Scattering Amplitudes and the Spinor-Helicity Formalism

Giuseppe Bogna*
Scuola Normale Superiore
Università di Pisa

Supervisor	Supervisor	Supervisor
Marc Montull	Christophe Grojean	Fady Bishara
DESY	DESY	DESY

September 4, 2019

Abstract

Spinor-helicity variables are introduced as solutions to the massless Dirac equation and are used as a tool for calculating several scattering amplitudes. The extension to the massive case is then examined and massless and massive variables are finally used to construct a UV completion of general relativity.

*giuseppe.bogna@sns.it

Contents

1. Introduction

The aim of this report is to give an introduction to the spinor-helicity formalism and its applications. This powerful tool allow to compute on-shell scattering amplitudes in a straightforward way, usually without getting involved in long and complicate calculations with Feynman diagrams. Its importance is then double: on one hand, spinor-helicity formalism can be efficiently used to compute cross section for various processes (and indeed some techniques are already implemented in numerical codes for particle physics experiments). On the other hand, it can give us some theoretical insights and novel ideas, as we will see later on in the report.

This work is divided as follows: in Section 2 we introduce the spinor-helicity formalism for massless particles, following the discussion in [1]. This section is the very heart of the report, as it contains many definitions, properties and also examples and explicit calculations: we start from the Weyl equation and construct the spinor-helicity variables for massless fermions. The first part of the section is pretty technical, but then we show how one can use the newly introduced variables in some calculation in a model with a massless spin 1/2 fermion and a massless real scalar interacting via a Yukawa coupling. After that, we introduce a proper definition of polarization vector for spin-1 massless particles and discuss little group scaling. Next, we develop, under certain assumptions, some recursion relations [2, 3] for the computation of many-particle scattering amplitudes at tree level. We use them to prove the Parke-Taylor formula [4] for maximally helicity violating (MHV) scattering amplitudes in QCD.[1] In Section 3 we extend some of these ideas to the case of massive particles, as done in [5]: in particular, we compute all three-particle scattering amplitudes and discuss some examples, among which Compton scattering between photons and massive scalars, fermions and vectors.

Finally, in Section 4 we use the machinery introduced so far to discuss a possible UV completion of gravity, following the analysis made in [6]. In detail, we start studying graviton mediated four-point tree level scattering amplitudes, in the case the external legs are SM massless particles and/or gravitons. We then summarize the discussion made in [6] about the high energy behaviour of these amplitudes, where one finds out that, under certain hypothesis of regularity, two infinite towers of increasing masses are required to satisfy perturbative unitarity. Those resonances modify graviton mediated scattering amplitudes by dressing them either with Veneziano or Virasoro-Shapiro amplitudes.

In future work, we hope to extend the analysis carried out in [6] to massive external states.

2. The spinor-helicity formalism: the massless case

Throughout the report, we use the metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

2.1. Spinor-helicity variables

Given a 4-vector p^μ, we associate two 2×2 matrices to it

$$p_{a\dot{b}} = p_\mu (\sigma^\mu)_{a\dot{b}}\,, \qquad p^{\dot{a}b} = p_\mu (\bar{\sigma}^\mu)^{\dot{a}b} \tag{1}$$

[1] i.e. scattering amplitudes for n gluons where 2 gluons have the same helicity and the other $n-2$ gluons have the opposite helicity.

where $(\sigma^\mu)_{a\dot{b}} = (1, \sigma^i)_{a\dot{b}}$ and $(\bar{\sigma}^\mu)^{\dot{a}b} = (1, -\sigma^i)^{\dot{a}b}$, σ^i being the usual Pauli matrices. Explicitly

$$p_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} \tag{2}$$

The map $p^\mu \mapsto p_{a\dot{b}}$ is clearly linear and has the nice property $\det(p_{a\dot{b}}) = -p^\mu p_\mu$. This fact implies that if p^μ is the 4-momentum of a massless particle, then the associate matrix $p_{a\dot{b}}$ has rank 1. We can thus represent $p_{a\dot{b}}$ as an outer product of two 2-component spinors, which we write as a square ket $|p]_a$ and an angle bra $\langle p|_{\dot{b}}$, i.e.

$$p_{a\dot{b}} = -|p]_a \langle p|_{\dot{b}} \tag{3}$$

We can also raise the indices and write

$$p^{\dot{a}b} = -|p\rangle^{\dot{a}} [p|^b \tag{4}$$

where $[p|^a = \varepsilon^{ab}|p]_b$ and $|p\rangle^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}$. Here ε^{ab} is an antisymmetric 2×2 matrix such that $\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = -\varepsilon_{12} = -\varepsilon_{\dot{1}\dot{2}} = 1$.

At this point, it is very important to realize that we can choose the spinors $|p]_a$ and $\langle p|_{\dot{b}}$ as solutions of the massless Dirac equation. In detail, we can focus on the wavefunctions of outgoing (anti-)fermions, which in the massless case satisfy the Weyl equation

$$\bar{u}_\pm(p)\not{p} = 0\,, \qquad \not{p}v_\pm(p) = 0 \tag{5}$$

The subscript $\pm$ refers, in the massless case, to the helicity of the particle. We look for solutions of the form

$$\bar{u}_+(p) = \begin{pmatrix} [p|^a & 0 \end{pmatrix}\,, \qquad \bar{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix}\,, \tag{6}$$

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}\,, \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix} \tag{7}$$

With this ansatz for the solutions, the Weyl equation can be written as

$$[p|^a p_{a\dot{b}} = 0\,, \qquad \langle p|_{\dot{a}} p^{\dot{a}b} = 0 \tag{8}$$

$$p^{\dot{a}b}|p]_b = 0\,, \qquad p_{a\dot{b}}|p\rangle^{\dot{b}} = 0 \tag{9}$$

The completeness relation $-\not{p} = u_+\bar{u}_+ + u_-\bar{u}_-$ is then equivalent to Eq. 3 and 4, via the crossing symmetry $u_\pm = v_\mp$. An explicit example of this construction can be found in Appendix A.

2.2. Basic properties of angle and square bras and kets

In applications, there are some useful and recurrent tricks with spinor-helicity variables. First, notice that in the previous section we can allow p^μ to have complex values. This possibility is clearly unphysical, as particles' 4-momenta must be real-valued. However, in some derivation a complex-valued 4-momentum is also important. We stress that

- if p^μ is complex-valued, then the square and angle bra and kets are independent of each other,

- on the other hand, if p^μ is real valued, then $p_{a\dot{b}}$ and $p^{\dot{a}b}$ are Hermitian and we have the constraints

$$\langle p|_{\dot{a}} = (|p]_a)^*\,, \qquad [p|^a = (|p\rangle^{\dot{a}})^* \tag{10}$$

which simply mean that we can restrict our attention to $v_\pm(p)$.

With this clarification, we can now explain the algebraic properties of the spinor-helicity variables. Given two lightlike 4-momenta p^μ and q^μ, we define the spinor products

$$\langle p\,q\rangle = \langle p|_{\dot{a}}|q\rangle^{\dot{a}}\,, \qquad [p\,q] = [p|^a|q]_a \tag{11}$$

These products are antisymmetric under the exchange $p \leftrightarrow q$, and in particular $\langle p\,p\rangle = [p\,p] = 0$. Moreover, also the "mixed" products $\langle p\,q]$ and $[p\,q\rangle$ vanish. If p^μ and q^μ are real-valued, then we also have

$$[p\,q] = (\langle q\,p\rangle)^* \tag{12}$$

A very useful identity is the following

$$\langle p\,q\rangle[p\,q] = 2p^\mu q_\mu \tag{13}$$

which, in the massless case, reduces to

$$\langle p\,q\rangle[p\,q] = (p+q)^2 \tag{14}$$

We also define the product

$$[p|\gamma^\mu|q\rangle = [p|^a(\sigma^\mu)_{a\dot{b}}|q\rangle^{\dot{b}} \tag{15}$$

and a similar definition holds for $\langle p|\gamma^\mu|q]$. We notice that $[p|\gamma^\mu|q] = \langle p|\gamma^\mu|q\rangle = 0$: in general, a square ket can be connected to another square bra only using an even number of γ-matrices, and it can be connected to an angle bra only using an odd number of γ-matrices. Other properties of the product $[p|\gamma^\mu|q\rangle$ are

