$\mathcal{N} = 4$ Super Yang Mills

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Abstract

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The Lagrangian for $\mathcal{N} = 4$ Super Yang Mills with $g = 0$:

$$\mathcal{L}_{g=0} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}_A\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} - \frac{1}{4}\partial_\mu X^{\dagger a}_{AB}\partial^\mu X^{ABa} + F^{Aa}_B F^{*Ba}_A. \tag{1}$$

The super-symmetry transformations of the fields up to constants.

$$\delta X^{ABa} = x_1\delta^{[A}_C\varepsilon^C\lambda^{B]a} + x_2\epsilon^{ABCD}\varepsilon^\dagger_C\lambda^{\dagger a}_D \tag{2}$$

$$\delta X^{\dagger a}_{AB} = x_2\delta^C_{[A}\varepsilon^\dagger_C\lambda^a_{B]} + x_1\epsilon_{ABCD}\varepsilon^C\lambda^{Da} \tag{3}$$

$$X^{\dagger a}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CDa} \implies x_2 = x_1^* \tag{4}$$

$$\delta\lambda^{Aa} = y_1\left(F^{Aa}_B\varepsilon^B + \frac{i}{2}F^a_{\mu\nu}\sigma^\mu\bar{\sigma}^\nu\varepsilon^A\right) + y_2\partial_\mu X^{ABa}\sigma^\mu\varepsilon^\dagger_B \tag{5}$$

$$\delta\lambda^{\dagger a}_A = y_1^*\left(F^{*Ba}_A\varepsilon^\dagger_B - \frac{i}{2}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\nu\sigma^\mu\right) + y_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu \tag{6}$$

$$\delta F^a_{\mu\nu} = z_1\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A - z_1\varepsilon^A\sigma_\mu\partial_\nu\lambda^{\dagger a}_A - z_2\partial_\nu\lambda^{Aa}\sigma_\mu\varepsilon^\dagger_A \tag{7}$$

$$\delta F^{Aa}_B = w_1\varepsilon^A\sigma^\mu\partial_\mu\lambda^{\dagger a}_B + w_2\partial_\mu\lambda^{Aa}\sigma^\mu\varepsilon^\dagger_B \tag{8}$$

$$\delta F^{*Ba}_A = w_1^*\partial_\mu\lambda^{Ba}\sigma^\mu\varepsilon^\dagger_A + w_2^*\varepsilon^B\sigma^\mu\partial_\mu\lambda^{\dagger a}_A \tag{9}$$

To compute

$$\begin{aligned}\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{ABa}\right)}\partial_\mu\delta X^{ABa} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{\dagger a}_{AB}\right)}\partial_\mu\delta X^{\dagger a}_{AB} \\ + \frac{\partial\mathcal{L}}{\partial F^a_{\mu\nu}}\delta F^a_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\lambda^{Aa}\right)}\partial_\mu\delta\lambda^{Aa} \\ + \delta\lambda^{\dagger a}_A\frac{\partial\mathcal{L}}{\partial\lambda^{\dagger a}_A} + \frac{\partial\mathcal{L}}{\partial F^{Aa}_B}\delta F^{Aa}_B + \frac{\partial\mathcal{L}}{\partial F^{*Ba}_A}\delta F^{*Ba}_A\end{aligned} \tag{10}$$

we need to compute

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu X^{ABa}\right)} = -\frac{1}{4}\partial^\mu X^{\dagger a}_{AB} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu X^{\dagger a}_{AB}\right)} = -\frac{1}{4}\partial^\mu X^{ABa} \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial F^a_{\mu\nu}} = -\frac{1}{2}F^{\mu\nu a} \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \lambda^{Aa}\right)} = i\lambda^{\dagger a}_A\bar{\sigma}^\mu \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda^{\dagger a}_A} = i\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial F^{Aa}_B} = F^{*Ba}_A \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial F^{*Ba}_A} = F^{Aa}_B. \tag{17}$$

3 $\delta\mathcal{L}$

3.1 Terms with no X^{ABa} and no F^{Aa}_B

$$\begin{aligned}-\frac{1}{2}y_1^*\varepsilon^\dagger_A\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\partial_\rho\lambda^{Aa}F^a_{\mu\nu} - \frac{1}{2}y_1\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\varepsilon^A \\ +z_1F^{\mu\nu a}\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2F^{\mu\nu a}\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A\end{aligned} \tag{18}$$

