On $\mathcal{N} = 4$ Super Yang-Mills

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Abstract

A super-symmetry transformation is constructed for the $\mathcal{N} = 4$ gauge multiplet and the 4 auxiliary fields present in the $\mathcal{N} = 4$ Super Yang-Mills theory. In the special case where the gauge coupling constant g is set to 0 we check that this symmetry is in fact a symmetry of the action. The appropriate closure of the extended super-symmetry algebra will then be shown on one of the fields.

Contents

1 Introduction

The Lagrangian for the theory can be constructed by first considering the $\mathcal{N}=1$ Lagrangians for a chiral multiplet and a gauge multiplet, since the field content of the $\mathcal{N}=4$ theory can be decomposed into 3 chiral and 1 gauge $\mathcal{N}=1$ multiplets. The Lagrangians for such theories can be found in [1]. The next step is to impose $SU(4)_R$ symmetry by writing the Lagrangian in terms of appropriate representations of $SU(4)_R$, in this case λ^{Aa}, X^{ABa}, A^a_μ and F^{Aa}_B where the index a runs over the adjoint representation of the gauge group. λ^{Aa} contains the four fermions in the theory, F^{Aa}_B the 4 auxiliary fields and X^{ABa} contains the 3 complex scalar fields. X^{ABa} is anti-symmetric and additionally satisfies a self-duality condition:

$$X^{\dagger a}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CDa}. \tag{1}$$

We can now write down the Lagrangian for the theory with $g=0$:

$$\mathcal{L}_{g=0} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}_A\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} - \frac{1}{4}\partial_\mu X^{\dagger a}_{AB}\partial^\mu X^{ABa} + F^{Aa}_B F^{*Ba}_A. \tag{2}$$

2 Super-Symmetry transformations

One is very restricted when trying to write down super-symmetry transformations for these fields. By considering dimensional analysis, representation theory and by additionally requiring the transformations reduce to the $\mathcal{N}=1$ transformations given in [1], the only transformations that exist up to constants are

$$\delta X^{ABa} = x_1\delta^{[A}_C\varepsilon^C\lambda^{B]a} + x_2\epsilon^{ABCD}\varepsilon^\dagger_C\lambda^{\dagger a}_D \tag{3}$$

$$\delta X^{\dagger a}_{AB} = x_2\delta^C_{[A}\varepsilon^\dagger_C\lambda^a_{B]} + x_1\epsilon_{ABCD}\varepsilon^C\lambda^{Da} \tag{4}$$

$$\delta\lambda^{Aa} = y_1\left(F^{Aa}_B\varepsilon^B + \frac{i}{2}F^a_{\mu\nu}\sigma^\mu\bar{\sigma}^\nu\varepsilon^A\right) + y_2\partial_\mu X^{ABa}\sigma^\mu\varepsilon^\dagger_B \tag{5}$$

$$\delta\lambda^{\dagger a}_A = y_1^*\left(F^{*Ba}_A\varepsilon^\dagger_B - \frac{i}{2}F^a_{\mu\nu}\varepsilon^\dagger_A\bar{\sigma}^\nu\sigma^\mu\right) + y_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu \tag{6}$$

$$\delta F^a_{\mu\nu} = z_1\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A - z_1\varepsilon^A\sigma_\mu\partial_\nu\lambda^{\dagger a}_A - z_2\partial_\nu\lambda^{Aa}\sigma_\mu\varepsilon^\dagger_A \tag{7}$$

$$\delta F^{Aa}_B = w_1\varepsilon^A\sigma^\mu\partial_\mu\lambda^{\dagger a}_B + w_2\partial_\mu\lambda^{Aa}\sigma^\mu\varepsilon^\dagger_B \tag{8}$$

$$\delta F^{*Ba}_A = w_1^*\partial_\mu\lambda^{Ba}\sigma^\mu\varepsilon^\dagger_A + w_2^*\varepsilon^B\sigma^\mu\partial_\mu\lambda^{\dagger a}_A \tag{9}$$

where the spinor indices have been suppressed and we already have

$$X^{\dagger a}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CDa} \implies x_2 = x_1^*. \tag{10}$$