- Symmetry of square and angle variables

$$[p|\gamma^\mu|q\rangle = \langle p|\gamma^\mu|q] \tag{16}$$

- Conjugation property for real 4-momenta

$$[p|\gamma^\mu|q\rangle = ([q|\gamma^\mu|p\rangle)^* \tag{17}$$

- Fierz identity: given four 4-momenta p_1, p_2, p_3 and p_4, and using the shorthand $|p_i] = |i]$, we have

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 1\,3\rangle[2\,4] \tag{18}$$

and in particular

$$\langle k|\gamma^\mu|k] = 2k^\mu \tag{19}$$

- for *any* 4-momentum P^μ we define

$$[p|P|q\rangle = P_\mu[p|\gamma^\mu|q\rangle \tag{20}$$

If P^μ is lightlike, then

$$[p|P|q\rangle = -[p\,P]\langle P\,q\rangle \tag{21}$$

Finally, for every four 4-momenta p_i, $i = 1, \ldots, 4$, the Schouten identity holds

$$\langle n\,i\rangle\langle j\,k\rangle + \langle n\,j\rangle\langle k\,i\rangle + \langle n\,k\rangle\langle i\,j\rangle = 0 \tag{22}$$

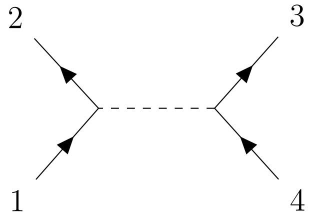
and a similar expression holds for square products.

2.3. First examples

Let us pause for a while and see how we can apply all the new objects we introduced so far. As a simple example, we consider a theory with Lagrangian

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + g\phi\bar{\psi}\psi \tag{23}$$

where ψ is a Dirac field and ϕ a real scalar field. Consider the s-channel for the four-fermion tree amplitude. The usual Feynman rules give then



$$= (ig)^2 \bar{u}_{h_3}(p_3)v_{h_4}(p_4)\frac{-i}{(p_1+p_2)^2}\bar{u}_{h_2}(p_2)v_{h_1}(p_1) \tag{24}$$

where h_i is the helicity of the i-th particle. Notice that we assumed that all the particles are outcoming. We can now convert the products $\bar{u}_{h_3}(p_3)v_{h_4}(p_4)$ and $\bar{u}_{h_2}(p_2)v_{h_1}(p_1)$ in terms of spinor-helicity variables, using Eq. 6 and 7: clearly, the amplitude is proportional to some product of spinor variables of particle 3 and 4, so the amplitude is non-vanishing only if $h_3 = h_4$. In the same way we deduce that the final result is non-zero only if $h_1 = h_2$. For example, if we consider the case $h_1 = h_2 = -1/2$ and $h_3 = h_4 = +1/2$, we find

$$\mathcal{A}_4[1^-2^-3^+4^+] = g^2[4\,3]\frac{1}{(p_1+p_2)^2}\langle 2\,1\rangle \tag{25}$$

We can simplify this expression using Eq. 14 and we find

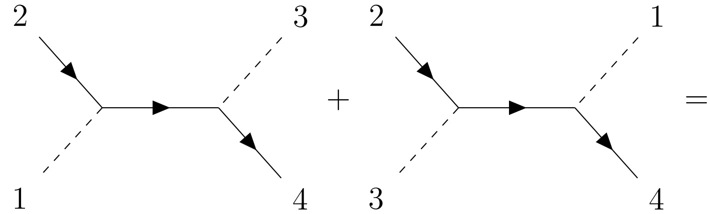
$$\mathcal{A}_4[1^-2^-3^+4^+] = g^2\frac{[3\,4]}{[1\,2]} \tag{26}$$

which is a very nice and compact result! Notice also that the form of the amplitude is not unique, i.e. we can also write the scalar propagator as $1/(p_3+p_4)^2$. In this case we find

$$\mathcal{A}_4[1^-2^-3^+4^+] = g^2\frac{\langle 1\,2\rangle}{\langle 3\,4\rangle} \tag{27}$$

As second example, consider the two-fermion two-scalar tree amplitude. Using the Feynman rules we obtain

$$i\mathcal{A}_4[\phi \bar{f}^{h_2}\phi f^{h_4}] =$$



$$= (ig)^2\bar{u}_{h_4}(p_4)\frac{-i(\not{p}_1+\not{p}_2)}{(p_1+p_2)^2}v_{h_2}(p_2) + (1 \leftrightarrow 3) \tag{28}$$

In this case we have a product of the form $\bar{u}_h \gamma^\mu v_{h'}$, which is non-vanishing only if the helicities of the two fermions are opposite. For instance, if $h_2 = -h_4 = +1/2$, we have

$$\mathcal{A}_4[\phi \bar{f}^+ \phi f^-] = g^2 \frac{\langle 4|p_1 + p_2|2]}{\langle 1\,2\rangle [1\,2]} + (1 \leftrightarrow 3) \tag{29}$$

Using Weyl equation (which reads $p_2|2] = 0$) and Eq. 21 we finally find

$$\mathcal{A}_4[\phi \bar{f}^+ \phi f^-] = g^2 \left(\frac{\langle 1\,4\rangle}{\langle 1\,2\rangle} + \frac{\langle 3\,4\rangle}{\langle 3\,2\rangle} \right) \tag{30}$$

These two examples are nice, as they show how we can express some simple scattering amplitudes in a compact form, but they don't display all the power of spinor-helicity formalism: after all, we have simply applied Feynman rules and *then* converted the result in the new formalism. Let's try now to compute the spin sum

$$\langle |\mathcal{A}_4[\phi \bar{f} \phi f]|^2 \rangle = \sum_{h_2, h_4} |\mathcal{A}_4[\phi \bar{f}^{h_2} \phi f^{h_4}]|^2 \tag{31}$$

The spin sum consists of only two terms, namely those where $h_2 = -h_4$. Let's start with $h_2 = -h_4 = +1/2$: since the particles have real 4-momenta, we can use the conjugation property Eq. 12 and write

$$|\mathcal{A}_4[\phi \bar{f}^+ \phi f^-]|^2 = g^4 \left(\frac{\langle 1\,4\rangle}{\langle 1\,2\rangle} + \frac{\langle 3\,4\rangle}{\langle 3\,2\rangle} \right) \left(\frac{[1\,4]}{[1\,2]} + \frac{[3\,4]}{[3\,2]} \right) \tag{32}$$

Notice now that we have $\langle 3\,4\rangle [1\,4] = -\langle 3\,2\rangle [1\,2]$. In terms of the usual Mandelstam variables, we simply find

$$|\mathcal{A}_4[\phi \bar{f}^+ \phi f^-]|^2 = g^4 \frac{(s-u)^2}{su} \tag{33}$$

The other term in the spin sum, i.e. the one with $h_2 = -h_4 = -1/2$, yields the same result, and finally

$$\langle |\mathcal{A}_4[\phi \bar{f} \phi f]|^2 \rangle = 2g^4 \frac{(s-u)^2}{su} \tag{34}$$

2.4. Polarization vectors

A common choice of polarization vectors for spin-1 massless particles with momentum $\mathbf{p} = E(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is

$$\tilde{\varepsilon}^\mu_\pm(p) = \pm \frac{e^{\pm i\phi}}{\sqrt{2}} (0, \cos\theta\cos\phi \pm i\sin\phi, \cos\theta\sin\phi \mp i\cos\phi, -\sin\theta) \tag{35}$$

They have the following properties

$$\tilde{\varepsilon}^\mu_\pm(p) \tilde{\varepsilon}_{\pm\,\mu}(p) = 0\,, \qquad \tilde{\varepsilon}^\mu_\pm(p) p_\mu = 0\,, \qquad \tilde{\varepsilon}^\mu_\pm(p) \tilde{\varepsilon}^*_{\pm\,\mu}(p) = 1 \tag{36}$$

In spinor-helicity formalism, polarization vectors are defined as

$$\varepsilon^\mu_+(p,q) = -\frac{1}{\sqrt{2}} \frac{\langle q|\gamma^\mu|p]}{\langle q\,p\rangle}\,, \qquad \varepsilon^\mu_-(p,q) = -\frac{1}{\sqrt{2}} \frac{\langle p|\gamma^\mu|q]}{[q\,p]} \tag{37}$$

Here q^μ is an additional 4-vector, called reference momentum, and it must be lightlike and not parallel to p^μ. Its presence is a consequence of gauge invariance, so we expect that all scattering amplitudes are independent of q. Notice that Eq. 37 is a suitable definition of polarization vectors, as one can show that

$$\varepsilon^\mu_\pm(p,q) = \tilde{\varepsilon}^\mu_\pm(p) + \lambda_\pm(p,q)p^\mu \tag{38}$$

for a proper choice of $\lambda_\pm(p,q)$. In particular, the polarization vectors in Eq. 37 satisfy Eq. 36 as well.

