3-1} (f_1 f_3)^{n_3}}{4b^4 f_1 \Gamma(n_3+1)^2 \Gamma(n_3+4)} \Bigg(2\sqrt{\pi} b^3 \Gamma(n_3+4) \left(1-\frac{f_2(Gm)^2}{b^2}\right)^{-2n_3-\frac{1}{2}} \times \\ &\times \Bigg(\pi b n_3 \csc(2\pi n_3) \sqrt{1-\frac{f_2(Gm)^2}{b^2}}\; {}_3\tilde{F}_2 \left(1, \frac{1-n_3}{2}, 1-\frac{n_3}{2}; \frac{3}{2}, 1-2n_3; \frac{4\left(f_2-\frac{b^2}{(Gm)^2}\right)}{f_1^2}\right) + \\ &+ f_1(Gm)\Bigg(\pi \sec(2\pi n_3)\, {}_2\tilde{F}_1\left(\frac{1-n_3}{2}, -\frac{n_3}{2}; \frac{1}{2}-2n_3; \frac{4\left(f_2-\frac{b^2}{(Gm)^2}\right)}{f_3^2}\right) - 2\Gamma\left(2n_3+\frac{1}{2}\right)\Bigg)\Bigg) + \\ &+ f_1^4 (Gm)^4 \Gamma(n_3+1)\Gamma(2n_3+2) \left(1-\frac{f_2(Gm)^2}{b^2}\right)^{-2(n_3+1)} \times \\ &\qquad \times {}_3F_2\left(1, \frac{3}{2}, 2n_3+2; \frac{n_3}{2}+2, \frac{n_3}{2}+\frac{5}{2}; -\frac{f_1^2(Gm)^2}{4\left(b^2-f_2(Gm)^2\right)}\right)\Bigg)\,.\end{aligned} \tag{3.3}$$

We believe this is the furthest we can get with standard techniques. More explicitly we tried to perform the double sum after substituting the definitions of the Hypergeometric functions or searching for recurrence identities after the use of contiguous relations, but to no avail. This may be, conservatively, a clue that a closed expression of named functions (if it exists at all) would not be shortly evaluated by computers, undermining the whole goal of the effort. We also conjecture that this same difficulty may arise not only for the complete expression but even for its coefficients in the Gm/b expansion. Indeed, by substituting $n = n_1 + 2n_2 + 3n_3$ as a sum variable we manage only to eliminate one index:

$$\chi_b = \sum_{u=-\infty,\, n=0}^{\infty} \frac{\sqrt{\pi} f_1^{-u} \Gamma\left(\frac{n+1}{2}\right) \left(\frac{(Gm)}{b}\right)^n f_2^{\frac{n+u}{2}} \, {}_2F_1\left(-\frac{n}{4} - \frac{u}{4}, -\frac{n}{4} + \frac{1}{2} - \frac{u}{4}; 1-u; \frac{4f_1 f_3}{f_2^2}\right)}{u\Gamma(1-u)\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{1}{2}(n+u+2)\right)} \,, \tag{3.4}$$

so even χ_n may only be implicitly expressed as a sum. Of course we get circularly the same conclusion if we try to collect for Gm/b in the expression summed only over n_3.

3.2 All-order results for tidal deformations

Now we want to investigate resummation identities for the tidal effects in the no recoil limit. We make the approximation in which these corrections are small, so that we may consider only the linear order, compute the various terms' contributions separately and then superpose them.