Using the identity

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho = -\eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\nu\rho}\bar{\sigma}^\mu + \eta^{\mu\rho}\bar{\sigma}^\nu + i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_\kappa \tag{19}$$

this reduces to

$$\begin{aligned}\frac{1}{2}y_1^*\eta^{\mu\nu}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\rho\partial_\rho\lambda^{Aa} + y_1^*F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\mu\partial^\nu\lambda^{Aa} - \frac{i}{2}y_1^*\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\kappa\partial_\rho\lambda^{Aa} \\ +y_1\partial^\nu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\rho\varepsilon^A + \frac{1}{2}y_1\eta^{\nu\rho}\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\mu\varepsilon^A - \frac{i}{2}y_1\epsilon^{\mu\nu\rho\kappa}\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\kappa\epsilon^A \\ +z_1F^{\mu\nu a}\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2F^{\mu\nu a}\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A\end{aligned} \tag{20}$$

where we have used the anti-symmetry of $F^a_{\mu\nu}$ to combine terms. This anti-symmetry also implies that the terms with $\eta^{\mu\nu}F^a_{\mu\nu}$ are 0 since $\eta^{\mu\nu}$ is 0 when $\mu \neq \nu$. We also have for each κ that $\epsilon^{\mu\nu\rho\kappa}\partial_\mu F^a_{\nu\rho}$ vanishes by the Jacobi identity. Using the product rule we may write

$$-\frac{i}{2}y_1^*\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\kappa\partial_\rho\lambda^{Aa} = -\frac{i}{2}y_1^*\partial_\rho\left(\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\kappa\lambda^{Aa}\right) + \frac{i}{2}y_1^*\epsilon^{\rho\mu\nu\kappa}\partial_\rho F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\kappa\lambda^{Aa} \tag{21}$$

where the first term is a total derivative which upon integration becomes a boundary term at infinity hence does not change the action assuming suitable decay conditions on the fields. Finally using the fact that $\psi\sigma^\mu\xi^\dagger = -\xi^\dagger\bar{\sigma}^\mu\psi$ for any Weyl spinors ψ, ξ we may simplify this expression to

$$y_1^* F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\mu\partial^\nu\lambda^{Aa} + y_1\partial^\nu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\rho\varepsilon^A - z_1 F^{\mu\nu a}\partial_\mu\lambda^{\dagger a}_A\bar{\sigma}_\nu\varepsilon^A - z_2 F^{\mu\nu a}\varepsilon^\dagger_A\bar{\sigma}_\nu\partial_\mu\lambda^{Aa}. \tag{22}$$

The first and last terms clearly cancel if $y_1^* = z_2$ and up to a total derivative $-z_1 F^{\mu\nu a}\partial_\mu\lambda^{\dagger a}_A\bar{\sigma}_\nu\varepsilon^A = z_1\partial_\mu F^{\mu\nu a}\lambda^{\dagger a}_A\bar{\sigma}_\nu\varepsilon^A$ hence the middle terms cancel if $y_1 = -z_1$.

3.2 X^{ABa} terms

$$\begin{aligned}
-\frac{x_1}{2}\partial^\mu X^{\dagger a}_{AB}\varepsilon^A\partial_\mu\lambda^{Ba} - \frac{x_2}{4}\epsilon^{ABCD}\partial^\mu X^{\dagger a}_{AB}\varepsilon^\dagger_C\partial_\mu\lambda^{\dagger a}_D \\
-\frac{x_2}{2}\partial^\mu X^{ABa}\varepsilon^\dagger_A\partial_\mu\lambda^{\dagger a}_B - \frac{x_1}{4}\epsilon_{ABCD}\partial^\mu X^{ABa}\varepsilon^C\partial_\mu\lambda^{Da} \\
+iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar{\sigma}^\nu\partial_\nu\lambda^{Aa} + iy_2\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa}
\end{aligned} \tag{23}$$