2.5. Little group scaling

Recall that the fundamental quantity we introduced is the matrix $p_{a\dot{b}}$, which is clearly invariant under the transformation

$$\begin{cases} |p\rangle \mapsto t|p\rangle \\ |p] \mapsto t^{-1}|p] \end{cases} \tag{39}$$

for every $t \in \mathbb{C}\backslash\{0\}$. If the momentum p is real-valued, we restrict to $t \in \mathrm{U}(1)$ because of the condition $|p\rangle^* = [p|$. This scaling is known as little group scaling and it represents the action of the little group of p^μ [2] on the spinors $|p\rangle$, $|p]$. Let's see now how the little group scaling acts on a scattering amplitude: using Feynman rules, we can trace back the amplitude to some diagrams made up of vertices, internal propagators and external legs. Vertices and internal propagators are invariant under the scaling, but the external legs are not. In particular

- a scalar external leg is invariant as well,
- a spin-1/2 external leg consists either of a square or an angle spinor, and thus scales as t^{-2h}, h being the helicity of the associated fermion,
- a spin-1 external leg is a polarization vector of the form of Eq. 37. Again, we find the scaling law t^{-2h}.

Hence we conclude that a generic n-point amplitude scales as follow

$$\mathcal{A}_n[t|1\rangle, t^{-1}|1], h_1; |2\rangle, |2], h_2; \ldots; |n\rangle, |n], h_n] = t^{-2h_1}\mathcal{A}_n[|1\rangle, |1], h_1; |2\rangle, |2], h_2; \ldots; |n\rangle, |n], h_n] \tag{40}$$

2.6. Further examples

Let's now apply little group scaling to the calculation of some amplitudes. First, we derive an important property of three-particle kinematics: if we have three lightlike 4-momenta p_1, p_2 and p_3 which satisfy $p_1 + p_2 + p_3 = 0$, then either $\langle i\,j\rangle = 0$ or $[i\,j] = 0$ for all i, j. The proof is simple: from 4-momentum conservation we have $\langle 1\,2\rangle[1\,2] = p_3^2 = 0$. Assume now $\langle 1\,2\rangle \neq 0$. Then $[1\,2] = 0$ and moreover

$$\langle 1\,2\rangle[2\,3] = \langle 1|p_1 + p_3|3] = 0 \tag{41}$$

[2] Which indeed for lightlike p^μ in four dimensions is $\mathrm{SO}(2) \cong \mathrm{U}(1)$.

and so $[2\,3]=0$. In the same way we prove $[1\,3]=0$, while if we assume $[1\,2]\neq 0$ then $\langle 1\,2\rangle = \langle 2\,3\rangle = \langle 1\,3\rangle = 0$. Notice also that if p_1, p_2, p_3 are real-valued then we have $\langle i\,j\rangle = [i\,j] = 0$ for all i, j. In other words, a three-point massless amplitude is always vanishing for real momenta. This is a conclusion that can be deduced in a simpler way using basic relativistic kinematics, for example considering the process $1+2\to 3$ in the CM reference frame. On the other hand, a non-vanishing three-point amplitude can depend either on square brackets or angle brackets, but not on both, and makes sense only for complex momenta. Hence it is reasonable to conjecture the following form for a generic three-point amplitude

$$\mathcal{A}_3[1^{h_1}2^{h_2}3^{h_3}] = \alpha\langle 1\,2\rangle^{\lambda_{12}}\langle 1\,3\rangle^{\lambda_{13}}\langle 2\,3\rangle^{\lambda_{23}} \tag{42}$$

Imposing the proper little group scaling for all the spinors we get

$$\mathcal{A}_3[1^{h_1}2^{h_2}3^{h_3}] = \alpha\langle 1\,2\rangle^{h_3-h_1-h_2}\langle 1\,3\rangle^{h_2-h_1-h_3}\langle 2\,3\rangle^{h_1-h_2-h_3} \tag{43}$$

Of course, we can also reasonably conjecture that the amplitude depends on square brackets, in this case we find

$$\mathcal{A}_3[1^{h_1}2^{h_2}3^{h_3}] = \alpha[1\,2]^{h_1+h_2-h_3}[1\,3]^{h_1+h_3-h_2}[2\,3]^{h_2+h_3-h_1} \tag{44}$$

The correct amplitude can be found using dimensional analysis. For example [6], if the third particle is a graviton, then $\alpha \propto M_{\mathrm{Pl}}^{-1}$. In this case the only non-vanishing amplitude occurs when $h_1=-h_2=h$ and we conclude

$$\mathcal{A}_3[1^{h}2^{-h}3^{h_3}] = \alpha\begin{cases} \dfrac{[1\,3]^{2h+2}[2\,3]^{-2h+2}}{[1\,2]^2}\,, & h_3 = 2 \\ \dfrac{\langle 1\,3\rangle^{-2h+2}\langle 2\,3\rangle^{2h+2}}{\langle 1\,2\rangle^2}\,, & h_3=-2 \end{cases} \tag{45}$$

As a second example, consider a Yang-Mills theory with gauge group SU(N). Little group scaling and dimensional analysis give the following form for the three-point amplitude

$$\mathcal{A}_3[1^-2^-3^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,, \qquad \mathcal{A}_3[1^+2^+3^-] = \frac{[1\,2]^4}{[1\,2][2\,3][3\,1]} \tag{46}$$

2.7. Recursion relations

In this section we develop some powerful recursion relations for the computation of generic n-point amplitudes. The main result is the generalization of Eq. 46 to n gluons

$$\mathcal{A}_n[1^-2^-3^+\dots n^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\dots\langle n\,1\rangle} \tag{47}$$

This amplitude is known as Parke-Taylor formula [4].

Consider n lightlike 4-momenta $\{p_i^\mu\}_{i=1}^n$ which satisfy 4-momentum conservation $\sum_{i=1}^n p_i^\mu = 0$ and suppose we want to calculate the n-point tree amplitude for n particle with those 4-momenta, i.e. $\mathcal{A}_n[p_1^{h_1},\dots,p_n^{h_n}]$. Moreover, consider a set of n 4-momenta $\{r_i^\mu\}_{i=1}^n$ with the following properties:

1. $\sum_{i=1}^{n} r_i^\mu = 0$,

2. $r_i^\mu r_{j\,\mu} = 0$ for all i, j,

3. $r_i^\mu p_{i\,\mu} = 0$ for all i (no sum over i).

We now introduce a set of shifted 4-momenta, defined by

$$\hat{p}_i^\mu(z) = p_i^\mu + z r_i^\mu \tag{48}$$

where $z \in \mathbb{C}$. Notice that

- the shifted momenta are lightlike,
- the shifted momenta satisfy 4-momentum conservation.