Substituting the higher derivative operators' contributions in (2.25), we note that for the corrections E^2, B^2 we obtain a (bi)quadratic equation. For the other operators we get instead a proper quartic of difficult solution, so for now we restrict our computation only to the former contributions.
Let $\lambda_E = \frac{1}{G^4 m_2^4 m_1} c_E$, $\lambda_B = \frac{1}{G^4 m_2^4 m_1} c_B$. We start by considering the E^2 contribution and find two solutions for p_r^2 from (2.25). Only one of these solution is physical, and it is the (non-divergent) one that gives back the known solution for the point particle in the non-tidal limit $\lambda_E \longrightarrow 0$:

$$p_r^2 = \frac{m_1^2\left(\gamma^2 + (2u-1)\left(j^2u^2+1\right)\right)}{(1-2u)^2} + \frac{12\lambda_E m_1^2 u^6\left(3j^4u^4 + 3j^2u^2 + 1\right)}{1-2u} \,. \tag{3.5}$$

So the expression, from (2.20), that we must integrate in y to find χ for the tidal correction is:

$$\frac{1}{\sqrt{\gamma^2 + 2\left(y^2+1\right)y\epsilon - y^2 - 1}} - \frac{6\lambda_E y^6 \epsilon^6 \left(6\gamma^2 + y\left(3y\left(4\gamma^2 + 6y^3\epsilon - 3y^2 + 10y\epsilon - 5\right) + 10\epsilon\right) - 5\right)}{\left(\gamma^2 + 2\left(y^2+1\right)y\epsilon - y^2 - 1\right)^{3/2}} \,, \tag{3.6}$$

where $1/j \equiv \epsilon$. The first term originates the known point particle result, while the second is responsible for the tidal correction $\Delta\chi_E$, which will be our focus. At this point we can expand in $1/j \equiv \epsilon$ the denominator, keep only the λ_E-linear part and integrate order by order in ϵ:

$$\begin{aligned}
\Delta\chi_E = &\sum_{n=0}^{\infty} \int_{y=0}^{\sqrt{\gamma^2-1}} dy \Bigg[3\lambda_E 2^{n-6} y^6 \left(18y^5 + 30y^3 + 10y\right) \binom{-\frac{3}{2}}{n-7} \left(y^3+y\right)^{n-7} \epsilon^n \left(\gamma^2 - y^2 - 1\right)^{\frac{11}{2}-n} + \\
&+ 3\lambda_E 2^{n-5} y^6 \binom{-\frac{3}{2}}{n-6} \left(y^3+y\right)^{n-6} \epsilon^n \left(6\gamma^2 - 9y^4 + 12\gamma^2y^2 - 15y^2 - 5\right) \left(\gamma^2 - y^2 - 1\right)^{\frac{9}{2}-n} \Bigg] = \\
= &\sum_{n=0}^{\infty} \frac{3\pi(n-5)(n-4)(n-3)(n-1) j^{-n} \left(\gamma^2-1\right)^{5-\frac{n}{2}} \lambda_E}{64(2n-11)\Gamma\left(\frac{n}{2}+2\right)} \left(\gamma^2\left(n\left(2n^3 - 71n + 27\right) + 72\right) - 3\gamma^4 \times\right. \\
&\times (n-1)(n+1)(2n-11)(2n-9) - 3((n-1)n+3)) \, {}_2\tilde{F}_1\left(7-n, \frac{n+1}{2}; 6-\frac{n}{2}; 1-\gamma^2\right) + \gamma^2(n+1) \times \\
&\times \left((n-1)\left(3\gamma^4(n(6n-55)+121) + 3\gamma^2 n(11-2n) + n\right) + 3\right) \, {}_2\tilde{F}_1\left(7-n, \frac{n+3}{2}; 6-\frac{n}{2}; 1-\gamma^2\right) \,.
\end{aligned} \tag{3.7}$$

