Schur Index of Various Non-Lagrangian Theories

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Abstract

Minahan and Nemeschansky theories and T_N theories are $\mathcal{N} = 2$ Quantum Field Theories without any Lagrangian description available. These theories are tightly bounded to simple quiver theories with a duality called Seiberg-Argyres duality and this relation is enhanced interest to these unconventional non-lagrangian theories. By examing their Schur Indeces we compute and compare dynamics of numerous such theories.

Contents

1 Introduction

In this study we mainly focus on the $\mathcal{N} = 2$ 4d Supersymmetric Quantum Field Theories which does not have a classical Lagrangian description. These are originally introduced by Minahan and Nemeschansky [1][2].In these papers Minahan and Nemeschansky discuss 4d Supersymmetric Quantum Field Theories regarding to particular global symmetry groups such as D_4 or E_n(MN theories).Since these theories does not fit into the general scheme they could have forgotten by now. However, it is realised by Seiberg and Argyres [3] that there is a close relation between MN theories and ordinary $\mathcal{N} = 2$ supersymmetric gauge theories.
In fact Argyres and Seiberg showed that the MN theory with flavor symmetry E_6 which is gauged with the SU(2) quiver theory is dual to the SU(3) quiver theory. Then for the ordinary SU(3) theory the strongly-coupled limit becomes the weakly-coupled limit of its dual theory. This is called the Argyres-Seiberg duality.
Before calculating various Argyres-Seiberg duality examples we study different non-Lagrangian theories including E_6 MN theory in terms of superconformal index to understand the dynamics of the superconformal theories. Also, while making the calculations we studied fundamental structures of physics and mathematics to have a genuine idea about the grand scheme, such as Supersymmetry Algebra, Lagrangian descriptions of the Quantum Field Theories and representation theory.

2 Theory

2.1 Supersymmetry Algebra and Representation Theory

In order to encode the symmetries in a Supersymmetric QFT we use the SuperPoincare algebra which is the extension of the Poincare Algebra with the supersymmetry generators Q. This operators perform the characteristic operation in the theory,

$$Q\,|boson\rangle = |fermion\rangle \tag{1}$$

such that $Q^I_\alpha = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$,$Q^I_\alpha = \begin{pmatrix} q_3 \\ q_4 \end{pmatrix}$ where $\alpha, I = 1, 2$ and $q_i \in \mathbb{C}$ with the anti-commutation relation $\{Q_\alpha, \bar{Q_\beta}\} = 2\sigma^\mu_{\alpha\beta}P_\mu$.
For $\mathcal{N} = 1$ 4d $\chi = (\psi, \varphi)$ is the chiral multiplet such that ψ is the spinor and φ is the scalar and (A_μ, λ) is the vector multiplet such that A_μ is representing the vector and λ is the spinor. Moreover, when $\mathcal{N} = 2$ the chiral multiplet becomes hypermultiplet $(\chi_1, \tilde{\chi_2})$ and vector multiplet also gets doubled. In terms of representation theory these multiplets are the representations of the space-time symmetries in terms of SuperPoincare Algebra. On the other hand for the internal symmetries, such as flavor symmetry we use different irreducible representations of various Lie Groups.[5]
Furthermore, by using this description we can write the Schur index of numerous Quiver Theories shown in the following $\mathcal{N} = 2$ quiver diagram form.

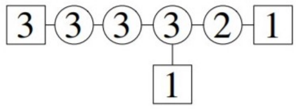

Figure 1: An example of $\mathcal{N}=2$ linear quiver diagram

In this linear quiver theory designation every node is representing a gauge multiplet with SU(n) group and every square is representing the flavor symmetry. The straigth lines between two nodes are two arrows which are pointing opposite directions to place hypermultiplets between different gauge groups. Related Schur index description is

$$\mathcal{I}(q)=\oint d\mu P.E[f^{V}(q)\chi_{G}(u)+f^{\frac{1}{2}H}(q)\chi_{R}(u)\chi_{F}(z)] \tag{2}$$

where f^{V} is the contribution from every, single gauge multiplet and $f^{\frac{1}{2}H}$ is the contribution from one arrow as defined in [6].