We can then study the shifted amplitude

$$\hat{\mathcal{A}}_n(z) = \mathcal{A}_n[\hat{p}_1(z)^{h_1}, \ldots, \hat{p}_n(z)^{h_n}] \tag{49}$$

and this amplitude is a proper, on-shell amplitude, although evaluated for complex momenta. As we are focusing on tree level, the amplitude is a rational function of $|\hat{p}_i(z)\rangle$ and $|\hat{p}_i(z)]$ and therefore is a meromorphic function of z. Now consider a pole of $\hat{A}_n$: this pole can only be generated by an internal propagator in the associated Feynman diagram. To be more precise, for a given subset $I \subseteq \{1, \ldots, n\}$ [3] define

$$P_I^\mu = \sum_{i \in I} p_i^\mu \,, \qquad \hat{P}_I^\mu(z) = \sum_{i \in I} \hat{p}_i{}^\mu(z) \,, \qquad R_I^\mu = \sum_{i \in I} r_i^\mu \tag{50}$$

An internal propagator is necessarily of the form $1/\hat{P}_I^2(z)$ for a proper I. Now notice that $\hat{P}_I^2(z)$ is linear in z

$$\hat{P}_I^2(z) = P_I^2 + 2z R_I^\mu P_{I\,\mu} \tag{51}$$

In particular, for a given I we obtain a pole located at

$$z_I = -\frac{P_I^2}{2R_I^\mu P_{I\,\mu}} \tag{52}$$

and the residue of the propagator is $-z_I/P_I^2$. If the pole associated to I is present in $\hat{\mathcal{A}}_n$, then necessarily the diagrams associated to $\mathcal{A}_n$ factorise in two subdiagrams, connected by the propagator $1/\hat{P}_I^2(z)$. Let's denote with $\hat{\mathcal{A}}_L^I(z)$ and $\hat{\mathcal{A}}_R^I(z)$ the corresponding subamplitudes. These amplitudes involve a lower number of particles and cannot contain themselves the internal propagator $1/\hat{P}_I^2(z)$, thus they are regular around the pole in z_I. We then conclude that

- the poles of $\hat{\mathcal{A}}_n$ are associated to some subsets of $\{1, \ldots, n\}$, and every subset of $\{1, \ldots, n\}$ is associated at most to one pole of $\hat{\mathcal{A}}_n$,
- every pole is simple,

[3] We restrict to the case $2 \leq |I| \leq n-2$ in order to avoid trivial cases.

- the residue at a pole located in z_I is

$$\text{Res}[\hat{\mathcal{A}}_n, z_I] = -\frac{z_I}{P_I^2}\hat{\mathcal{A}}_L^I(z_I)\hat{\mathcal{A}}_R^I(z_I) \tag{53}$$

Finally, consider the function $f_n(z) = \hat{\mathcal{A}}_n(z)/z$. Using the residue theorem we find

$$\mathcal{A}_n = \hat{\mathcal{A}}_n(0) = B_n + \sum \hat{\mathcal{A}}_L^I(z_I)\frac{1}{P_I^2}\hat{\mathcal{A}}_R^I(z_I) \tag{54}$$

where we denoted with B_n the residue at infinity of the function f_n. The sum is extended to all the shifted, internal propagators. This relation is precisely a recursion relation, because $\hat{\mathcal{A}}_L^I$ and $\hat{\mathcal{A}}_R^I$ are k-point tree amplitudes with $k < n$. Therefore, if we manage to compute B_n in an independent way we can calculate the desired n-point tree amplitude. This task is often very difficult, but there are several examples in which $B_n = 0$, as we will see soon.

2.7.1. BCFW relation

Now we set a particular choice of the auxiliary vectors r_i, known as $[i, j\rangle$-shift. Consider the following shift in the spinors

$$|\hat{i}] = |i] + z|j]\,, \qquad |\hat{i}\rangle = |i\rangle \tag{55}$$

$$|\hat{j}\rangle = |j\rangle - z|i\rangle\,, \qquad |\hat{j}] = |j] \tag{56}$$

while the other spinors are left unchanged. This shift corresponds to the choice of the auxiliary vectors

$$r_i^\mu = -r_j^\mu = \frac{1}{2}\langle i|\gamma^\mu|j]\,, \qquad r_k^\mu = 0 \tag{57}$$

for $k \neq i, j$. In this case it can be shown [3] that in a Yang-Mills theory $\hat{\mathcal{A}}_n(z)$ has the following asymptotic behaviour:

- if $(h_i, h_j) \in \{(-,-), (-,+), (+,-)\}$, then $\hat{\mathcal{A}}_n(z) = \mathcal{O}(z^{-1})$ when $z \to \infty$,
- if $(h_i, h_j) = (+,-)$, then $\hat{\mathcal{A}}_n(z) = \mathcal{O}(z^3)$ when $z \to \infty$.

In the first three cases we have then $B_n = 0$ and the $[i, j\rangle$-shift is a suitable choice for a proper recursion relation. This kind of recursion relation is known as BCFW relation [2]. In particular, let's now prove the Parke-Taylor formula. We proceed by induction: the formula is true for the case $n = 3$, as already shown by little group scaling arguments. We now use a $[1, 2\rangle$-shift and use the BCFW relation to write

$$\mathcal{A}_n[1^-2^-3^+\dots n^+] = \sum_{k=4}^{n}\sum_{h=\pm}\hat{\mathcal{A}}_{n-k+3}[\hat{1}^-, \hat{P}_I^h, k^+, \dots, n^+]\frac{1}{P_I^2}\hat{\mathcal{A}}_{k-1}[-\hat{P}_I^{-h}, \hat{2}^-, 3^+, \dots, (k-1)^+] \tag{58}$$

It can be shown that the n-gluon tree amplitude is vanishing if all the gluons have the same helicity or all the gluons but one have the same helicity. Thus in the previous sum only two terms can be non-vanishing and we deduce

$$\mathcal{A}_n[1^-2^-3^+\dots n^+] = \hat{\mathcal{A}}_{n-1}[\hat{1}^-, \hat{P}_{23}^-, 4^+, \dots, n^+]\frac{1}{P_{23}^2}\hat{\mathcal{A}}_3[-\hat{P}_{23}^+, \hat{2}^-, 3^+]+ \\ + \hat{\mathcal{A}}_3[\hat{1}^-, -\hat{P}_{1n}^+, n^+]\frac{1}{P_{1n}^2}\hat{\mathcal{A}}_{n-1}[\hat{P}_{1n}^-, \hat{2}^-, 3^+, \dots, (n-1)^+] \tag{59}$$

In the previous equation, we also defined $P_{ij} = p_i + p_j$. The first term must be evaluated at the point z_{23} defined by $\hat{P}_{23}^2(z_{23}) = 0$. Similarly, the second one must be evaluated at z_{1n}, defined by $\hat{P}_{1n}^2(z_{1n}) = 0$. Consider the latter term: the three-point amplitude is [4]

$$\hat{\mathcal{A}}_3[\hat{1}^-, -\hat{P}_{1n}^+, n^+] = \frac{[\hat{P}_{1n}\, n]^3}{[\hat{1}\,\hat{P}_{1n}][n\,\hat{1}]} \tag{60}$$

At the point z_{1n} we have $0 = \hat{P}_{1n}^2 \propto \hat{p}_1 \cdot p_n$, so we find

$$0 = \langle \hat{1}\, n\rangle [\hat{1}\, n] = \langle 1\, n\rangle[\hat{1}\, n] \tag{61}$$

where we used the fact that under a $[1, 2\rangle$-shift the angle bra $\langle 1|$ is left unchanged. Thus we deduce $[\hat{1}\, n] = 0$, which implies

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n}\, n] = -\hat{P}_{1n}|n] = -\hat{p}_1|n] = |\hat{1}\rangle[\hat{1}\, n] = 0 \tag{62}$$

and finally $[\hat{P}_{1n}\, n] = 0$. Hence we conclude that the amplitude 60 vanishes and, by induction hypothesis,

$$\mathcal{A}_n[1^-2^-3^+\dots n^+] = \frac{\langle \hat{1}\,\hat{P}_{23}\rangle^3}{\langle \hat{P}_{23}\, 4\rangle\langle 4\, 5\rangle \dots \langle n\,\hat{1}\rangle} \cdot \frac{1}{\langle 2\, 3\rangle[2\, 3]} \cdot \frac{[\hat{P}_{23}\, 3]^3}{[\hat{P}_{23}\,\hat{2}][\hat{2}\, 3]} \tag{63}$$

with the shifted momenta evaluated at the point z_{23}. Using now the relations

$$[\hat{2}\, 3] = [2\, 3]\,, \qquad \langle \hat{P}_{23}\, 4\rangle[\hat{P}_{23}\, 2] = \langle 3\, 4\rangle[3\, 2]\,, \qquad \langle \hat{1}\,\hat{P}_{23}\rangle[3\,\hat{P}_{23}] = \langle 1\, 2\rangle[2\, 3] \tag{64}$$

Eq. 63 finally reduces to the Parke-Taylor formula.