Using the relations

$$X^{\dagger a}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CDa} \tag{24}$$

and

$$X^{ABa} = \frac{1}{2}\epsilon^{ABCD}X^{\dagger a}_{CD} \tag{25}$$

this reduces to

$$\begin{aligned}
-x_1\partial^\mu X^{\dagger a}_{AB}\varepsilon^A\partial_\mu\lambda^{Ba} - x_2\partial^\mu X^{ABa}\varepsilon^\dagger_A\partial_\mu\lambda^{\dagger a}_B \\
+iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar{\sigma}^\nu\partial_\nu\lambda^{Aa} + iy_2\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa}.
\end{aligned} \tag{26}$$

First let's consider the term with y_2^*. Using the identity

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = -2\eta^{\mu\nu}I \tag{27}$$

we may write

$$\begin{aligned}
iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar{\sigma}^\nu\partial_\nu\lambda^{Aa} = \frac{1}{2}iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar{\sigma}^\nu\partial_\nu\lambda^{Aa} \\
-\frac{1}{2}iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\nu\bar{\sigma}^\mu\partial_\nu\lambda^{Aa} - iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\partial^\mu\lambda^{Aa}.
\end{aligned} \tag{28}$$

Using our total derivative trick twice on the second term allows us to swap the derivatives at the price of two minus signs, leading to a cancellation of the first two terms up to a total derivative. Using the anti-symmetry of $X^{\dagger a}_{AB}$ we see that the remaining term cancels with the first term in 26 provided that $x_1 = iy_2^*$. Similarly

$$iy_2\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa} = \frac{1}{2}iy_2\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa} - \frac{1}{2}iy_2\lambda^{\dagger a}_A\bar{\sigma}^\nu\sigma^\mu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa} \\ -iy_2\lambda^{\dagger a}_A\varepsilon^\dagger_B\partial^\mu\partial_\mu X^{ABa} \tag{29}$$

using the identity

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = -2\eta^{\mu\nu}I. \tag{30}$$

Now the first two terms immediately cancel by the commutativity of the derivatives and up to a total derivative the last term cancels with the second term in 26 using anti-symmetry of X^{ABa} if $iy_2 = -x_2$ which is consistent with $x_2 = x_1^*$.

3.3 F^{Aa}_B terms

$$iy_1^*F^{*Ba}_A\varepsilon^\dagger_B\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} + iy_1\partial_\mu F^{Aa}_B\lambda^{\dagger a}_A\bar{\sigma}^\mu\varepsilon^B + w_1F^{*Ba}_A\varepsilon^A\sigma^\mu\partial_\mu\lambda^{\dagger a}_B + w_2F^{*Ba}_A\partial_\mu\lambda^{Aa}\sigma^\mu\varepsilon^\dagger_B \\ +w_1^*F^{Aa}_B\partial_\mu\lambda^{Ba}\sigma^\mu\varepsilon^\dagger_A + w_2^*F^{Aa}_B\varepsilon^B\sigma^\mu\partial_\mu\lambda^{\dagger a}_A \tag{31}$$

Immediately there is a cancellation between the y_1^* term and the w_2 term provided $w_2 = iy_1^*$. Similarly after swapping the derivative and the spinors in the w_2^* term it cancels with the y_1 term provided $w_2^* = -iy_1$ which is consistent. We are left with two terms who appear not to cancel

$$w_1F^{*Ba}_A\varepsilon^A\sigma^\mu\partial_\mu\lambda^{\dagger a}_B + w_1^*F^{Aa}_B\partial_\mu\lambda^{Ba}\sigma^\mu\varepsilon^\dagger_A \tag{32}$$

4 Closure of the Super-symmetry Algebra on Shell

We will now show that the extended super-symmetry algebra relation

$$\{Q^A_\alpha, Q^\dagger_{\dot{\beta}B}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta^A_B \tag{33}$$

holds on shell where we have the equations of motion

$$\partial^2 X^{ABa} = \partial^2 X^{\dagger a}_{AB} = 0 \tag{34}$$

$$\partial_\mu F^{\mu\nu a} = 0 \tag{35}$$

$$F^{Aa}_B = F^{*Ba}_A = 0 \tag{36}$$

$$i\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} = i\partial_\mu\lambda^{\dagger a}_A\bar{\sigma}^\mu = 0. \tag{37}$$

References

[1] Study of ... *Author name*