2.2 Calculations

For the T_3 theory the Schur index is defined as

$$I_{q,n}(q)=\sum_{\lambda}\frac{\prod_{i}N(a_i)\chi_{\lambda}(a_i)}{N^{n+2g-2}\chi_{\lambda}(q^{\rho})^{n+2g-2}} \tag{3}$$

where $a_i\in SU(N)$ is the flavor symmetry for the i-th puncture, $\sum_{\lambda}$ represents the sum over different irreducible representations of the flavor symmetry group and $\chi_{\lambda}(a)$ is the character. Since the theory defined on a Reimann surface C_g, g represents the word gen us, n is the number of punctures on this surface and $q^{\rho}:=(q^{(N-1)/2},\ldots,q^{(1-N)/2})$ from [7].
For the T_3 theory we take $g=0$, $n=3$, $N=3$, $\lambda=\{[0,0],[1,0],[0,1],[1,1],[2,0],[0,2]\}$ in the Dynkin label notation of SU(3). We observed that the irreducible representations in the set λ is closely related to the term that index will be truncated. Since we only focus on the first and the second order of q in the index we use the representations which have the Dynkin label at most two.And we obtain for the theory $S_{SU(3)}\langle C_{0,3}\rangle$,

$$I_{0,3}(q)=1+78q+2511q^{2}+O[q^{3}] \tag{4}$$

In order to generalize this index for spheres with regular punctures other than full punctures we use the following description.

$$I_{q,n}(a_1,...,a_n,q)=\sum_{\lambda}\frac{\prod_{i}K_{Y_i}(a_i)\chi_{\lambda}(a_iq^{\rho Y_i})}{N^{n+2g-2}\chi_{\lambda}(q^{\rho})^{n+2g-2}} \tag{5}$$

where K(a) is the generalized version of the function N(a).

$$K_Y(a)^{-1} = \prod_d \prod_\omega \prod_{n\geq 0} (1 - q^{n+(d+1)/2} a^\omega) \tag{6}$$

where d is the dimension of irreducible representation of SU(2) and ω is the weights of the representation. While writing these functions and characters we use the embedding of SU(2) into the maximal flavor symmetry group as described in [**?**]. For example for the E_7 Theory which described with two full punctures (SU(4)) flavor symmetry) and one with SU(2) has the following index.

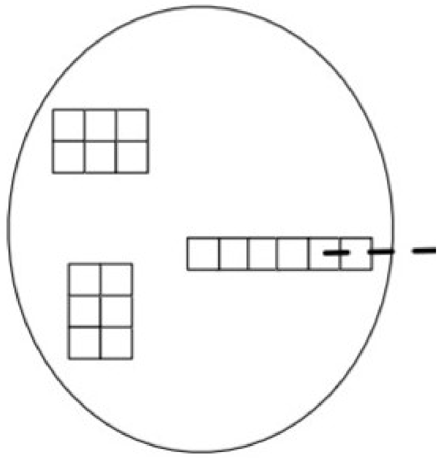

Figure 2: Minahan-Nemeschanskys E_8 Theory

$$I_{E_7}(q) = 1 + 133q + O[q]^{3/2} \tag{7}$$

It is important to highlight that given calculations are the unrefined versions of the index such that every flavor fugacity is taken to be one. On the other hand including all these different fugacities to index calculation makes the function very interesting. By examining the appearing combinations of representations for various orders of q, we gain information about the dynamics of the theory. The refined versions of the two given MN theories are

$$
\begin{aligned}
I_{0,3}(x,y,z;q) = 1 + (6 + \frac{x_1}{x_2^2} + \frac{1}{x_1x_2} + \frac{x_1^2}{x_2} + \frac{x_2}{x_1^2} + x_1x_2 + \frac{x_2^2}{x_1} + \frac{y_1}{y_2^2} + \frac{1}{y_1y_2} + \frac{y_1^2}{y_2} + \frac{y_2}{y_1^2} + \frac{y_2^2}{y_1} + y_1y_2 \\
+ \frac{z_1}{z_2^2} + \frac{1}{z_1z_2} + \frac{z_1^2}{z_2} + \frac{z_2}{z_1^2} + z_1z_2 + \frac{z_2}{z_1^2} + (\frac{1}{x_1} + \frac{x_1}{x_2} + x_2)(\frac{1}{y_1} + \frac{y_1}{y_2} + y_2) \\
(\frac{1}{z_1} + \frac{z_1}{z_2} + z_2) + (x_1 + \frac{x_2}{x_1} + \frac{1}{x_2})(y_1 + \frac{y_2}{y_1} + \frac{1}{y_2})(z_1 + \frac{z_2}{z_1} + \frac{1}{z_2}))q
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
I_{E_7}(x,y,z;q) = 1 + (7 + \frac{x_1^2}{x_2} + \frac{x_2}{x_1^2} + \frac{x_2}{x_3^2} + \frac{1}{x_1x_3} + \frac{x_1}{x_2x_3} + \frac{x_1x_2}{x_3} + \frac{x_2^2}{x_1x_3} + x_1x_3 + \frac{x_1x_3}{x_2^2} + \frac{x_3}{x_1x_2} \\
+ \frac{x_2x_3}{x_1} + \frac{x_3^2}{x_2} + + \frac{y_1^2}{y_2} + \frac{y_2}{y_1^2} + \frac{y_2}{y_3^2} + \frac{1}{y_1y_3} + \frac{y_1}{y_2y_3} + \frac{y_1y_2}{y_3} + \frac{y_2^2}{y_1y_3} + y_1y_3 + \frac{y_1y_3}{y_2^2} + \frac{y_3}{y_1y_2} \\
+ \frac{y_2y_3}{y_1} + \frac{y_3^2}{y_2} + (\frac{1}{x_2} + x_2 + \frac{x_1}{x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1} + \frac{x_1x_3}{x_2})(\frac{1}{y_2} + y_2 + \frac{y_1}{y_3} + \frac{y_2}{y_1y_3} \\
+ \frac{y_3}{y_1} + \frac{y_1y_3}{y_2}) + \frac{1}{z^2} + z^2 + (\frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + x_3)(\frac{1}{y_1} + \frac{y_1}{y_2} + \frac{y_2}{y_3} + y_3)(\frac{1}{z} + z) \\
+ (x_1 + \frac{x_2}{x_1} + \frac{1}{x_3} + \frac{x_3}{x_2})(y_1 + \frac{y_2}{y_1} + \frac{1}{y_3} + \frac{y_3}{y_2})(\frac{1}{z} + z))q
\end{aligned}
\tag{9}
$$