3. Extension to the massive case

3.1. Massive spinor-helicity variables

The main difference between the massive and the massless case is the fact that the matrix $p_{a\dot{b}}$ has rank 2. In particular, in the massive case $p_{a\dot{b}}$ is invertible and we can't write it as an exterior product anymore. However, linearity allows us to write $p_{a\dot{b}}$ as a sum of two rank-1 matrices, i.e.

$$p_{a\dot{b}} = -|p^I]_a\langle p_I|_{\dot{b}} \tag{65}$$

with $I = 1, 2$. In this case, the Weyl equation reads

$$p_{a\dot{b}}|p^I\rangle^{\dot{b}} = m|p^I]_a\,, \qquad p^{\dot{a}b}|p^I]_b = m|p^I\rangle^{\dot{a}} \tag{66}$$

[4]Notice that $|-p\rangle \propto |p\rangle$ and $|-p] \propto |p]$. A common convention is $|-p\rangle = -|p\rangle$ and $|-p] = |p]$.

or, in other words, $u^I = \begin{pmatrix} |p^I]^t_a & |p^I\rangle^{t\,\dot a} \end{pmatrix}^t$ is a Dirac spinor. We can also arrange $|p^I]_a$ and $\langle p_I|_{\dot b}$ in two 2×2 matrices, respectively λ and $\tilde\lambda$. Binet theorem gives then the constraint

$$\det\lambda\;\det\tilde\lambda = m^2 \tag{67}$$

If we impose separately $\det\lambda = \det\tilde\lambda = m$, then the action of the little group of p^μ is now given by

$$\lambda^I \mapsto W^I_J\lambda^J\,, \qquad \tilde\lambda_I \mapsto (W^{-1})^J_I\tilde\lambda_J \tag{68}$$

where, in general, $W\in \mathrm{SL}(2,\mathbb{C})$. Again, if p^μ is real-valued we must restrict the possible little group transformations to $W\in\mathrm{SU}(2)$. For this reason, we refer to I, J as SU(2) indices, while we refer to a, $\dot b$ as Lorentz indices.

Unfortunately, in the massive case the little group action does not imply particular scaling laws. However, if we represent a spin-S massive state as a symmetric rank-$2S$ SU(2) tensor, then a scattering amplitude will be a SU(2) tensor as well. To give an example, consider a 4-point amplitude where the first particle is massive and has spin S_1, the second one is massive and has spin S_2, and finally the third and fourth particle are massless, with helicities respectively h_3 and h_4. Then the scattering amplitude will be an object $\mathcal{A}$ with $2S_1+2S_2$ SU(2) indices and 2 U(1) indices, which transforms under the little group as

$$\begin{aligned}\mathcal{A}^{\{I_1,\dots,I_{2S_1}\}\{J_1,\dots,J_{2S_2}\}\{h_3\}\{h_4\}} \mapsto& (W_1)^{I_1}_{I'_1}\dots(W_1)^{I_{2S_1}}_{I'_{2S_1}}\times(W_2)^{J_1}_{J'_1}\dots(W_2)^{J_{2S_2}}_{J'_{2S_2}}\times(t_3)^{-2h_3}\times\\ &\times(t_4)^{-2h_4}\times\mathcal{A}^{\{I'_1,\dots,I'_{2S_1}\}\{J'_1,\dots,J'_{2S_2}\}\{h_3\}\{h_4\}}\end{aligned} \tag{69}$$

Notice now that a SU(2) index I can be generated only by the presence of a square or an angle massive variable, but Weyl equation allows us to interchange the two kind of variables. So we can assume without loss of generality that a massive spin-S amplitude is of the form

$$\mathcal{A}^{\{I_1,\dots,I_{2S}\}} = |p^{I_1}]_{a_1}\dots|p^{I_{2S}}]_{a_{2S}}\mathcal{M}^{\{a_1,\dots,a_{2S}\}} \tag{70}$$

where $\mathcal{M}$ is a symmetric tensor with $2S$ Lorentz indices. In this way, we can look for an appropriate base (u_a, v_a) of $\mathbb{C}^2$, with a being a Lorentz index, and write in that base the most general form of $\mathcal{M}$, which is now a homogeneous polynomial of degree $2S$ in the variables u_a, v_a.

Finally, before we characterize all possible three-point amplitudes, we introduce a compact notation for massive variables: we suppress SU(2) indices and use boldface for massive spinors. For example, later on we will derive the amplitude for the Compton scattering with spin-1/2 fermion

$$\mathcal{A}_{\text{Compton}}[\mathbf{1}2^-3^+\mathbf{4}] = \langle 2|(p_1-p_4)|3]\frac{\langle\mathbf{1}\,2\rangle[\mathbf{4}\,3]+\langle\mathbf{4}\,2\rangle[\mathbf{1}\,3]}{(s-m^2)(u-m^2)} \tag{71}$$

where, for instance, the notation $\langle\mathbf{1}\,2\rangle[\mathbf{4}\,3]$ means $\langle 1^I\,2\rangle[4^J\,3]$.

3.2. Three-point amplitudes

In this section we write down the most general form for all possible three-point amplitudes. The final form is different depending on the number of massive legs and on the values of the masses, so we separate the different cases.

3.2.1. Three massless particles

We already studied this case using the little group scaling. We simply recall that if all coupling constants have non-negative mass dimension, then

$$\mathcal{A}_3[1^{h_1}2^{h_2}3^{h_3}] = g\begin{cases}[1\,2]^{h_1+h_2-h_3}[2\,3]^{h_2+h_3-h_1}[1\,3]^{h_1+h_3-h_2}\,, & h_1+h_2+h_3>0\\ \langle 1\,2\rangle^{h_3h_1-h_2}\langle 2\,3\rangle^{h_1-h_2-h_3}\langle 1\,3\rangle^{h_2-h_1-h_3}\,, & h_1+h_2+h_3<0\end{cases} \tag{72}$$

3.2.2. Two massless particles, one massive particle

In this case, if the first two particles are massless we pick the basis $(|1]_a, |2]_a)$. If the massive particle has spin S and mass m, then little group scaling fixes the symmetric tensor $\mathcal{M}$ to be equal to

$$\mathcal{M}^{\{a_1,\ldots,a_{2S}\}\{h_1\}\{h_2\}} = \frac{g}{m^{2S-h_1-h_2-1}}\left([1|^{S+h_1-h_2}[2|^{S+h_2-h_1}\right)^{\{a_1\ldots a_{2S}\}}\langle 1\,2\rangle^{S-h_1-h_2} \tag{73}$$

We introduced a proper power of m in order to have a dimenionless coupling g. The corresponding amplitude is therefore

$$\mathcal{A}[1^{h_1}2^{h_2}\mathbf{3}] = \frac{g}{m^{2S-h_1-h_2-1}}[1\,\mathbf{3}]^{S+h_1-h_2}[2\,\mathbf{3}]^{S+h_2-h_1} \tag{74}$$

3.2.3. One massless particle, two massive particles (different masses)

In this case we take the first particle to be massless, while the second particle has mass m_2 and spin S_2 and the third one has mass $m_3 \neq m_2$ and spin S_3. We pick the basis $(|1]_a, (p_2|1\rangle)_a)$. The little group scaling is no longer enough to completely determine the tensor $\mathcal{M}$, which now has the general form

$$\mathcal{M}^{\{a_1,\ldots,a_{2S_2}\}\{b_1,\ldots,b_{2S_3}\}\{h_1\}} = \sum_i g_i\left([1|^{S_2+S_3+h_1}(\langle 1|p_2)^{S_2+S_3-h_1}\right)^{\{a_1,\ldots,a_{2S_2}\}\{b_1,\ldots,b_{2S_2}\}} \tag{75}$$

Notice that we expect to have a different coupling for every possible term. The sum is extended to all the possible ways of distributing the $2S_2+2S_3$ indices between the two sets $\{a_1,\ldots,a_{2S_2}\}$ and $\{b_1,\ldots,b_{2S_2}\}$, and therefore $\mathcal{M}$ contains in the most general case $S_2+S_3-|S_2-S_3|$ terms. In other words, the amplitude is the sum of $2S+1$ terms, with $S=\min\{S_2,S_3\}$. As an example, if $S_2=2$, $S_3=1$ and the first particle is a photon with helicity $+1$, the amplitude is

$$\mathcal{A}_3[1^+\mathbf{2}\,\mathbf{3}] = g\langle 1\,\mathbf{2}\rangle^2[1\,\mathbf{2}]^2[1\,\mathbf{3}]^2 + g'\langle 1\,\mathbf{2}\rangle[1\,\mathbf{2}]^3\langle 1\,\mathbf{3}\rangle[1\,\mathbf{3}] + g''[1\,\mathbf{2}]^4\langle 1\,\mathbf{3}\rangle^2 \tag{76}$$

Here we used the properties $p_2|\mathbf{2}] \propto |\mathbf{2}\rangle$ and $\langle 1|p_2|\mathbf{3}] \propto \langle 1\,\mathbf{3}\rangle$ and we absorbed the proportionality constants into the couplings.