3 Computing $\delta\mathcal{L}$

To compute

$$\begin{aligned}\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{ABa}\right)}\partial_\mu\delta X^{ABa} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{\dagger a}_{AB}\right)}\partial_\mu\delta X^{\dagger a}_{AB} \\ +\frac{\partial\mathcal{L}}{\partial F^a_{\mu\nu}}\delta F^a_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\lambda^{Aa}\right)}\partial_\mu\delta\lambda^{Aa} \\ +\delta\lambda^{\dagger a}_A\frac{\partial\mathcal{L}}{\partial\lambda^{\dagger a}_A} + \frac{\partial\mathcal{L}}{\partial F^{Aa}_B}\delta F^{Aa}_B + \frac{\partial\mathcal{L}}{\partial F^{*Ba}_A}\delta F^{*Ba}_A\end{aligned} \tag{11}$$

we need to compute

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{ABa}\right)} = -\frac{1}{4}\partial^\mu X^{\dagger a}_{AB} \tag{12}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu X^{\dagger a}_{AB}\right)} = -\frac{1}{4}\partial^\mu X^{ABa} \tag{13}$$

$$\frac{\partial\mathcal{L}}{\partial F^a_{\mu\nu}} = -\frac{1}{2}F^{\mu\nu a} \tag{14}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\lambda^{Aa}\right)} = i\lambda^{\dagger a}_A\bar{\sigma}^\mu \tag{15}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda^{\dagger a}_A} = i\bar{\sigma}^\mu\partial_\mu\lambda^{Aa} \tag{16}$$

$$\frac{\partial\mathcal{L}}{\partial F^{Aa}_B} = F^{*Ba}_A \tag{17}$$

$$\frac{\partial\mathcal{L}}{\partial F^{*Ba}_A} = F^{Aa}_B. \tag{18}$$

We now organise the resulting terms into collections of terms that must mutually cancel.

3.1 Terms with no X^{ABa} and no F^{Aa}_B

The terms are

$$\begin{aligned}-\frac{1}{2}y_1^*\varepsilon^\dagger_A\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\partial_\rho\lambda^{Aa}F^a_{\mu\nu} - \frac{1}{2}y_1\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho\varepsilon^A \\ +z_1F^{\mu\nu a}\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2F^{\mu\nu a}\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A.\end{aligned} \tag{19}$$

Using the identity

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho = -\eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\nu\rho}\bar{\sigma}^\mu + \eta^{\mu\rho}\bar{\sigma}^\nu + i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_\kappa \tag{20}$$

this reduces to

$$\begin{aligned}\frac{1}{2}y_1^*\eta^{\mu\nu}F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\rho\partial_\rho\lambda^{Aa} + y_1^*F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\mu\partial^\nu\lambda^{Aa} - \frac{i}{2}y_1^*\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\kappa\partial_\rho\lambda^{Aa}\\ +y_1\partial^\nu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar\sigma^\rho\varepsilon^A + \frac{1}{2}y_1\eta^{\nu\rho}\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar\sigma^\mu\varepsilon^A - \frac{i}{2}y_1\epsilon^{\mu\nu\rho\kappa}\partial_\mu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar\sigma^\kappa\epsilon^A\\ +z_1F^{\mu\nu a}\varepsilon^A\sigma_\nu\partial_\mu\lambda^{\dagger a}_A + z_2F^{\mu\nu a}\partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon^\dagger_A\end{aligned} \tag{21}$$

where we have used the anti-symmetry of $F^a_{\mu\nu}$ to combine terms. This anti-symmetry also implies that the terms with $\eta^{\mu\nu}F^a_{\mu\nu}$ are 0 since $\eta^{\mu\nu}$ is 0 when $\mu \neq \nu$. We also have for each κ that $\epsilon^{\mu\nu\rho\kappa}\partial_\mu F^a_{\nu\rho}$ vanishes by the Jacobi identity. Using the product rule we may write

$$-\frac{i}{2}y_1^*\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\kappa\partial_\rho\lambda^{Aa} = -\frac{i}{2}y_1^*\partial_\rho\left(\epsilon^{\mu\nu\rho\kappa}F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\kappa\lambda^{Aa}\right) + \frac{i}{2}y_1^*\epsilon^{\rho\mu\nu\kappa}\partial_\rho F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\kappa\lambda^{Aa} \tag{22}$$