We realise that the fugacities appearing in the first order of the index $I_{E_7}(x,y,z;q)$ is the adjoint representation of E_7 decomposed into $SU(4)\times SU(4)\times SU(2)$ with the branching rules and the first order of the index, $I_{0,3}(x,y,z;q)$ is the adjoint representation of E_6 decomposed to $SU(3)^3$. After observing this relations between flavor groups and particular algebras we look for the certain transformations between these algebras by finding necessary change of variables. For the E_8 MN theory we follow the same procedure and after obtaining the refined index we make the transformation from $SU(6)\times SU(3)\times SU(2)$ to E_8 and then E_8 to $E_6 \times SU(3)$ by using the branching rules and the transformation matrices provided by LieArt. After gauging the SU(3) part we obtained a theory with the global symmetry E_6.

$$
I'_{E_6}(a;q) = \oint_{|x|=1} d\mu PE[f^V(q)\chi_G(u)] I_{E_8}(a,x;q) \tag{10}
$$

where $d\mu$ is the Haar measure and PE is the plethystic exponential and f^V is the vector multiplet as described in [6]. On the other hand since we are gauging the SU(3) subgroup $\chi_G(u)$ is the adjoint representation of SU(3).

By using the gauging tool we can also get much complicated non-Lagrangian theories and by using the too called partially closing we can change the flavor symmetries of the already existing punctures on the Riemann surface. Furthermore, by gauging a SU(2)

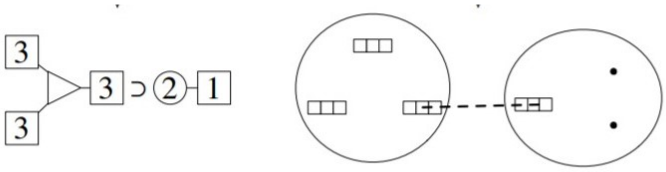

Figure 3: Gauging the E_6 MN Theory

subalgebra of the E_6 MN theory with SU(2) supersymmetric quiver theory (with flavor symmetry U(1)) we obtain the SU(3) supersymmetric gauge theory with flavor symmetry SU(6) and this is the emergence of Argyres-Seiberg Duality in our project as shown in Figure 2.This duality is also known as the weak-strong duality because the SU(3) theory with the coupling constant τ which in the strongly-coupled limit is the weakly-coupled limit of the E_6 MN theory gauged with SU(2).

$$\begin{aligned}&\oint_{|x|=1} d\mu PE[f^V(q)\chi_G(u) + f^{\frac{1}{2}H}(q)\chi_G(u)\chi_R(v)]I_{E_8}(a,u;q)\\&\qquad = \oint_{|x|=1} d\mu PE[f^V(q)\chi_G(u) + f^{\frac{1}{2}H}(q)\chi_G(u)\chi_R(v)]\end{aligned} \tag{11}$$

3 Conclusion

In this project, by using plentiful tools such as gauging, partially closing or branching rules we examine the nature of strange Supersymmetric Quantum Field Theories. By computing different Argyres-Seiberg duality examples we also examine the connection between different quantum field theories. After these calculations, we realize that the grand scheme of quantum field theories are very interesting and promising and we need many further discoveries on the field to fully understand it.

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