3.2.4. One massless particle, two massive particles (equal masses)

In this case the previous basis is no longer available, since we have

$$\langle 1|p_2|1] \propto p_1^\mu p_{2\,\mu} \tag{77}$$

If $m_2 = m_3 = m$, then

$$m^2 = p_3^2 = (p_1 + p_2)^2 = m^2 + 2p_1^\mu p_{2\,\mu} \tag{78}$$

and thus $\langle 1|p_2|1] = 0$, which means that the spinors $|1]$ and $p_2|1\rangle$ are not linear independent. We denote with x the proportionality constant between them, i.e.

$$x = \frac{\langle k|p_2|1]}{m\langle k\, 1\rangle} \tag{79}$$

where k is an auxiliary spinor, in the same way q was an auxiliary spinor in the definition of the polarization vectors $\varepsilon_\pm^\mu(p, q)$. Notice that under the little group of p_1 we have the scaling law $x \mapsto t^{-2}x$. Finally, let's write $\mathcal{M}$. We can use $|1]$, $p_2|1\rangle$ and the matrix ε as basis, so now the general form of $\mathcal{M}$ is

$$\mathcal{M}^{\{a_1 \ldots a_{2S_2}\}\{b_1 \ldots b_{2S_3}\}\{h_1\}} = \sum_{i=|S_2-S_3|}^{S_2+S_3} \sum_j g_{i,j} x^{h_1} \left([1|^i \left(\frac{\langle 1|p_2}{m} \right)^i \varepsilon^{S_1+S_2-i} \right)^{\{a_1 \ldots a_{2S_2}\}\{b_1 \ldots b_{2S_3}\}} \tag{80}$$

where, again, the second sum runs over the possible ways of distributing the indices between particle 2 and particle 3. In particular, we *define* minimal coupling of a massive particle to a photon as the choice $g_{i,j} = 0$ for $i > 0$. This means that the three-point amplitude for a photon and two massive particle of mass m and spin S

$$\mathcal{A}_3[1^+ \mathbf{2}\,\mathbf{3}] = x \frac{\langle \mathbf{2}\,\mathbf{3}\rangle^{2S}}{m^{S-1}}\,, \qquad \mathcal{A}_3[1^- \mathbf{2}\,\mathbf{3}] = x^{-1} \frac{[\mathbf{2}\,\mathbf{3}]^{2S}}{m^{S-1}} \tag{81}$$

3.2.5. Three massive particles

Finally, if all the external legs are massive we do not have an obvious choice of the basis for $\mathcal{M}$. Indeed, it is easier if we write $\mathcal{M}$ in terms of the tensor

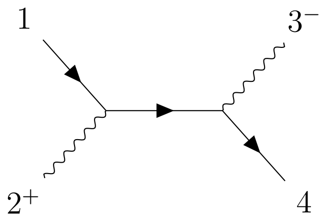
$$\mathcal{T}_{ab} = (p_1)_{\{a\,\dot{b}}(p_2)_{b\}}{}^{\dot{b}} \tag{82}$$

Now the form of $\mathcal{M}$ is given by

$$\mathcal{M}^{\{a_1 \ldots a_{2S_1}\}\{b_1 \ldots b_{2S_2}\}\{c_1 \ldots c_{2S_3}\}} = \sum_i \sum_j g_{i,j} \left(\mathcal{T}^{S_1+S_2+S_3-i} \varepsilon^i \right)^{\{a_1 \ldots a_{2S_1}\}\{b_1 \ldots b_{2S_2}\}\{c_1 \ldots c_{2S_3}\}} \tag{83}$$

3.3. Compton scattering

Let's see how the massive spinor-helicity formalism works with an example. Consider the Compton scattering $p\gamma^+ \to p\gamma^-$, where p is a particle of mass m and spin S. In the s channel



$$= m^2 \frac{x_{12}}{x_{34}} \frac{1}{s - m^2} \left(\frac{\langle \mathbf{1}\, p^I \rangle [p_I\, \mathbf{4}]}{m^2} \right)^S \tag{84}$$

with $p = p_1 + p_2$. If we choose the auxiliary spinors to be $k_{12} = p_3$ and $k_{34} = p_2$, then we have

$$m^2 \frac{x_{12}}{x_{34}} = \frac{[2|p_1|k_{12}\rangle [k_{34}|p_4|3\rangle}{\langle k_{12}\, 2\rangle [k_{34}\, 3]} = -\frac{(\langle 3|p_1|2]^2)}{t} \tag{85}$$

Notice that the s channel is dominant in the limit $s \to 0$, where $t \to -u + m^2$. Moreover, the matrix $|p^I\rangle [p_I|$ can be explicitly computed[5] and is found to be equal to

$$|p^I]\langle p_I| = \frac{-m^2 |3]\langle 2| + (p_1|2\rangle)([3|p_4)}{\langle 3|p_1|2]} \tag{87}$$

We then get the amplitude with the correct residues in s and u

$$\mathcal{A}^S_{\text{Compton}}[\mathbf{1}2^+3^-\mathbf{4}] = \frac{1}{(s-m^2)(u-m^2)} \begin{cases} (\langle 3|p_1|2])^2\,, & S = 0 \\ \langle 3|p_1|2](\langle \mathbf{1}\, 3\rangle [\mathbf{4}\, 2] + [\mathbf{1}\, 2]\langle \mathbf{4}\, 3\rangle)\,, & S = 1/2 \\ (\langle \mathbf{1}\, 3\rangle [\mathbf{4}\, 2] + [\mathbf{1}\, 2]\langle \mathbf{4}\, 3\rangle)^2\,, & S = 1 \end{cases} \tag{88}$$

If $S > 1$ then spurious poles appear and the discussion must be modified, but we don't examine this case in detail.

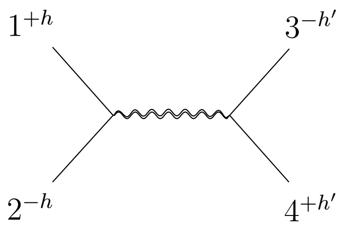
4. A possible UV completion for gravity

In this section, we construct all possible scattering amplitudes for particles of spin 0, 1/2, 1 and 2, i.e. SM particles and gravitons. We restrict to the case of massless particles, or equivalently to sufficiently high energy. Then, we study these amplitudes in order to construct a UV completion of general relativity. More precisely, we will construct all possible four-point scattering amplitudes mediated either by graviton or generic massive particles.

As done in [6], the following discussion is restricted to weak coupling and tree level, so the new massive resonances should lay below the Planck mass.