The terms of the sum above are the closed expressions for $\Delta\chi_n$, where $\Delta\chi = \sum_{n=0}^{\infty} \Delta\chi_n / j^n$, that we were searching for. We proceed analogously now for the B^2 correction finding:

$$\Delta\chi_B = \sum_{n=0}^{\infty} \frac{9\pi\lambda_B(n-5)(n-4)(n-3)(n-2)(n-1)^2 j^{-n}\left(\gamma^2-1\right)^{5-\frac{n}{2}}}{64(2n-11)\Gamma\left(\frac{n}{2}+2\right)}\Big(\gamma^2(n+1)\left(\gamma^2(2n-11)+1\right)\times$$
$$\times\ {}_2\tilde{F}_1\left(6-n,\frac{n+3}{2};6-\frac{n}{2};1-\gamma^2\right)+\left(\gamma^2(11-2(n-4)n)-1\right)\ {}_2\tilde{F}_1\left(6-n,\frac{n+1}{2};6-\frac{n}{2};1-\gamma^2\right)\Big)\,. \tag{3.8}$$

These formulae for the E^2 and B^2 corrections reproduce every known PM coefficient for the tidal corrections to the scattering angle (computed in [4]). As a reference we report the first non-null four tidal corrections (starting from $n=6$), superposition of the E^2 and B^2 contributions:

$$\begin{aligned}\Delta\chi_E^{(6)} =&\frac{1}{512}(-15)\pi\Big(-420\gamma^8(\lambda_B+\lambda_E)+\gamma^6(1200\lambda_B+1200\lambda_E-17017)+\\&+3\gamma^4(-360\lambda_B-424\lambda_E+5005)+\gamma^2(240\lambda_B+624\lambda_E-3003)+60\lambda_B-132\lambda_E+77\Big)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Delta\chi_E^{(7)} =&\frac{1}{35\left(\gamma^2-1\right)^{7/2}}\Big(15360\gamma^{16}(\lambda_B+\lambda_E)-3072\gamma^{14}(31\lambda_B+31\lambda_E-80)+\\&+1344\gamma^{12}(185\lambda_B+188\lambda_E-832)-224\gamma^{10}(1572\lambda_B+1665\lambda_E-9152)+\\&3360\gamma^8(86\lambda_B+99\lambda_E-572)-6720\gamma^6(20\lambda_B+27\lambda_E-143)+336\gamma^4(92\lambda_B+172\lambda_E-715)+\\&-24\gamma^2(80\lambda_B+388\lambda_E-1001)-192\lambda_B+480\lambda_E-429\Big)\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Delta\chi_E^{(8)} =&\frac{3\pi}{32768}\Big(448\gamma^{10}(27027\lambda_B+27027\lambda_E)+\gamma^8(-32598720\lambda_B-32598720\lambda_E+\\&+111546435)+140\gamma^6(211680\lambda_B+227808\lambda_E-969969)-350\gamma^4(28224\lambda_B+38976\lambda_E-138567)+\\&+700\gamma^2(1008\lambda_B+3312\lambda_E-7293)+7(4032\lambda_B-11328\lambda_E+10725)\Big)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\Delta\chi_E^{(9)} =&\frac{1}{9009\left(\gamma^2-1\right)^{9/2}}\Big(-18\gamma^2(226304\lambda_B+1022528\lambda_E-1738165)+\\&+32\gamma^4(-252\gamma^6(881712\lambda_B+977379\lambda_E-2781064)+36\gamma^4(2788968\lambda_B+3374553\lambda_E-9733724)+\\&-6\gamma^2(4455360\lambda_B+6293196\lambda_E-17034017)+8\gamma^8(144\gamma^8(5824\lambda_B+5824\lambda_E)+\\&-8\gamma^6(760032\lambda_B+760032\lambda_E-654368)+9\gamma^4(2134080\lambda_B+416(5163\lambda_E-8228))+\\&-9\gamma^2(3837600\lambda_B+3923400\lambda_E-8557120)+21(1843452\lambda_B+3536(549\lambda_E-1430)))+\\&+9(393120\lambda_B+753480\lambda_E-1738165))-119808\lambda_B+374400\lambda_E-347633\Big)\,.\end{aligned} \tag{3.12}$$

We could attempt to resum also the last index, and investigate if there is a closed form for tidal corrections of the whole scattering angle. Standard methods do not yield results, however we can bring the computation in the f_i's space. This is heuristically motivated by the fact that in general f_n's are nicer than χ_n's, as we can see in the non tidal case (2.22). At this point the procedure would be to build a matrix with columns entries the coefficients in v of the f_n's. In this way we can search for recurrence identities in more generality than both in the PN and PM expansions, for example going along the diagonals. We leave open this possibility for future work.