where the first term is a total derivative which upon integration becomes a boundary term at infinity hence does not change the action assuming suitable decay conditions on the fields. Finally using the fact that $\psi\sigma^\mu\xi^\dagger = -\xi^\dagger\bar\sigma^\mu\psi$ for any Weyl spinors ψ, ξ we may simplify this expression to

$$y_1^*F^a_{\mu\nu}\varepsilon^\dagger_A\bar\sigma^\mu\partial^\nu\lambda^{Aa} + y_1\partial^\nu F^a_{\nu\rho}\lambda^{\dagger a}_A\bar\sigma^\rho\varepsilon^A - z_1F^{\mu\nu a}\partial_\mu\lambda^{\dagger a}_A\bar\sigma_\nu\varepsilon^A - z_2F^{\mu\nu a}\varepsilon^\dagger_A\bar\sigma_\nu\partial_\mu\lambda^{Aa}. \tag{23}$$

The first and last terms clearly cancel if $y_1^* = z_2$ and up to a total derivative $-z_1F^{\mu\nu a}\partial_\mu\lambda^{\dagger a}_A\bar\sigma_\nu\varepsilon^A = z_1\partial_\mu F^{\mu\nu a}\lambda^{\dagger a}_A\bar\sigma_\nu\varepsilon^A$ hence the middle terms cancel if $y_1 = -z_1$.

3.2 X^{ABa} terms

The terms are

$$\begin{aligned}-\frac{x_1}{2}\partial^\mu X^{\dagger a}_{AB}\varepsilon^A\partial_\mu\lambda^{Ba} - \frac{x_2}{4}\epsilon^{ABCD}\partial^\mu X^{\dagger a}_{AB}\varepsilon^\dagger_C\partial_\mu\lambda^{\dagger a}_D\\ -\frac{x_2}{2}\partial^\mu X^{ABa}\varepsilon^\dagger_A\partial_\mu\lambda^{\dagger a}_B - \frac{x_1}{4}\epsilon_{ABCD}\partial^\mu X^{ABa}\varepsilon^C\partial_\mu\lambda^{Da}\\ +iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar\sigma^\nu\partial_\nu\lambda^{Aa} + iy_2\lambda^{\dagger a}_A\bar\sigma^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa}.\end{aligned} \tag{24}$$

Using the relations

$$X^{\dagger a}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CDa} \tag{25}$$

and

$$X^{ABa} = \frac{1}{2}\epsilon^{ABCD}X^{\dagger a}_{CD} \tag{26}$$

this reduces to

$$\begin{aligned}-x_1\partial^\mu X^{\dagger a}_{AB}\varepsilon^A\partial_\mu\lambda^{Ba} - x_2\partial^\mu X^{ABa}\varepsilon^\dagger_A\partial_\mu\lambda^{\dagger a}_B\\ +iy_2^*\partial_\mu X^{\dagger a}_{AB}\varepsilon^B\sigma^\mu\bar\sigma^\nu\partial_\nu\lambda^{Aa} + iy_2\lambda^{\dagger a}_A\bar\sigma^\mu\sigma^\nu\varepsilon^\dagger_B\partial_\mu\partial_\nu X^{ABa}.\end{aligned} \tag{27}$$

First let's consider the term with y_2^*. Using the identity

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = -2\eta^{\mu\nu} I \tag{28}$$

we may write

$$\begin{aligned} iy_2^* \partial_\mu X^{\dagger a}_{AB} \varepsilon^B \sigma^\mu \bar{\sigma}^\nu \partial_\nu \lambda^{Aa} &= \frac{1}{2} iy_2^* \partial_\mu X^{\dagger a}_{AB} \varepsilon^B \sigma^\mu \bar{\sigma}^\nu \partial_\nu \lambda^{Aa} \\ -\frac{1}{2} iy_2^* \partial_\mu X^{\dagger a}_{AB} \varepsilon^B \sigma^\nu \bar{\sigma}^\mu \partial_\nu \lambda^{Aa} &- iy_2^* \partial_\mu X^{\dagger a}_{AB} \varepsilon^B \partial^\mu \lambda^{Aa}. \end{aligned} \tag{29}$$