4.1. Graviton mediated amplitudes

Starting from the graviton mediated amplitudes, using Eq. 45, we see that in the s channel we have the amplitude



$$= \kappa^2 \frac{[1\,p]^{2h+2}[2\,p]^{-2h+2}}{[1\,2]^2} \frac{1}{p^2} \frac{\langle 3\,p\rangle^{2h'+2}\langle 4\,p\rangle^{-2h'+2}}{\langle 3\,4\rangle^2} \tag{89}$$

[5] From the definition, we have to solve

$$|p^I]\langle p_I\, 2\rangle = -m x_{12}|2]\,, \qquad |p^I]\langle p_I 3\rangle = m x_{34}|3] \tag{86}$$

where $\kappa = \sqrt{8\pi}/M_{\mathrm{Pl}}$, $p = p_1 + p_2$ and where we supposed that the exchange graviton emerges from the left vertex with positive helicity. Define now $s_{ij} = (p_i + p_j)^2$: the amplitude just written is valid only in the limit $s_{12} \to 0$, which also means $s_{13} \to -s_{14}$. Therefore, we can always multiply the amplitude by a factor $s_{13}^r s_{14}^{-r}$ without affecting the on-shell amplitude. All in all, we obtain the following expression for the amplitude

$$\mathcal{A}^{\mathrm{GR}}_{h,h'} = \kappa^2 \frac{s_{13}^{1-h'-r} s_{14}^{1-h'+r}}{s_{12}} ([1\,4]\langle 2\,3\rangle)^{2h} (\langle 3|p_{12}|4])^{2h'-2h} \tag{90}$$

where $p_{ij} = (p_i - p_j)/2$. Notice that r must be (half-)integer if h' is (half-)integer. Moreover, scaling arguments require the constraint $-1 + h' \leq r \leq 1 - h'$, which means

- if $h = h' = 0$, then r originates a contact term proportional to s_{12}. In other words, the scattering amplitude for *distinguishable* scalars is

  $$\mathcal{A}^{\mathrm{GR}}_{0,0,\mathrm{dist}}(s_{12}, s_{13}, s_{14}) = \kappa^2 \left(\frac{s_{13} s_{14}}{s_{12}} - a s_{12} \right) \tag{91}$$

  On the other hand, the scattering amplitude for identical scalars is

  $$\mathcal{A}^{\mathrm{GR}}_{0,0,\mathrm{id}}(s_{12}, s_{13}, s_{14}) = \mathcal{A}^{\mathrm{GR}}_{0,0,\mathrm{dist}}(s_{12}, s_{13}, s_{14}) + \mathcal{A}^{\mathrm{GR}}_{0,0,\mathrm{dist}}(s_{13}, s_{14}, s_{12}) + \mathcal{A}^{\mathrm{GR}}_{0,0,\mathrm{dist}}(s_{14}, s_{12}, s_{13}) \tag{92}$$

  and thus is independent of a.

- Similarly, for spin-1/2 particles we get

  $$\mathcal{A}^{\mathrm{GR}}_{1/2,1/2,\mathrm{dist}}(s_{12}, s_{13}, s_{14}) = \kappa^2 [1\,4]\langle 2\,3\rangle \left(\frac{s_{13}}{s_{12}} + \frac{b}{2} \right) \tag{93}$$

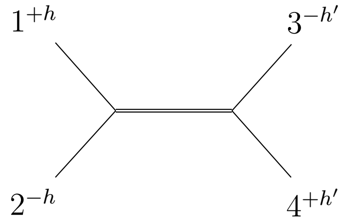
  and for identical fermions

  $$\mathcal{A}^{\mathrm{GR}}_{1/2,1/2,\mathrm{id}}(s_{12}, s_{13}, s_{14}) = \kappa^2 [1\,4]\langle 2\,3\rangle \left(\frac{s_{13}}{s_{12}} + \frac{s_{12}}{s_{13}} + b \right) \tag{94}$$

- In the other cases, we are forced to take $r = 0$.

4.2. Massive spin J mediated amplitudes

Consider now the amplitudes mediated by massive particles: the s channel in the case of a mediator of mass M and spin J can be obtained by gluing together the three-point amplitudes discussed in Eq. 74. The amplitude takes finally the form



$$= g_J^2 \frac{M^2}{s_{12} - M^2} \times \left(\frac{\langle \mathbf{p}|p_{12}|\mathbf{p}]}{M^2} \right)^{J-2h} \times \left(\frac{\langle \mathbf{p}\,2\rangle[1\,\mathbf{p}]}{M^2} \right)^{2h} \times$$

$$\times \left(\frac{\langle \mathbf{p}|p_{43}|\mathbf{p}]}{M^2} \right)^{J-2h'} \times \left(\frac{\langle \mathbf{p}\,3\rangle[4\,\mathbf{p}]}{M^2} \right)^{2h'} \tag{95}$$

The main difference between the graviton mediated amplitudes and the massive spin J mediated amplitudes is the presence of the massive spinors $|\mathbf{p}\rangle$, $|\mathbf{p}]$. In the latter case, we need to sum over all possible contractions of SU(2) indices. After the sum, the amplitude is

$$\mathcal{A}^J_{h,\,h'} = g_J^2 \frac{(2J)!!}{(2J-1)!!4^J}\left(\frac{[1\,4]\langle 3\,2\rangle}{M^2}\right)^{2h}\left(\frac{\langle 3|p_{12}|4]}{M^2}\right)^{2h'-2h}\frac{M^2}{s_{12}-M^2}\binom{J+2h'}{J}\binom{J}{2h'}^{-1}P^{(2h'-2h,2h'+2h)}_{J-2h'}(x) \tag{96}$$

where $P^{(a,b)}_M$ are Jacobi polynomials and $x = 1+2s_{13}/M^2$. Notice that for scalars the amplitude $\mathcal{A}^J_{0,\,0}$ is proportional to the Legendre polynomial $P_J(x)$.

4.3. Partial wave analysis and UV completion of gravity

The presence of Jacobi and Legendre polynomials suggests then a partial wave discussion: for the process $1^h 2^{-h} \to 3^{h'} 4^{-h'}$, we also have

$$x = 1 - \frac{s}{M^2}(1-\cos\theta)\,, \qquad \langle 3\,1\rangle = \sqrt{s}\sin\frac{\theta}{2}\,, \qquad [1\,4] = \langle 3\,2\rangle = \sqrt{s}\cos\frac{\theta}{2} \tag{97}$$

where θ is the scattering angle in the CM reference frame. To be more precise, we can write a decomposition of the amplitude

$$\mathcal{A}_{1^h 2^{-h}\to 3^{h'}4^{-h'}} = 16\pi \sum_{J\geq \max 2h,2h'} (2J+1)a_J(s)d^J_{2h,2h'}(\theta) \tag{98}$$

where $d^J_{a,b}$ are the Wigner d-functions. The coefficients of the previous decomposition can be evaluated numerically: for example, the decomposition of the graviton mediated amplitude for distinguishable scalars is simply

$$a_0(s) = \frac{s(1-6a)}{96M^2_{\rm Pl}}\,, \qquad a_2(s) = -\frac{s}{480M^2_{\rm Pl}}\,, \qquad a_K(s) = 0 \tag{99}$$

if $K \neq 0, 2$. The growth of a_2 with energy can be mitigated by introducing a spin J massive resonance, with $J \geq 2$. However, as discussed in [6], if $J = 2$ the divergence is cured only if there are at most two different species of scalars. On the other hand, if $J > 2$ we are forced to introduce an infinite tower of resonances of increasing higher spin. One way to solve the problem is proposed in [6], where under certain hypothesis of regularity (which ultimately come from unitarity, locality and causality) the proposed scattering amplitude for distinct scalars is[6]

$$\mathcal{A}_{0,\,0,\,\rm dist} \propto \kappa^2\left(\frac{tu}{s} - as\right)\frac{\prod_{n=1}^{\infty}(M^2 t + \hat{M}^2 s - M^2(n\hat{M}^2 + \hat{M}_0^2))}{\prod_{k=1}^{\infty}(s-kM^2)\prod_{\ell=1}^{\infty}(t-\ell\hat{M}^2-\hat{M}_0^2)} \tag{100}$$

Here M^2, $\hat{M}^2$ and $\hat{M}_0^2$ are parameters, which for the moment are unconstrained. The main feature of this modified amplitude is the presence of two towers of infinite resonances in s and t, with masses

$$M_n^2 = nM^2\,, \qquad \hat{M}_n^2 = \hat{M}_0^2 + n\hat{M}^2 \tag{101}$$

[6]Since we are focusing on the process $1\,2 \to 3\,4$, we use usual Mandelstam variables instead of s_{ij} variables.