Lastly we point out that we could go beyond the linear order, up to arbitrary order in λ_E, as the integrand is in general always a sum of terms with known integral. We computed the quadratic (starting from $n=12$) and cubic (from $n=18$) contributions, here as an example we report the $\Delta\chi_E$

at order λ_E^2:

$$\frac{3\pi\lambda_E^2(n-11)(n-10)(n-9)(n-8)(n-7)(n-5)(n-3)(n-1)\left(\gamma^2-1\right)^{10-\frac{n}{2}}\epsilon^n}{4096(2n-23)(2n-21)\Gamma\left(\frac{n}{2}+3\right)}\times \tag{3.13}$$
$$\times\Bigg(\gamma^2(n+1)(81\gamma^{10}(n-7)(n-1)(n+1)(2n-23)(2n-21)(3n-25)(3n-23)+$$
$$-9\gamma^8(n-1)(2n-23)(2n-21)(3n-23)(n(3n(10n-69)-404)-839)+$$
$$+9\gamma^6(n-1)(2n-23)(2n-21)(n(n(n(27n-88)-947)-3226)-1386)+$$
$$-3\gamma^4(n-1)(2n-23)(2n-21)(n(n(n(6n+107)+519)+1072)+486)+$$
$$+\gamma^2(n(n(n(n(n(4n(30n-781)+21657)+33341)-597891)-152473)+1991229)+14391)+$$
$$-5(n(n(n(n(40(n-18)n+1159)+22254)-5087)-77604)+1053))\,{}_2\tilde{F}_1\left(14-n,\frac{n+3}{2};11-\frac{n}{2};1-\gamma^2\right)+$$
$$-\Bigg(81\gamma^{10}(n-7)(n-1)(n+1)^2(2n-23)(2n-21)(2n-19)(3n-23)+$$
$$-9\gamma^8(n-1)(n+1)(2n-23)(2n-21)(2n-19)(n(3n(8n-29)-652)-1165)+$$
$$+9\gamma^6(n-1)(n+1)(2n-23)(2n-21)(2n-19)(n(n(5n+71)+264)+330)+$$
$$-3\gamma^4(n(n(n(n(2n(2n(n(n+31)-433)-10297)+117689)+914716)+1460497)-976146)-1570158)+$$
$$+5\gamma^2(n(n(n(n(n(2n(4n(2n-53)+1405)+7949)-88327)-135103)+218035)+358473)+53001)+$$
$$-35(n(n(n(n(40(n-18)n+1159)+22254)-5087)-77604)+1053)\Bigg)\,{}_2\tilde{F}_1\left(14-n,\frac{n+1}{2};11-\frac{n}{2};1-\gamma^2\right)\Bigg)\,.$$

The cubic contribution has the same form of sum of regularized gaussian Hypergeometrics, albeit more complicated. A possibility of future development would be to compute enough of these terms to conjecture the form of the generic order contribution. We remark however that before using any of these (non-linear) results we should check that all contributions from the other operators are null in the problem of interest. Indeed a rigorous approach at non-linear orders would require to consider the effect of the tidal operators together as we do not have superposition anymore.

Acknowledgements

I'm deeply grateful to my supervisors Gregor and Zhengwen for the useful insights and Mathematica support they provided. I would also like to thank Rafael Porto, for proposing the project for the DESY summer student program, and the rest of the GWAT group for the lovely time spent together.
At last I wanted to acknowledge the financial support from Scuola Normale Superiore during the research period.

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