Using our total derivative trick twice on the second term allows us to swap the derivatives at the price of two minus signs, leading to a cancellation of the first two terms up to a total derivative. Using the anti-symmetry of $X^{\dagger a}_{AB}$ we see that the remaining term cancels with the first term in 27 provided that $x_1 = iy_2^*$. Similarly

$$\begin{aligned} iy_2 \lambda^{\dagger a}_A \bar{\sigma}^\mu \sigma^\nu \varepsilon^\dagger_B \partial_\mu \partial_\nu X^{ABa} = \frac{1}{2} iy_2 \lambda^{\dagger a}_A \bar{\sigma}^\mu \sigma^\nu \varepsilon^\dagger_B \partial_\mu \partial_\nu X^{ABa} - \frac{1}{2} iy_2 \lambda^{\dagger a}_A \bar{\sigma}^\nu \sigma^\mu \varepsilon^\dagger_B \partial_\mu \partial_\nu X^{ABa} \\ -iy_2 \lambda^{\dagger a}_A \varepsilon^\dagger_B \partial^\mu \partial_\mu X^{ABa} \end{aligned} \tag{30}$$

using the identity

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = -2\eta^{\mu\nu} I. \tag{31}$$

Now the first two terms immediately cancel by the commutativity of the derivatives and up to a total derivative the last term cancels with the second term in 27 using anti-symmetry of X^{ABa} if $iy_2 = -x_2$ which is consistent with $x_2 = x_1^*$.

3.3 F^{Aa}_B terms

The terms are

$$\begin{aligned} iy_1^* F^{*Ba}_A \varepsilon^\dagger_B \bar{\sigma}^\mu \partial_\mu \lambda^{Aa} + iy_1 \partial_\mu F^{Aa}_B \lambda^{\dagger a}_A \bar{\sigma}^\mu \varepsilon^B + w_1 F^{*Ba}_A \varepsilon^A \sigma^\mu \partial_\mu \lambda^{\dagger a}_B + w_2 F^{*Ba}_A \partial_\mu \lambda^{Aa} \sigma^\mu \varepsilon^\dagger_B \\ + w_1^* F^{Aa}_B \partial_\mu \lambda^{Ba} \sigma^\mu \varepsilon^\dagger_A + w_2^* F^{Aa}_B \varepsilon^B \sigma^\mu \partial_\mu \lambda^{\dagger a}_A. \end{aligned} \tag{32}$$

Immediately there is a cancellation between the y_1^* term and the w_2 term provided $w_2 = iy_1^*$. Similarly after swapping the derivative and the spinors in the w_2^* term it cancels with the y_1 term provided $w_2^* = -iy_1$ which is consistent. We are left with two terms who appear not to cancel

$$w_1 F^{*Ba}_A \varepsilon^A \sigma^\mu \partial_\mu \lambda^{\dagger a}_B + w_1^* F^{Aa}_B \partial_\mu \lambda^{Ba} \sigma^\mu \varepsilon^\dagger_A \tag{33}$$

however this problem can simply be solved by setting $w_1 = 0$. Hence we see that the action is invariant under the super-symmetry transformations

$$\delta X^{ABa} = ix_1\delta_C^{[A}\varepsilon^C\lambda^{B]a} - i\epsilon^{ABCD}\varepsilon_C^\dagger\lambda_D^{\dagger a} \tag{34}$$

$$\delta X_{AB}^{\dagger a} = -i\delta_{[A}^C\varepsilon_C^\dagger\lambda_{B]}^a + i\epsilon_{ABCD}\varepsilon^C\lambda^{Da} \tag{35}$$

$$\delta\lambda^{Aa} = \left(F_B^{Aa}\varepsilon^B + \frac{i}{2}F_{\mu\nu}^a\sigma^\mu\bar{\sigma}^\nu\varepsilon^A\right) + \partial_\mu X^{ABa}\sigma^\mu\varepsilon_B^\dagger \tag{36}$$