A similar discussion can be made for the other amplitudes: in each case, the amplitude given by Eq. 95 is modified by a proper multiplicative factor, which depends generically on the symmetry properties that we require from the corresponding amplitude. The infinite products in Eq. 100 can be related to the Gamma function via Euler's definition of Γ. Therefore, if we define

$$\tilde{s} = \frac{s}{M^2}\,, \qquad \tilde{t} = \frac{t}{M^2}\,, \qquad \eta = \frac{M^2}{\hat{M}^2}\,, \qquad \gamma_0 = \frac{\hat{M}_0^2}{M^2} \tag{102}$$

$$\mathcal{A}_{\mathrm{VZ}}^{\eta,\gamma_0}(s,t) = \frac{\Gamma(1-\tilde{s})\Gamma(1+\eta\gamma_0-\eta\tilde{t})}{\Gamma(1+\eta\gamma_0-\eta\tilde{t}-\tilde{s})}\,, \qquad \mathcal{A}_{\mathrm{VZ}}^{\eta=1,\gamma_0=0}(s,t) = \mathcal{A}_{\mathrm{VZ}}(s,t) \tag{103}$$

$$\mathcal{A}_{\mathrm{VS}}^{\gamma_0}(s,t,u) = \frac{\Gamma(1+2\gamma_0)\Gamma(1+\gamma_0-\tilde{u})\Gamma(1+\gamma_0-\tilde{t})\Gamma(1-\tilde{s})}{\Gamma(1+\tilde{u}+\gamma_0)\Gamma(1+\tilde{t}+\gamma_0)\Gamma(1+\tilde{s}+2\gamma_0)}\,, \qquad \mathcal{A}_{\mathrm{VS}}^{\gamma_0}(s,t,u) = \mathcal{A}_{\mathrm{VS}}^{\gamma_0=0}(s,t,u) \tag{104}$$

then the UV-completed amplitudes are summarized in Tab. 1. The presence of the Gamma

$\mathcal{A}^{\mathrm{UV}}$	Scalar	Fermion	Vector	Graviton
Scalar	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}^{\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}^{\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Fermion		$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}^{\eta,\gamma_0}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Vector			$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VZ}}$	$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$
Graviton				$\mathcal{A}^{\mathrm{GR}}\mathcal{A}_{\mathrm{VS}}$

Table 1: UV completion of graviton mediated amplitudes.

functions allows the amplitudes to be in agreement with causality, both in the Regge limit[7] and in the hard scattering limit[8]. A numerical partial wave analysis can be done on these new amplitudes. The main results are

- there are infinite resonances, with both integer and half-integer spin,
- unitarity imposes positivity constraints on the coefficients $a_J(s)$, which imply

$$0 \le a \le 2\,, \qquad \frac{2}{3} \le b \le \frac{22}{5}\,, \qquad \eta = 1\,, \qquad \gamma_0 < \frac{3}{2} \tag{105}$$

On the other hand, if we consider the amplitudes computed using the Feynman rules obtained from general relativity minimally coupled to matter, we find

$$a = 0\,, \qquad b = \frac{1}{2} \tag{106}$$

So either we have to modify Einstein-Hilbert action or we need to modify the UV completion here proposed. A possible change in Einstein-Hilbert action is presented at the end of [6] and the deviation is possible if space-time has torsion in addition to curvature.

[7] i.e. $s \to \infty$, $\theta \to 0$, with $t = -s\sin^2\theta/2$ fixed. In this case the amplitude should be $o(s^2)$.

[8] i.e. $s \to \infty$, $t \to \infty$, with θ fixed. In this case the amplitude decays exponentially in s.

5. Conclusion

In this report, we introduced spinor-helicity formalism, both in the massless and in the massive case. We presented all possible three-point amplitudes and then we showed how they can be used to construct a UV completion of tree level GR scattering amplitudes for massless external particles. In future, we hope to extend the discussion to external massive particles. An important question about the uniqueness of the bottom-up construction of UV completed scattering amplitudes is still unanswered, but it's beyond the purpose of this report.

Acknowledgments

I'm deeply grateful to my supervisors Marc Montull, Christophe Grojean and Fady Bishara, for the useful discussions we had and the support. I also would like to thank my thesis supervisor, prof. Augusto Sagnotti, for many helpful insights.

Last, but not least, I would like to thank the other Summer Students - and in particular my roommate Alessandro - for the lovely time we spent together.

A. Explicit example of spinor-helicity massless variables

Let us consider the most general lightlike, real-valued 4-momentum

$$p^\mu = E(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{107}$$

Then Eq. 1 reads

$$p_{a\dot{b}} = -2E \begin{pmatrix} s_{\theta/2}^2 & -s_{\theta/2}c_{\theta/2}e^{-i\phi} \\ -s_{\theta/2}c_{\theta/2}e^{i\phi} & c_{\theta/2}^2 \end{pmatrix} \tag{108}$$

$$p^{\dot{a}b} = -2E \begin{pmatrix} c_{\theta/2}^2 & s_{\theta/2}c_{\theta/2}e^{-i\phi} \\ s_{\theta/2}c_{\theta/2}e^{i\phi} & -s_{\theta/2}^2 \end{pmatrix} \tag{109}$$

where we used the shorthands $\sin\alpha = s_\alpha$ and $\cos\alpha = c_\alpha$. If we write

$$|p\rangle^{\dot{a}} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{110}$$

then the Weyl equation $p_{a\dot{b}}|p\rangle^{\dot{b}} = 0$ reads

$$\begin{pmatrix} As_{\theta/2}^2 - Bs_{\theta/2}c_{\theta/2}e^{-i\phi} \\ -As_{\theta/2}c_{\theta/2}e^{i\phi} + Bc_{\theta/2}^2 \end{pmatrix} = 0 \tag{111}$$

We can pick A and B such that

$$|p\rangle^{\dot{a}} = \sqrt{2E} \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}e^{i\phi} \end{pmatrix} \tag{112}$$

In exactly the same way, we can solve the other Weyl equations and write

$$[p|^a = \sqrt{2E} \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}e^{-i\phi} \end{pmatrix}^T, \quad |p]_a = \sqrt{2E} \begin{pmatrix} -s_{\theta/2}e^{-i\phi} \\ c_{\theta/2} \end{pmatrix}, \quad \langle p|_{\dot{a}} = \sqrt{2E} \begin{pmatrix} -s_{\theta/2}e^{i\phi} \\ c_{\theta/2} \end{pmatrix}^T \tag{113}$$

From Eq. 112 and 113 it now clear that $|p]_a = (\langle p|_{\dot{a}})^*$ and $|p\rangle^{\dot{a}} = ([p|^a)^*$. Moreover

$$\begin{aligned} |p]_a\langle p|_{\dot{b}} &= 2E \begin{pmatrix} -s_{\theta/2}e^{-i\phi} \\ c_{\theta/2} \end{pmatrix} \begin{pmatrix} -s_{\theta/2}e^{i\phi} & c_{\theta/2} \end{pmatrix} \\ &= -p_{a\dot{b}} \end{aligned} \tag{114}$$

and a similar calculation shows that $|p\rangle^{\dot{a}}[p|^b = -p^{\dot{a}b}$.

References

[1] H. Elvang and Y. t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

[2] R. Britto, F. Cachazo and B. Feng, "New recursion relations for tree amplitudes of gluons," Nucl. Phys. B **715** (2005) 499 doi:10.1016/j.nuclphysb.2005.02.030, arXiv:hep-th/0412308.

[3] R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory," Phys. Rev. Lett. **94** (2005) 181602 doi:10.1103/PhysRevLett.94.181602, arXiv:hep-th/0501052.

[4] S. J. Parke and T. R. Taylor, "An Amplitude for n Gluon Scattering," Phys. Rev. Lett. **56** (1986) 2459. doi:10.1103/PhysRevLett.56.2459

[5] N. Arkani-Hamed, T. C. Huang and Y. t. Huang, "Scattering Amplitudes For All Masses and Spins," arXiv:1709.04891 [hep-th].

[6] R. Alonso and A. Urbano, "On amplitudes, resonances and the ultraviolet completion of gravity," arXiv:1906.11687 [hep-ph].