$$\delta\lambda_A^{\dagger a} = \left(F_A^{*Ba}\varepsilon_B^\dagger - \frac{i}{2}F_{\mu\nu}^a\varepsilon_A^\dagger\bar{\sigma}^\nu\sigma^\mu\right) + \partial_\mu X_{AB}^{\dagger a}\varepsilon^B\sigma^\mu \tag{37}$$

$$\delta F_{\mu\nu}^a = -\varepsilon^A\sigma_\nu\partial_\mu\lambda_A^{\dagger a} + \partial_\mu\lambda^{Aa}\sigma_\nu\varepsilon_A^\dagger + \varepsilon^A\sigma_\mu\partial_\nu\lambda_A^{\dagger a} - \partial_\nu\lambda^{Aa}\sigma_\mu\varepsilon_A^\dagger \tag{38}$$

$$\delta F_B^{Aa} = i\partial_\mu\lambda^{Aa}\sigma^\mu\varepsilon_B^\dagger \tag{39}$$

$$\delta F_A^{*Ba} = -i\varepsilon^B\sigma^\mu\partial_\mu\lambda_A^{\dagger a} \tag{40}$$

where we had the freedom to set $y_1 = y_2 = 0$.

4 Closure of the Super-symmetry Algebra

We will now show that the extended super-symmetry algebra anti-commutation relation [2]

$$\{Q_\alpha^A, Q_{\dot{\beta}B}^\dagger\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu\delta_B^A \tag{41}$$

holds for the field A_μ^a. To get the transformation in terms of Q and $Q^\dagger$ we use that

$$\delta = \varepsilon Q + \varepsilon^\dagger Q^\dagger. \tag{42}$$

$$\left[\varepsilon_1 Q, \varepsilon_2^\dagger Q^\dagger\right] A_\mu^a = -\varepsilon_1^A\sigma_\mu(\varepsilon_2^\dagger Q^\dagger)\lambda^{Aa} + (\varepsilon_1 Q)\lambda^{Aa}\sigma_\mu\varepsilon_{2A}^\dagger \tag{43}$$

$$= \frac{i}{2}\eta_{\mu\rho}F_{\gamma\nu}^a\varepsilon_1^A\sigma^\rho\bar{\sigma}^\gamma\sigma^\nu\varepsilon_{2A}^\dagger + \frac{i}{2}\eta_{\mu\rho}F_{\gamma\nu}^a\varepsilon_1^A\sigma^\nu\bar{\sigma}^\gamma\sigma^\rho\varepsilon_{2A}^\dagger \tag{44}$$

$$\tag{45}$$

Using

$$\sigma^\nu\bar{\sigma}^\gamma\sigma^\rho = -\eta^{\nu\gamma}\sigma^\rho - \eta^{\gamma\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\gamma - i\epsilon^{\nu\gamma\rho\kappa}\sigma_\kappa \tag{46}$$

this simplifies to

$$-2iF_{\mu\nu}^a\varepsilon_1^A\sigma^\nu\varepsilon_{2A}^\dagger + \frac{1}{2}\eta_{\mu\rho}\epsilon^{\rho\gamma\nu\kappa}F_{\gamma\nu}^a\varepsilon_1^A\sigma_\kappa\varepsilon_{2A}^\dagger + \frac{1}{2}\eta_{\mu\rho}\epsilon^{\nu\gamma\rho\kappa}F_{\gamma\nu}^a\varepsilon_1^A\sigma_\kappa\varepsilon_{2A}^\dagger. \tag{47}$$

After swapping ν and ρ in the last term's Levi-Civita symbol the last two terms cancel and we are left with

$$-2i\partial_\mu A_\nu^a\varepsilon_1^A\sigma^\nu\varepsilon_{2A}^\dagger + 2i\partial_\nu A_\mu^a\varepsilon_1^A\sigma^\nu\varepsilon_{2A}^\dagger \tag{48}$$

where the second term is the desired translation of the A_μ^a field and the first term corresponds to a gauge transformation $A_\mu^a \to A_\mu^a + \partial_\mu V$ for some scalar field V which in this case is $A_\nu^a\varepsilon_1^A\sigma^\nu\varepsilon_{2A}^\dagger$.

References

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[2] J. D. Lykken, “Introduction to supersymmetry,” hep-th/9612114.